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MONOTONE RETRACTIONS AND DEPTH OF CONTINUA

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Abstract. It is shown that for every two countable ordinals α and β with $\alpha > \beta$ there exist λ-dendroids X and Y whose depths are α and β respectively, and a monotone retraction from X onto Y. Moreover, the continua X and Y can be either both arclike or both fans.

1. Introduction

For each hereditarily decomposable and hereditarily unicoherent continuum X an ordinal number $k(X)$ (called the depth of X) is defined in [6], and it is shown that for each continuous mapping f we have $k(f(X)) \leq k(X)$. Since the depth of a continuum is a topological invariant, homeomorphisms do not change the depth of a continuum. Some other mappings, which are – roughly speaking – close enough to homeomorphisms, have the same property. The aim of this paper is to show that monotone retractions are very far from having the property: for every two countable ordinal numbers α and β with $\alpha > \beta$ we construct continua X of depth α and Y of depth β such that there exists a monotone retraction from X onto Y.

S. D. Iliadis has constructed in [6] an uncountable family of hereditarily decomposable arclike continua $A(\alpha)$ numbered with ordinals $\alpha < \omega_1$ having the property that

(1.1) for each $\alpha < \omega_1$ the depth of the α-th member of the family is just α.

We use these continua $A(\alpha)$ to define our continua X and Y mentioned above being arclike.

Another family of continua $F[\alpha]$ which satisfies (1.1) is constructed in [3] such that each its member is a countable plane fan. This family is exploited to show that we can also take countable plane fans for the continua X and Y.

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2. Preliminaries

A *continuum* means a compact connected metrix space. A continuum is said to be *hereditarily unicoherent* provided that the intersection of any two its subcontinua is connected. A continuum X is said to be *hereditarily decomposable* provided that every subcontinuum of X is the union of two its proper subcontinua. A hereditarily decomposale and hereditarily unicoherent continuum is called a *λ-dendroid.* Given a λ-dendroid X we denote by $\mathscr{P}(X)$ the family of all subcontinua S of X such that for each finite cover of X the elements of which are subcontinua of X, the continuum S is contained in a member of the cover. A (transfinite) well-ordered sequence (numbered with ordinals α) of nondegenerate subcontinua X_α of a λ-dendroid X is said to be *normal* provided that the following conditions are satisfied:

(2.1) $X_1 = X$;
(2.2) $X_{\alpha+1} \in \mathscr{P}(X_\alpha)$;
(2.3) $X_\beta = \cap\{X_\alpha : \alpha < \beta\}$ for each limit ordinal β.

The *depth* $k(X)$ of a λ-dendroid X is defined as the minimum ordinal number γ such that the order type of each normal sequence of subcontinua of X is not greater than γ. The reader is referred to [6], [10] and [13] for an additional information related to this concept. The following three important facts concerning the depth will be needed in the present paper. For their proofs see [6], Theorems 1, 2 and 3, p. 94 and 95.

Fact 2.4. *For every two λ-dendroids X and Y if $Y \subset X$, then $k(Y) \le k(X)$.*

Fact 2.5. *A λ-dendroid X is locally connected (i.e., it is a dendrite) if and only if $k(X) = 1$.*

Fact 2.6. *If a λ-dendroid Y is a continuous image a λ-dendroid X, then $k(Y) \le k(X)$.*

Recal that a point c of a continuum X is called a *cut point* of X provided that $X\setminus\{c\}$ is not connected.

Proposition 2.7. *Let a λ-dendroid X contain a cut point c such that the number of components of $X\setminus\{c\}$ is finite, and let $C_1, C_2, \dots, C_n$ denote the closures of the components. Then*

$$k(X) = \max\{k(C_1), k(C_2), \dots, k(C_n)\}\,.$$

Proof. It is enough to observe that since the family $\{C_1, C_2, \dots, C_n\}$ is a finite covering of X by its subcontinua, for each normal sequence $X_1 = X$, $X_2, \dots$ of subcontinua of X the second term of the sequence, X_2, must be contained in C_i for some $i \in \{1, 2, \dots, n\}$.

Let a finite family of spaces $\{X(i) : i \in \{1, \dots, n\}\}$ with fixed points $x(i) \in X(i)$ be given. Then the space obtained from the free union $X(1) + \dots + X(n)$ by identifying all points $x(i)$ to one point v (then called the *glue point*) is called the *one-point-union* of spaces $X(1),\dots,X(n)$. In particular, if $n = 2$, the one-point-union is called the *wedge* of the two spaces and is denoted by $X(1) \vee X(2)$ (see e.g. [7], Example 4, p. 42). Note that if the spaces $X[i]$ are continua, then the one-point-union is a continuum for which v is a cut point. As a consequence of Proposition 2.7 we have the following result. □

Corollarry 2.8. *Let a one-point-union X of λ-dendroids $X(i)$ for $i \in \{1,\dots,n\}$ be given. Then*

$$k(X) = \max\{k(X(i)) : \ i \in \{1,\dots,n,\}\}\,. \tag{2.9}$$

Remark 2.10. It is essential that the number n of components in Proposition 2.7 (and of λ-dendroids in Corollary 2.8) is finite. Indeed, for each straight line segment I we have $k(I) = 1$ by Fact 2.5, while it can be verified that for the harmonic fan F_H (i.e. the cone over the set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$) as well as for the Cantor fan F_C (i.e. the cone over the Cantor ternary set) we have $k(F_H) = k(F_C) = 2$. However, the harmonic fan can be considered as a one-point-union of straight line segments, and we see that in this case formula (2.9) for infinite n is not true; and the Cantor fan can be considered as one-point-union of countably many Cantor fans and of one straight line limit segment, and now formula (2.9) is still valid (for infinite n). So the following question seems to be interesting.

Question 2.11. Let a λ-dendroid X be a one-point-union of infinitely many λ-dendroids $X(i)$, where $i \in I$, and let the depth $k(X(i))$ be given for each $i \in I$. How to calculate $k(X)$?

A *mapping* means a continuous function. A mapping $f : X \to Y$ between continua X and Y is said to be:

a *local homeomorphism*, provided that for each point of the domain space X there is an open neighbourhood U such that $f(U)$ is an open subset of Y and that the partial mapping $f|U : U \to f(U)$ is a homeomorphism;

open, provided that it maps open subsets of the domain onto open subsets of the range;

monotone, provided that for each point $y \in Y$ its inverse image $f^{-1}(y)$ is connected;

a *retraction*, provided that $Y \subset X$ and that each point of Y is a fixed point under f (i.e., the partial mapping $f|Y$ is the identity).

Since the notion of the depth of a continuum is formulated by means of topological concepts, it is a topological invariant, i.e., homeomorphisms do not change the depth of a continuum. Further, it is known that every local homeomorphism from a continuum onto a tree-like continuum (see [1], p. 656, for the definition) is a homeomorphism ([9], (6.1), p. 50), and that each λ-dendroid is tree-like ([5], Corollary, p. 21). Therefore local homeomorphism of λ-dendroids do not change the depth either. Each local homeomorphism is an open mapping (see e.g. [9], (4.26), p. 20). Open mappings can change the depth, as it can be seen from the natural projection of the harmonic fan onto its limit segment.

In connection with the above remarks the following problem seems to be interesting and natural.

Problem 2.12. Let a family $\mathscr{D}$ of λ-dendroids and a class $\mathscr{M}$ of mappings between members of $\mathscr{D}$ be given. Is it true that for every two ordinal numbers α and β with $\alpha > \beta$ there are two members X and Y of $\mathscr{D}$ such that $k(X) = \alpha$ and $k(Y) = \beta$ for which there exists a surjection $f : X \to Y$ belonging to $\mathscr{M}$?

We will show a positive solution to 2.12 in case when $\mathscr{M}$ is the class of monotone retractions and $\mathscr{D}$ is either the family of arclike hereditarily decomposable continua (Section 3), or the family of countable plane fans (Section 4).

3. Arclike continua

A continuum X is said to be *arclike* (also called chainable or snake-like) provided that for each positive number ε there is a chain in X covering X such that each its link has diameter less than ε. It is well-known that each arclike continuum is planable ([1], Theorem 4, p. 654) and hereditarily unicoherent ([1], Theorem 11, p. 660). A point p of an arclike continuum X is called an *end point* of X provided that for each positive number ε there is a chain in X covering X such that each its link has diameter less than ε and that only the first link of the chain contains p.

The following fact is obvious.

Fact 3.1. *It two arclike continua A and B are given with end points a and b respectively, then their wedge $A \vee B$ obtained by identification a and b is arclike.*

The next result is due to S. D. Iliadis (see [6]), Theorem 5, p. 96, and compare also [12], Theorem 24, p. 24).

Theorem 3.2. *For each ordinal number $\gamma < \omega_1$ there exists an arclike hereditarily decomposable continuum $A(\gamma)$ such that $k(A(\gamma)) = \gamma$.*

The main result of this section is the following.

Theorem 3.3. *For every two ordinal numbers α and β with $\omega_1 > \alpha > \beta$ there are hereditarily decomposable arclike continua X and Y such that $k(X) = \alpha$ and $k(Y) = \beta$, for which there exists a monotone retraction from X onto Y.*

Proof. Take the family $\{A(\gamma) : \gamma < \omega_1\}$ of Theorem 3.2. Define X as the wedge $A(\alpha) \vee A(\beta)$ where an end point of $A(\alpha)$ is identified with and end point of $A(\beta)$ to a glue point $p \in X$. Then, according to Fact 3.1, X is an arclike continuum which is hereditarily decomposable (because $A(\alpha)$ and $A(\beta)$ are) and, by Corollary 2.8 we have $k(X) = \max\{k(A(\alpha)), k(A(\beta))\} = \alpha$. Put $Y = A(\beta) \subset X$ and define $f : X \to Y$ by $f(x) = p$ if $x \in Y$ and $f(x) = x$ if $x \in X \setminus Y$. Then f is a monotone retraction we need. The argument is complete. $\square$

4. Fans

A continuum that is hereditarily unicoherent and arcwise connected is called a *dendroid*. It is well-known that every dendroid is hereditarily decomposable, so it is a λ-dendroid. A point p in a dendroid X is called a *ramification point* of X if p is the vertex of a simple triod contained in X (i.e. if there are three points a, b

and c in X such that any two of the three arcs pa, pb and pc have the point p in common only). A dendroid having exactly one ramification points is called a *fan*, and the point is called the *vertex* of the fan. A fan is said to be *countable* provided that the set of all its end points is countable.

The following fact is obvious.

Fact 4.1. *If two fans A and B are given with vertices a and b respectively, then their wedge $A \vee B$ obtained by identification a and b is a fan having the glue point as its vertex. Furthermore, if the fans A and B are countable, then the fan $A \vee B$ is countable, too.*

A proof of the next theorem is given in [3].

Theorem 4.2. *For each ordinal number $\gamma < \omega_1$ there exists a countable plane fan $F[\gamma]$ such that $k(F[\gamma]) = \gamma$.*

Now we are ready to prove the main result of this section of the paper.

Theorem 4.3. *For every two ordinal numbers α and β with $\omega_1 > \alpha > \beta$ there are countable fans X and Y such that $k(X) = \alpha$ and $k(Y) = \beta$, for which there exists a monotone retraction from X onto Y.*

Proof. Take the family $\{F[\gamma] : \gamma < \omega_1\}$ of Theorem 4.2. Define X as the wedge $F[\alpha] \vee F[\beta]$ where the vertex of $F[\alpha]$ is identified with the vertex of $F[\beta]$ to a glue point $p \in X$. Then, according to Fact 4.1, X is a fan having p as its vertex. By Corollary 2.8 we have

$$k(X) = \max\{k(F[\alpha]), k(F[\beta])\} = \alpha\,.$$

Putting $Y = F[\beta] \subset X$ and defining $f : X \to Y$ by the same conditions as in the proof of Theorem 3.3 we complete our argument. □

5. Final remarks and questions

Among various classes of mappings only monotone retractions were considered in the previous two sections. For other classes $\mathscr{M}$ of mappings Problem 2.12 remains open. Below we discuss some particular results and questions related to this problem.

A mapping $f : X \to Y$ between continua X and Y is said to be *confluent* provided that for each subcontinuum Q of Y and for each component C of $f^{-1}(Q)$ we have $f(C) = Q$. All monotone and all open mappings are confluent (see e.g. [9], (3.2) and (3.6), p. 13). It is known that an open, as well as a confluent image of the sin $1/x$-curve S is homeomorphic to the closed unit interval, to S, or to a singleton (see [11], (16.30), p. 555). Thus, in general, confluent mappings of arclike continua do not preserve depth.

Questions 5.1. Does there exist a hereditarily decomposable arclike continuum X of depth $k(X) > 1$ such that for each a) open b) confluent mapping $f : X \to Y$ onto a nondegenerate hereditarily decomposable arclike continuum Y we have $k(X) = k(Y)$?

If an answer to one of these questions is positive, the next questions can be asked.

Questions 5.2. For what ordinal numbers $\alpha < \omega_1$ do there exist hereditarily decomposable arclike continua X of depth $k(X) = \alpha$ such that for each a) open b) confluent mapping $f : X \to Y$ onto a nondegenerate hereditarily decomposable arclike continuum Y we have $k(X) = k(Y)$?

As regards the inequality $\alpha < \omega_1$ let us recall that, answering a question of Iliadis in [6], Remark 3, p. 98, Mohler has proved in [10], Section 3, p. 719 that the depth of arclike λ-dendroids is less than ω_1.

In the rectangular Cartesian coordinate system in the plane denote (for each $n \in \mathbb{N}$) by S_n the straight line segment joining the origin $v = (0,0)$ with the point $(1/n, 1/n^2)$. Then the union $F^\omega = \bigcup\{S_n : n \in \mathbb{N}\}$ is a countable plane fan. Since it is locally connected, we infer from Fact 2.5 that $k(F^\omega) = 1$. It is known that each image of F^ω under an open mapping is homeomorphic to F^ω. Another fan having this property (even for confluent mappings) is the Lelek fan F_L being a subcontinuum of the Cantor fan F_C (defined and studied in [8], Section 9, p. 314-318; for its properties see [4], Corollary, p. 33; compare also [2]). Since $k(F_C) = 2$ (see Remark 2.10), we infer from Fact 2.4 that $k(F_L) = 2$. Thus for $n = 1$ and $n = 2$ there are fans $F(n)$ of depth n having the property that every open image of $F(n)$ is homeomorphic to $F(n)$, so their depth is not changed under open mappings. Then above examples are related to the following questions.

Questions 5.3. For what natural numbers n do there exist fans $F(n)$ of depth n such that **a)** every open image of $F(n)$ is homeomorphic to $F(n)$, **b)** their depth is not changed under open mappings?

Question 5.4. For what ordinal numbers α do there exist λ-dendroids of the depth α such that their depth is not changed under open mappings?

All the above questions are particular cases of the following problem, which is – in a sense – opposite to Problem 2.12, and which can be considered as a program of a future research.

Problem 5.5. For what families $\mathscr{D}$ of λ-dendroids, for what classes $\mathscr{M}$ of mappings between members of $\mathscr{D}$ and for what ordinal numbers α do there exist λ-dendroids in $\mathscr{D}$ of the depth α such that their depth is not changed under mappings belonging to $\mathscr{M}$?

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