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ON THE STRUCTURE OF SOLUTIONS OF A SYSTEM OF THREE DIFFERENTIAL INEQUALITIES

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Abstract. The aim of this paper is to study the global structure of solutions of three differential inequalities with respect to their zeros. New information for the differential equation of the third order with quasiderivatives is obtained, too.

1. Introduction

The aim of this paper is to investigate a global structure of solutions with respect to zeros of a system of differential inequalities

$$\begin{aligned} &\alpha_i y_i'(t) y_{i+1} \geq 0\,, \\ &y_{i+1}(t) = 0 \Rightarrow y_i'(t) = 0, \quad i = 1,2,3, \quad t \in J \end{aligned} \tag{1}$$

where $\alpha_i \in \{-1, 1\}$, $y_4 = y_1$, $J = (a, b)$, $-\infty \leq a < b \leq \infty$.

$y = (y_1, y_2, y_3)$ is called a solution of (1) if $y_i : J \to R$, $R = (-\infty, \infty)$ is locally absolute continuous and (1) holds for all $t \in J$ such that y_i' exists.

Put $y_{i+3k} = y_i$, $i = 1, 2, 3$, $k \in Z$, $Z = \{\dots, -1, 0, 1, \dots\}$.

Two special cases of (1) which are often studied.

(a) A system of three differential equations

$$y_i' = f_i(t, y_1, y_2, y_3), \quad i = 1, 2, 3, \tag{2}$$

$$\begin{aligned} &\alpha_i f_i(t, x_1, x_2, x_3) x_{i+1} \geq 0, \\ &x_{i+1} = 0 \;\Rightarrow\; f_i(t, x_1, x_2, x_3) = 0 \text{ in } D, \quad i = 1, 2, 3 \end{aligned} \tag{3}$$

where $\alpha_i \in \{-1, 1\}$, $x_4 = x_1$, $f_i : D = R \times R^3 \to R$ satisfies the local Carathéodory conditions, $i = 1, 2, 3$. See e.g. [2,5] and the references herein. $y = (y_1, y_2, y_3)$, defined in J, is called a solution of (2) if it is locally absolute continuous and (2) holds for almost all $t \in J$.

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(b) The differential equation with the quasi-derivatives of the third order

$$L_3 x(t) = f(t, x, x', x''), \tag{4}$$

$$\alpha f(t, x_1, x_2, x_3) x_1 \geq 0, \quad f(t, 0, x_2, x_3) = 0 \tag{5}$$

where $\alpha \in \{-1, 1\}$, $f : R \times R^3 \to R$ fulfils the local Carathéodory conditions, $a_j : R \to R$ are continuous, $a_j(t) > 0$ for $t \in R$, $j = 0, 1, 2, 3$ and $L_j x$ is the j-th quasi-derivative of x : $L_0 x = a_0(t) x$, $L_i x = a_i(t)(L_{i-1} x)'$, $i = 1, 2, 3$. Further, suppose that $a_0 \in C^1(R)$ if $f(t, x_1, x_2, x_3) \equiv f(t, x_1, x_2)$ and $a_0 \in C^2(R)$, $a_1 \in C^1(R)$ if f depends in x_3, too.

By the use of a standard transformation we can see that (4), (5) is equivalent to (2), (3): $y_j = L_{j-1} x$, $j = 1, 2, 3$,

$$\begin{aligned} y_1'(t) &= \frac{y_2(t)}{a_1(t)}, \quad y_2'(t) = \frac{y_3(t)}{a_2(t)}, \\ y_3'(t) &= \frac{1}{a_3(t)} f\left(t, \frac{y_1}{a_0}, \frac{y_2}{a_1 a_0} - \frac{a_0' y_1}{a_0^2}, \frac{y_3}{a_0 a_1 a_2} - \frac{y_2}{a_0 a_1}\left(\frac{a_1'}{a_1} + \frac{2a_0'}{a_0}\right) \right. \\ &\quad \left. - \frac{y_1}{a_0^3 a_1}\left(a_0 a_0'' a_1 - 2 a_1 a_0'^2\right)\right) \doteq \bar{f}(t, y_1, y_2, y_3). \end{aligned} \tag{6}$$

Note that $\alpha \bar{f}(t, y_1, y_2, y_3) y_1 \geq 0$, $\bar{f}(t, 0, y_2, y_3) = 0$, $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = \alpha$.

In [2, 4] the structure of oscillatory solutions (defined in the usual sense) has been studied for the differential equation (4), (5) and its special forms. It is shown that there exists a relation between zeros of the derivatives of a solution. Further, in [1] oscillatory solutions of (1) are investigated under the validity of the relation

$$y_i'(t) = 0 \Rightarrow y_{i+1}(t) = 0, \quad i = 1, 2, 3. \tag{7}$$

Especially, it is proved that zeros of y_i, $i = 1, 2, 3$ are simple in some neighbourhood of its cluster point, i.e. if $y_i(t) = 0$, then $y_{i+1}(t) \neq 0$. In [6] non-oscillatory solutions of differential inclusions for which (1) holds are studied.

In the present paper a generalization and an extension of these results to the system (1) are going to be made. Some new results for (4) are gained, too.

Note that (7) is fulfilled if it is supposed that

$$\begin{aligned} \alpha_i f_i(t, x_1, x_2, x_3) x_{i+1} &> 0 \text{ for } x_{i+1} \neq 0, \\ &= 0 \text{ for } x_{i+1} = 0, \quad i = 1, 2, 3 \end{aligned} \tag{8}$$

is valid for (2) instead of (3). Similarly, (7) is valid if

$$\alpha f(t, x_1, x_2, x_3) x_1 > 0 \text{ for } x_1 \neq 0, \ f(t, 0, x_2, x_3) = 0 \tag{9}$$

holds for (4) instead of (5).

Definition. Let $y : (a, b) \to R^3$ be a solution of (1). Then y is called non-continuable if two following relations are valid

(i) either $a = -\infty$ or, if $-\infty < a$, then $\lim\limits_{t\to a^+} \sup \sum\limits_{i=1}^{3} |y_i(t)| = \infty$

(ii) either $b = \infty$ or, if $b < \infty$, then $\lim\limits_{t\to b^-} \sup \sum\limits_{i=1}^{3} |y_i(t)| = \infty$.

y is called trivial if $y_i(t) = 0$ in (a, b), $i = 1, 2, 3$.

In our further considerations the points of the zero initial conditions will play an important role. The "non-trivial" ones are defined in the following

Definition. Let $y : J = (a, b) \to R^3$ be a solution of (1). Let c be such a point that $c \in J$, $y_i(c) = 0$, $i = 1, 2, 3$ holds, and in any neighbourhood I of c there exists $\tau \in I$ such that $\sum\limits_{j=1}^{3} |y_j(\tau)| > 0$. Then c is called Z-point of y.

Let $i \in \{1, 2, 3\}$, $J_1 \subset J$ be either $J_1 = [a_1, b)$ or $J_1 = (a, b_1]$ or $J_1 = [a_1, b_1]$, $a_1, b_1 \in J$. J_1 is called Z-interval of y_i if $y_i = 0$ in J_1 and two following relations hold:

(i) y_i is non-trivial in any left neighbourhood of $t = a_1$ if $J_1 = [a_1, b)$ or $J_1 = [a_1, b_1]$;

(ii) y_i is non-trivial in any right neighbourhood of $t = b_1$ if $J_1 = (a, b_1]$ or $J_1 = [a_1, b_1]$.

Property V is valid in the interval J_1 if there exists an index $i \in \{1, 2, 3\}$ such that J_1 is Z-interval of both y_i, y_{i+1} and $y_{i+2} \neq 0$ in J_1.

Notation. Let y be a solution of (1). Put $Y_1 = y_1$, $Y_2 = \alpha_1 y_2$, $Y_3 = \alpha_1\alpha_2 y_3$, $Y_{i+3k} = Y_i$, $k \in Z$, $i = 1, 2, 3$.

2. Case $\alpha_1\alpha_2\alpha_3 = -1$

In this chapter the case

$$\alpha_1\alpha_2\alpha_3 = -1 \tag{10}$$

will be studied. The validity of (10) will be supposed in all the considerations.

For the study of the structure of solutions of (1) the following types will be defined. Let $y : J = (c, d) \to R^3$.

Type I. Sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $i = 1, 2, 3$, $k = k_i, k_i + 1, \ldots$ exist such that $k_1 = 1$, $k_2 \in \{0, 1\}$, $k_3 \in \{0, k_2\}$, $t_k^i \in J$, $\lim\limits_{k\to\infty} t_k^i = d$ and

$$\begin{cases} t_k^1 \le \bar{t}_k^1 < t_k^3 \le \bar{t}_k^3 < t_k^2 \le \bar{t}_k^2 < t_{k+1}^1, \\ Y_i(t) = 0 \quad \text{for} \quad t \in [t_k^i, \bar{t}_k^i], \quad Y_i(t) \neq 0 \quad \text{for} \quad t \bar{\in} [t_k^i, \bar{t}_k^i], \\ Y_j(t)Y_1(t) > 0 \quad \text{for} \quad t \in (\bar{t}_k^1, t_k^j), \\ \qquad\qquad\quad < 0 \quad \text{for} \quad t \in (\bar{t}_k^j, t_{k+1}^1), \\ j = 2, 3, \quad i = 1, 2, 3, \text{ for all admissible } k \end{cases} \tag{11}$$

hold. Moreover, $\beta_i\gamma_i = -1$ where $\beta_i(\gamma_i)$ is $\operatorname{sign} Y_i$ in the interval $(c, t^i_{k_i})$ (in $(\bar{t}^i_{k_i}, t^i_{k_i+1})$), $i = 1, 2, 3$.

Type II. Sequences $\{t^i_k\}$, $\{\bar{t}^i_k\}$, $i = 1, 2, 3$, $k = k_i, k_i - 1, k_i - 2, \dots$, exist such that $k_1 = 1$, $k_3 \in \{0, 1\}$, $k_2 \in \{0, k_3\}$, $t^i_k \in J$, $\lim\limits_{k\to-\infty} t^i_k = c$ and (11) hold. Moreover, $\beta_i\gamma_i = -1$ where $\beta_i(\gamma_i)$ is $\operatorname{sign} Y_i$ in the interval $(\bar{t}^i_{k_i-1}, t^i_{k_i})((\bar{t}^i_{k_i}, d))$, $i = 1, 2, 3$.

Type III. Sequences $\{t^i_k\}$, $\{\bar{t}^i_k\}$, $i = 1, 2, 3$, $k \in Z$ exist such that $t^i_k \in J$, $\lim\limits_{k\to-\infty} t^i_k = c$, $\lim\limits_{k\to\infty} t^i_k = d$ and (11) holds for $k \in Z$.

Type IV. There exists $\tau \geq c$ such that

$$\left\{\begin{array}{l} Y_i,\ i \in \{1,2,3\} \quad \text{has a finite number of } Z-\text{intervals } [t^i_k, \bar{t}^i_k] \text{ in} \\ (c, \tau) \text{ and (11) holds until } c < \tau\,; \end{array}\right. \tag{12}$$

$|Y_1|$, $|Y_2|$ are non-decreasing, $|Y_3|$ is non-increasing and $Y_1(t)Y_2(t) > 0$, $Y_1(t)Y_3(t) \geq 0$ holds in (τ, d).

Type V. $\sum\limits_{i=1}^{3} |Y_i(t)| > 0$; $|Y_i|$, $i = 1, 2, 3$ are non-increasing,

$$Y_1(t)Y_2(t) \leq 0\,, \quad Y_1(t)Y_3(t) \geq 0\,, \quad Y_2(t)Y_3(t) \leq 0\,, \quad t \in J\,. \tag{13}$$

Type VI. There exists $\tau \geq c$ such that (12) holds; $|Y_1|$, $|Y_3|$ are non-decreasing, $|Y_2|$ is non-increasing,

$$Y_1(t)Y_2(t) \geq 0\,, \quad Y_1(t)Y_3(t) < 0 \quad \text{in} \quad (\tau, b)\,.$$

Type VII. There exists $\tau \geq c$ such that (12) holds; $|Y_1|$ is non-increasing, $|Y_2|$, $|Y_3|$ are non-decreasing and

$$Y_1(t)Y_2(t) \leq 0\,, \quad Y_2(t)Y_3(t) > 0 \quad \text{in} \quad (\tau, b)\,.$$

Type VIII. y is trivial in J.

Remark 1. The solutions of Type either I or III are ussually called oscillatory, the ones of Types IV-VII are non-oscillatory. The solution of Type V, $b = \infty$ is called Kneser solution.

Definition. Let $y : (a, b) \to R^3$ be a solution of (1), A_i be one of Types I-VIII, $i = 0, 1, 2, \dots, s$. Then y is of Type $\{A_1, A_2, \dots, A_s\}$ in (a, b) if the index j, $j \in \{1, 2, \dots, s\}$ exists such that y is of Type A_j in (a, b). Y is successively of Types $A_1, A_2, \dots, A_{s-1}$ and A_s if numbers $\tau_0, \dots \tau_s$ exist such that $a = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{s-1} \leq \tau_s = b$, y is of Type A_j in (τ_{j-1}, τ_j), $j = 1, 2, \dots, s$. At the same time if y is of Type A on (τ, τ), then Type A is missing.

Let us start with some lemmas.

Lemma 1. *Let y be a solution of* (1) *defined in an interval J.*

(a) *Let $j \in \{2,3\}$ and $Y_j(t) \geq 0 \quad (\leq 0)$ on J. Then Y_{j-1} is non-decreasing (non-increasing) on J.*

(b) *If $Y_1 \geq 0 \quad (Y_1 \leq 0)$ in J, then Y_3 is non-increasing (non-decreasing) in J.*

Proof. Let $j = 2$, $Y_2(t) = \alpha_1 y_2(t) \geq 0$ in J. As, by the use of (1) $\alpha_1 y_1'(t) y_2(t) \geq 0$, we have $y_1' = Y_1' \geq 0$ for almost all $t \in J$. In the other cases the proof is similar (in (b) the assumption (10) must be used, too). □

Remark 2. The following conclusions follow directly from Lemma 1.

Let $y : (c,d) \to R^3$ be a solution of (1).

(i) Let y be either of Type IV, $i = 3$ or of the Type VI, $i = 2$ or of Type VII, $i = 1$. If t_0, $t_0 \in (c,d)$ exists such that $y_i(t_0) = 0$, then $y_i(t) = 0$ in $[t_0, d)$.

(ii) Let y be of Type V, $i \in \{1,2,3\}$ and let t_0, $t_0 \in (c,d)$ exist such that $y_i(t_0) = 0$. Then $y_i(t) = 0$ in $[t_0, d)$.

Lemma 2. *Let $y : [t_1, t_2] \to R^3$ be a solution of* (1), *$i \in \{1,2,3\}$, $Y_i(t_1) = Y_{i+1}(t_1) = 0$, $Y_{i+2}(t) \neq 0$ in $[t_1, t_2]$. Then either*

$$Y_i \equiv Y_{i+1} \equiv 0 \text{ in } [t_1, t_2]\,, \tag{14}$$

or there exists a number τ such that $t_1 \leq \tau < t_2$, $Y_i(t) = Y_{i+1}(t) = 0$ in $[t_1, \tau]$, $(-1)^{i+1} Y_{i+1}(t) Y_{i+2}(t) > 0$ in $(\tau, t_2]$.

Proof. Suppose that (14) is not valid, $i = 1$ and $Y_3(t) > 0$ in $[t_1, t_2]$ holds for the simplicity. Then by the use of Lemma 1 the function Y_2 is non-decreasing, $Y_2 \geq 0$ in $[t_1, t_2]$ and Y_1 is non-decreasing, too. Thus, there exists τ, $t_1 \leq \tau < t_2$ such that $Y_1(t) = Y_2(t) = 0$ on $[t_1, \tau]$ and

$$Y_1^2(t) + Y_2^2(t) > 0 \quad \text{in} \quad (\tau, t_2]\,. \tag{15}$$

Suppose that $Y_2 = 0$ at some right neighbourhood J of τ. By the use of (1) we have $y_1'(t) = 0$ in J, and thus $y_1(t) = 0$, $Y_1(t) = 0$ in J. This contradiction to (15) proves the statement for $i = 1$. For the other i the proof is similar. □

Lemma 3. *Let* (10) *be valid, $y : J = [a,b) \to R^3$, $b \leq \infty$ be a solution of* (1) *such that $\sum\limits_{i=1}^{3} |y_i(t)| \neq 0$ in J and let the following relation be not valid for $t = a$:*

$$\alpha_1 y_1 y_2 < 0\,, \quad \alpha_1 \alpha_2 y_1 y_3 > 0\,. \tag{16}$$

Then y is successively of Types V and {I, IV, VI, VII}.

Proof. Let us investigate y under the validity of all possible Cauchy initial conditions at $t = a$. These conditions will be expressed by the use of the functions Y_i, $i = 1,2,3$.

1° $Y_1(a)Y_3(a) \geq 0, \quad Y_2(a)Y_3(a) > 0$
2° $Y_1(a)Y_2(a) > 0, \quad Y_1(a)Y_3(a) \leq 0$
3° $Y_1(a)Y_2(a) \leq 0, \quad Y_1(a)Y_3(a) < 0$
4° $Y_1(a) = 0, \quad Y_2(a)Y_3(a) < 0$
5° $Y_2(a) = 0, \quad Y_1(a)Y_3(a) > 0$
6° $Y_1(a)Y_2(a) < 0, \quad Y_3(a) = 0$
7° $Y_1(a) = Y_2(a) = 0, \quad Y_3(a) \neq 0$
8° $Y_1(a) \neq 0, \quad Y_2(a) = Y_3(a) = 0$
9° $Y_1(a) = Y_3(a) = 0, \quad Y_2(a) \neq 0\,.$

The conditions $Y_i(a) = 0$, $i = 1,2,3$ and $Y_1(a)Y_2(a) < 0$, $Y_1(a)Y_3(a) > 0$ cannot be valid with respect to the assumptions of the lemma.

Ad 1°. Suppose that $Y_1(a) \geq 0$, $Y_2(a) > 0$, $Y_3(a) > 0$ (the opposite case can be studied similarly). Then we have either $Y_1(t) \equiv 0$, $Y_j(t) > 0$, $j = 2,3$ in J (Type IV), or there exists $\bar{t}_0$ such that (see Lemma 1)

$$
\begin{aligned}
&Y_1(t) = 0, \quad Y_2(t) > 0, \quad Y_3(t) > 0 \quad \text{in} \quad [a, \bar{t}_0] \\
&Y_1(t) > 0 \quad \text{in some right neighbourhood of} \quad \bar{t}_0\,.
\end{aligned}
$$

In this case, according to Lemma 1, $Y_1 > 0$, $Y_2 > 0$ are non-decreasing and $Y_3 > 0$ is non-increasing for $t > \bar{t}_0$ until $Y_2 > 0$. Thus y is either of Type IV ($Y_j > 0$, $j = 1,2$, $Y_3 \geq 0$ in J) or there exists a number t_3, $t_3 > \bar{t}_0$ such that

$$Y_1(t_3) > 0, \quad Y_2(t_3) > 0, \quad Y_3(t_3) = 0\,.$$

By the repetition of the considerations the following conclusions can be proved in the same way: either y is one of Types IV, VI, VII or numbers $\bar{t}_3$, t_2, $\bar{t}_2$, t_0 exist such that $t_3 \leq \bar{t}_3 < t_2 \leq \bar{t}_2 < t_0$,

$$
\left\{
\begin{aligned}
&Y_1(t) > 0, \quad Y_2(t) > 0, \quad Y_3(t) = 0 \quad \text{in} \quad [t_3, \bar{t}_3] \\
&Y_1(t) > 0, \quad Y_2(t) > 0, \quad Y_3(t) < 0 \quad \text{in} \quad (\bar{t}_3, t_2)
\end{aligned}
\right. \tag{17}
$$

$$
\begin{gathered}
\left\{
\begin{aligned}
&Y_1(t) > 0, \quad Y_2(t) = 0, \quad Y_3(t) < 0 \quad \text{in} \quad [t_2, \bar{t}_2] \\
&Y_1(t) > 0, \quad Y_2(t) < 0, \quad Y_3(t) < 0 \quad \text{in} \quad (\bar{t}_2, t_0)
\end{aligned}
\right. \\
Y_1(t_0) = 0, \quad Y_2(t_0) < 0, \quad Y_3(t_0) < 0\,.
\end{gathered} \tag{18}
$$

It is evident that the same Cauchy initial conditions (with respect to signs) at t_0 are valid as in $t = a$. Thus by the repetition of these considerations we can see that the statement of the lemma is valid in the case 1°. According to the assumption $\sum\limits_{i=1}^{3} |y_i(t)| > 0$ of the lemma $\lim\limits_{k\to\infty} t_k^i = b$ must be valid if y is of Type I.

Ad 2°, 3°. The conditions are met in the case 1°, see (17), (18).

Ad 4°. For the simplicity, let $Y_2(a) > 0$ be valid. According to Lemma 1 $Y_3 < 0$, Y_3 is constant, Y_2 is non-increasing for $t \geq a$ until $Y_1 \equiv 0$. Thus either $Y_1 \equiv 0$, $Y_2 > 0$, $Y_3 < 0$ in J (Type VI) or there exists τ, $a < \tau < b$ such that

$$Y_1(\tau) = 0, \quad Y_2(\tau) = 0, \quad Y_3(\tau) < 0\,, \tag{19}$$

or there exists τ, $a < \tau < b$ such that

$$Y_1 > 0, \quad Y_2 > 0, \quad Y_3 < 0 \tag{20}$$

holds in some right neighbourhood of $t = \tau$. The case (20) is studied in 2°. Let (19) be valid. Then according to Lemma 2 either $Y_1 \equiv Y_2 \equiv 0$, $Y_3 < 0$ in J (Type IV) or there exists τ_1, $\tau \le \tau_1 < b$ such that we have $Y_1(t) = Y_2(t) = 0$, $Y_3(t) < 0$ in $[\tau, \tau_1], Y_1(t) \le 0$, $Y_2(t) < 0$, $Y_3(t) < 0$ in some right neighbourhood of $t = \tau_1$. But this case is studied in 1°.

Ad 5°, 6°. These cases can be studied similarly to 4°.

Ad 7°, 8°, 9°. The case 7° is met in 4°, see (19). Similarly, the cases 8°, 9° are investigated in 5°, 6°, respectively. The lemma is proved. □

Remark 3. (i) It is seen from the proof of the Lemma 3 that the following statement is valid.

Let $i \in \{1, 2, 3\}$ and (7) be valid only for i. Then $t_k^i = \bar{t}_k^i$, $k \in N$.

(ii) Let $y : [a, b) \to R^3$ be a non-continuable solution of the Type {I, IV, VI, VII}. Then b may be also finite as it is seen from the following example. For such solutions for (2) see [5].

Example. $y_1' = 0$, $y_2' = y_2^2 y_3$, $y_3' = 0$. Thus we can put $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = -1$. The solution $y_1 \equiv 0$, $y_2 = \frac{1}{1-t}$, $y_3 \equiv 1$, defined in $(-\infty, 1)$ is non-continuable.

Lemma 4. *Let $y : (a, b] \to R^3$ be a solution of* (1), (13) *hold at b and* $\sum_{i=1}^{3} |y_i(b)| > 0$. *Then y is of Type V on (a, b).*

Proof. Let $i \in \{1, 2, 3\}$, $Y_i(t) = 0$ on $J = (\tau_i, b]$, $a \le \tau_i$, $Y_i \ne 0$ in (a, τ_i). If such τ_i do not exist, put $\tau_i = b$. According to Lemma 1 Y_{i+2} is constant and by the use of (13) and Lemma 1 $Y_{i+1}(t)Y_{i+2}(t) \le 0$ in J. Thus the statement is valid in $(\tau, b]$, $\tau = \min_{1 \le i \le 3} \tau_i$. Let $\tau > a$. Thus (13) holds at $t = \tau$ and $Y_i \ne 0$, $i = 1, 2, 3$ in some left neighbourhood J_1 of $t = \tau$. From this, according to Lemma 1, $Y_1(t)Y_2(t) < 0$, $Y_1(t)Y_3(t) > 0$ in J_1. We prove indirectly that these inequalities hold in (a, τ). Thus suppose that there exists $\tau_1 \in (a, \tau)$ such that

$$Y_1(\tau_1) = 0, \quad Y_1(t) > 0, \quad Y_2(t) \le 0 \quad \text{for} \quad t \in (\tau_1, \tau)\,. \tag{21}$$

In the other cases the proof is similar. From this and from Lemma 1 Y_1 is non-increasing in (τ_1, τ) that contradicts to (21). □

Lemma 5. *Let $y : [c, d] \to R^3$, $c < d$ be a solution of* (1),

$$\sum_{i=1}^{3} |y_i(t)| > 0 \text{ in } (c, d)\,. \tag{22}$$

(i) *If c is the Z-point of y then y is of the Type II in some right neighbourhood of c.*

(ii) *If d is the Z-point of y then y is of the Type either I or V in some left neighbourhood of d.*

Proof. (i) At first, we consider that y_1 do not change its sign in some right neighbourhood J of $t = c$, e.g.

$$y_1 \geq 0 \quad \text{in} \quad J\,. \tag{23}$$

Then, by the use of Lemma 1, we have successively: Y_i is non-increasing, $Y_i \leq 0$, $i = 3, 2, 1$ in J. From this and from (23) $y_1 \equiv 0$ holds in J. As $y_2(c) = y_3(c) = 0$, it follows from Lemma 1 that $y_2 \equiv y_3 \equiv 0$ in J. The contradiction to (22) proves that there exists a sequence $\{t_k^1\}$, $k = 0, -1, -2, \dots$ of zeros of y_1 tending to c. The behaviour of y in $[t_k^1, t_0^1]$, $k \in N$ is studied in Lemma 3 and thus y is of Type II in J.

(ii) If y is not of Type V in a left neighbourhood of $t = d$, then there exists τ, $c \leq \tau < d$ such that (16) does not hold at $t = \tau$. With respect to (22) the behaviour of y in $[\tau, b)$ is studied by Lemma 3. As $y_i(d) = 0$, $i = 1, 2, 3$ Types IV, VI, VII are impossible and y must be of Type I in some left neighbourhood of $t = d$. The lemma has been proved. □

Theorem 1. *Let* (10) *be valid and let* $y : (a, b) \to R^3$ *be a non-trivial solution of* (1).

(i) Let Z-points of y *do not exist in* (a, b). *Then* y *is successively of Types {V, II, IV, VI, VII} and {I, IV, VI, VII} in* (a, b).

(ii) Let τ, $\tau \in (a, b)$ *be Z-point of* y *and* (22) *hold in* (a, τ). *Then* y *is either of the Type V in* (a, τ) *or there exists* τ_1, $a \leq \tau_1 < \tau$ *such that* y *is of Type I in* (τ_1, τ) *and of Type {II, IV, V, VI, VII} in* (a, τ_1). *Moreover, if* y *is of Type V in* (a, τ) *then the inequalities* (13) *are sharp.*

(iii) Let τ, τ_1, $a < \tau < \tau_1 < b$ *be Z-points of* y *such that* (22) *holds in* (τ, τ_1). *Then* y *is of Type III in* (τ, τ_1).

(iv) Let τ, $a < \tau < b$ *be Z-point of* y *such that* (22) *holds in* (τ, b). *Then* τ_1, $a < \tau_1 \leq b$ *exists such that* y *is of Type II in* (τ, τ_1) *and of Type {I, IV, VI, VII} in* (τ_1, b).

(v) Then there exists at most one maximal interval $J \subset (a, b)$ *with Property V.*

Proof. (i) According to the assumptions (22) holds for $t \in (a, b)$. Let $c \in (a, b)$. If the Cauchy initial conditions at $t = c$ do not fulfil (16), then by the use of Lemma 3 y is successively of Types V and {I, IV, VI, VII} in $[c, b)$. Let (16) be valid at c. Then y is either of the Type V in $[c, b)$, or τ, $\tau > c$ exists such that $y_1(\tau)y_2(\tau)y_3(\tau) = 0$. As (22) is valid at τ, the structure of y in $[\tau, b)$ is studied by Lemma 3. Thus y is successively of Types V and {I, IV, VI, VII} in $[c, b)$. The considerations about the structure of y in $(a, c]$ can be made similarly to Lemma 3 (use also Lemma 4).

(ii) The first statement follows from the proved part (i) and Lemma 5(ii). Let y be of Type V in (a, τ). We prove by the indirect proof that the inequalities (13) are sharp. Thus suppose that there exists a left neighbourhood J of τ such that $y_1(t) = 0$ in J (see Remark 2, (ii), too). Then $Y_2(\tau) = Y_3(\tau) = 0$, $Y_2 \leq 0$, $Y_3 \geq 0$

in J. By the use of Lemma 1 and from this, we have successively: $Y_3 \equiv 0$, $Y_2 \equiv 0$ in J. The contradiction to (22) proves this part.

(iii) The statement is a consequence of Lemmas 4 and 5.

(iv) The conclusion follows directly from the proved part (i) and Lemma 5(i).

(v) The interval with Property V may exist only in the Type V. The result follows from this and from Remark 2, (ii). □

Remark 4. If y is of Type {IV, VI, VII} in some right neighbourhood of a in the cases (i), (ii) of Theorem 1, then the number τ from the definition of these cases is equal to a.

Theoretically, an infinite number of Z-points may exist. The following theorem gives some conditions for the system of differential equations (2) under which Z-points do not exist. Thus it solves the problem of uniqueness of the Cauchy problem with zero conditions.

Theorem 2. *Let $\varepsilon > 0$, $\bar{\varepsilon} > 0$, $K > 0$ and y be a non-trivial solution of* (2), (3), (10) *defined in (a, b). Let continuons functions $a_i : R \times [0, \varepsilon]^2 \to R_+$, $g_i : [0, \varepsilon] \to R_+$, $i = 1, 2, 3$ exist such that g_i are non-decreasing,*

$$(24)\quad |f_i(t, x_1, x_2, x_3)| \leqq a_i(t, |x_i|, |x_{i+2}|) g_i(|x_{i+1}|) \quad \textit{in} \quad R \times [-\varepsilon, \varepsilon]^3, \quad i = 1, 2, 3$$

and

$$(25)\qquad g_1(\bar{\varepsilon} g_2(\bar{\varepsilon} g_3(z))) \leq K z, \quad z \in [0, \varepsilon]$$

hold. Then y has no Z-point in (a, b) and the statement of Theorem 1, (i) is valid.

Proof. On the contrary, suppose that Z-point $\tau \in (a, b)$ exists. Without loss of generality we can suppose that there exists a right neighbourhood of τ in which y is not trivial (for the left neighbourhood the proof is similar).

As $y_i(\tau) = 0$, $i = 1, 2, 3$ then an interval $J_1 = [\tau, \tau + \delta]$, $\delta > 0$ exists such that

$$(26)\qquad |y_i(t)| \leq \varepsilon, \quad t \in J_1 .$$

By the use of Lemma 1 y_1 is not trivial in any right neighbourhood of τ. Let ε_1 and $J = [\tau, \tau + \delta_1]$, $0 < \delta_1 \leq \delta$ be such that $0 < \varepsilon_1 \leq \bar{\varepsilon}$,

$$(27)\quad \varepsilon_1 K < 1, \quad \varepsilon_1 \max_{\substack{0 \leq s \leq \varepsilon \\ j=1,2,3}} g_j(s) \leq \varepsilon, \quad \max_{j=1,2,3} \int_J \max_{0 \leq x_1, x_2 \leq \varepsilon} a_j(t, x_1, x_2) dt \leq \varepsilon_1 .$$

Then by the use of (24), (26)

$$\begin{aligned} |y_i(t)| &\leqq \int_\tau^t |f_i(t, y_i(t), y_{i+1}(t), y_{i+2}(t))| \, dt \leq \int_J a_i(t, |y_i(t)|, |y_{i+2}(t)|) \, dt \; \times \\ &\times \; g_i(\max_{s \in J} |y_{i+1}(s)|), \quad t \in J, \quad i = 1, 2, 3 . \end{aligned}$$

Thus
$$\max_{s\in J}|y_i(s)| \le \varepsilon_1 g_i(\max_{s\in J}|y_{i+1}(s)|), \quad i=1,2,3\,.$$
From this and by the use of (25), (27) we get: $\nu = \max_{s\in J}|y_1(s)| > 0$,
$\nu \le \varepsilon_1 g_1(\varepsilon_1 g_2(\varepsilon_1 g_3(\nu))) \le \varepsilon_1 g_1(\bar\varepsilon g_2(\bar\varepsilon g_3(\nu))) \le \varepsilon_1 K\nu < \nu$. This contradiction proves the theorem. □

The structure of solutions of (4), (5) (or (6)) can be described more precisely. Note that $\alpha_1=\alpha_2=1$, $\alpha_3=\alpha=-1$.

Theorem 3. *Let $\alpha=-1$, $x:(a,b)\to R$ be a non-trivial solution of* (4), (5).

(i) Let $\varepsilon>0$, $K>0$ and let continuous functions $d: R_+\times[0,\varepsilon]^2\to R_+$, $g:[0,\varepsilon]\to R_+$ exist such that g is non-decreasing, $|f(t,x_1,x_2,x_3)|\le d(t,|x_2|,|x_3|)g(|x_1|)$, $t\in R$, $|x_i|\le\varepsilon$, $i=1,2,3$, and $g(x_1)\le Kx_1$ for $x_1\in[0,\varepsilon]$ hold. Let $y_i=L_{i-1}x$, $i=1,2,3$. Then x has no Z-point on (a,b) and the statement of Theorem 1, (i) is valid.

(ii) Let $\frac{a_2}{a_1}\in C^1(R)$. Then x has at most one Z-interval and one of the two following relations holds:

1° *y is successively of Types {V, II, IV, VI, VII} and {I, IV, VI, VII} in (a,b),*
2° *y is successively of Types V, VIII, II and {I, IV, VI, VII} in (a,b); if y is not of Type VIII in some left neighbourdhood of b, then Types VIII and II are both present.*

Proof. (i) It follows from (6) that the relation
$$x(\tau)=x'(\tau)=x''(\tau)=0 \Longleftrightarrow y_i(\tau)=0, \quad i=1,2,3$$
holds. Put $g_1(x)=g_2(x)=x$, $g_3\equiv g$. From this the statement is a consequence of Theorem 2.

(ii) Let $\tau\in(a,b)$ be Z-point such $\sum_{i=1}^{3}|y_i(t)|>0$ in some left neighbourhood J of $t=\tau$. According to Lemma 5 y is either of Type I or of Type V. We prove by the indirect proof that it is of Type V. Thus suppose that y is of Type I in J. Let $J_1=[\alpha,\tau]$, $J_1\subset\bar J$ be such interval that

$$\frac{3}{2}\min_{s\in J}\frac{1}{a_1(s)} - \frac{1}{2}\int_\alpha^\tau \frac{ds}{a_2(s)}\max_{s\in J}\left|\left(\frac{a_2(s)}{a_1(s)}\right)'\right| \ge 0\,. \tag{28}$$

Let us define for $t\in J_1$
$$F(t) = -\int_\alpha^t \frac{ds}{a_2(s)}y_3(t)y_1(t) + \frac{1}{2}\frac{a_2(t)}{a_1(t)}\int_\alpha^t\frac{ds}{a_2(s)}y_2^2(t) + y_1(t)y_2(t)\,.$$
Then, by the use of (4), (5), (6) and (28) we have for $t\in J_1$
$$F'(t) = -\int_\alpha^t\frac{ds}{a_2(s)}y_3'y_1 + \left[\frac{3}{2a_1(t)} + \frac{1}{2}\left(\frac{a_2(t)}{a_1(t)}\right)'\int_\alpha^t\frac{ds}{a_2(s)}\right]y_2^2(t) \ge 0\,.$$

Thus F is non-decreasing. It follows from (11) that we have for an arbitrary zero $\beta = t_k^1$ of y_1, $\beta \in (\alpha, \tau)$

$$F(\beta) = \frac{1}{2}\frac{a_2(\beta)}{a_1(\beta)}\int_\alpha^\beta \frac{ds}{a_2(s)} y_2^2(\beta) > 0, \quad F(\tau) = 0\,,$$

and we receive the contradiction to F being non-decreasing. Thus y is of Type V in J and by use of Lemma 4 y is of Type V in (a,τ). From this there exists at most one Z-interval in (a,b) and the statement follows from Theorem 1. □

Remark 5. (i) Let y be a solution of (4) of Type {I, II, III, IV, VI, VII}. Then it follows from Remark 3(i) that $t_k^i = \bar{t}_k^i$, $k \in N$, $i = 1,2$ holds (see (11)). Moreover, if (9) is valid, then $t_k^3 = \bar{t}_k^3$, too.

(ii) Theorem 1 generalizes and enlarges some results of [1]. Theorem 3 generalizes some results of [4] (for (4)) and of [2] (for the differential equation of the third order).

(iii) Theorem 2 generalizes the well-known condition for the non-existence of Z points, see [2, 5]:

$$\varepsilon > 0, \quad |f_i(t, x_1, x_2, x_3)| \le d_i(t)\sum_{j=1}^{3}|x_j|, \quad t \in R, \quad |x_i| \le \varepsilon, \quad i = 1,2,3.$$

(iv) Some conditions are given for (4) in [6] under which solutions of Types VI, VII do not exist. The paper [3] contains conditions under which solutions of (2) of Types III, VI, VII, $b = \infty$ do not exist (so called Property A of (2)).

(v) Let $y : (a,b) \to R^3$ be non-continuable solution of (4), (5) and be of Type IV in some left neighbourhood of b. Then $b = \infty$ (use (6)).

3. Case $\alpha_1\alpha_2\alpha_3 = 1$

This chapter is devoted to the case

$$\alpha_1\alpha_2\alpha_3 = 1\,. \tag{29}$$

The results will be only given. The proofs are similar to Chapter 2, or we can use the transformation of the independent variable $T = -t$, $t \in (a,b)$, $y(t) = \bar{y}(T)$. Then (1) is transformed into $-\alpha_i \bar{y}_i'(T)\bar{y}_{i+1}(T) \ge 0$, $\bar{y}_{i+1}(T) = 0 \Rightarrow \bar{y}_i'(T) = 0$, $i = 1,2,3$, $T \in (-b,-a)$. Thus the system has the same form as (1), the formula (10) is transformed into (29). This transformation conserves zeros, Z-points and Z-intervals.
Let us consider the following types of solutions of (1). Let $y : J = (c,d) \to R^3$.

Type I. Sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $i=1,2,3$, $k=k_i,k_i-1,k_i-2,\dots$ exist such that $k_1=1$, $k_2\in\{0,1\}$, $k_3\in\{0,k_2\}$, $t_k^i\in J$, $\lim\limits_{k\to-\infty} t_k^i=c$ and

$$(30)\qquad \begin{cases} \bar{t}_{k-1}^1 < t_k^2 \le \bar{t}_k^2 < t_k^3 \le \bar{t}_k^3 < t_k^1 \le \bar{t}_k^1\,, \\ Y_i(t)=0 \quad \text{for} \quad t\in[t_k^i,\bar{t}_k^i], \quad Y_i(t)\ne 0 \quad \text{for} \quad t\,\bar{\in}\,[t_k^i,\bar{t}_k^i]\,, \\ (-1)^{j-1}Y_j(t)Y_1(t)>0 \quad \text{for} \quad t\in(\bar{t}_{k-1}^1,t_k^j) \\ \qquad\qquad\qquad\qquad <0 \quad \text{for} \quad t\in(\bar{t}_k^j,t_k^1)\,, \\ j=2,3; \quad i=1,2,3, \quad \text{for all admissible} \quad k \end{cases}$$

holds. Moreover $\beta_i\gamma_i=-1$ where $\beta_i(\gamma_i)$ is $\operatorname{sign} Y_i$ in the interval $(\bar{t}_{k_{i-1}}^i,t_{k_i}^i)$ (in $(\bar{t}_{k_i}^i,d)$), $i=1,2,3$.

Type II. Sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $i=1,2,3$, $k=k_i,k_{i+1},\dots$ exist such that $k_1=1$, $k_3\in\{0,1\}$, $k_2\in\{0,k_2\}$, $t_k^i\in J$, $\lim\limits_{k\to\infty} t_k^i=d$ and (30) hold. Moreover $\beta_i\gamma_i=-1$ where $\beta_i(\gamma_i)$ is $\operatorname{sign} Y_i$ in the interval $(c,t_{k_i}^i)$ (in $(\bar{t}_{k_i}^i,t_{k_{i+1}}^i)$); $i=1,2,3$.

Type III. Sequences $\{t_k^i\}$, $\{\bar{t}_k^i\}$, $i=1,2,3$, $k\in Z$ exist such that $t_k^i\in J$, $\lim\limits_{k\to-\infty} t_k^i=c$, $\lim\limits_{k\to\infty} t_k^i=d$ and (30) holds for $k\in Z$.

Type IV. $\tau\le d$ exists such that

$$(31)\qquad \begin{cases} Y_i,\, i\in\{1,2,3\} \quad \text{has a finite number of } Z-\text{intervals} \quad [t_k^i,\ \bar{t}_k^i] \quad \text{in} \\ (\tau,d),\ (30) \quad \text{holds until} \quad \tau<d\,, \end{cases}$$

$|Y_1|$, $|Y_2|$ are non-increasing, $|Y_3|$ is non-decreasing and

$$Y_1(t)Y_2(t)<0, \qquad Y_1(t)Y_3(t)\ge 0 \quad \text{in} \quad (c,\tau).$$

Type V. $\sum\limits_{i=1}^{3}|y_i(t)|>0$ in J; $|Y_1|$, $|Y_2|$, $|Y_3|$ are non-decreasing and

$$Y_1(t)Y_2(t)\ge 0, \qquad Y_1(t)Y_3(t)\ge 0, \qquad Y_2(t)Y_3(t)\ge 0, \quad t\in J\,.$$

Type VI. There exists $\tau\le d$ such that (31) holds; $|Y_1|$, $|Y_3|$ are non-increasing, $|Y_2|$ is non-decreasing and

$$Y_1(t)Y_2(t)\le 0, \qquad Y_1(t)Y_3(t)<0 \quad \text{in} \quad (c,\tau)\,.$$

Type VII. There exists $\tau\le d$ such that (31) holds; $|Y_1|$ is non-decreasing, $|Y_2|$, $|Y_3|$ are non-increasing and

$$Y_1(t)Y_2(t)\ge 0, \qquad Y_2(t)Y_3(t)<0 \quad \text{in} \quad (c,\tau)\,.$$

Type VIII. y is trivial in J.

Theorem 4. *Let (29) be valid and let $y : (a, b) \to R^3$ be a non-trivial solution of (1).*

(i) Let Z-points of y do not exist in (a, b). Then y is successively of Types {I, IV, VI, VII} and {II, V, IV, VI, VII} in (a, b).

(ii) Let τ, $\tau \in (a, b)$ be Z-point of y and (22) hold in (a, τ). Then τ_1, $a \le \tau_1 < b$ exists such that y is of Type II in (τ_1, b) and of Type {I, IV, VI, VII} in (a, τ_1).

(iii) Let τ, τ_1, $a < \tau < \tau_1 < b$ be Z-points of y such that (22) holds in (τ, τ_1). Then y is of Type III in (τ, τ_1).

(iv) Let τ, $\tau \in (a, b)$ be Z-points of y and (22) hold in (τ, b). Then y is either of the Type V in (τ, b) or there exists τ_1, $\tau < \tau_1 \le b$ such that y is of Type I in (τ, τ_1) and of Type {II, IV, V, VI, VII} in (τ_1, b).

(v) Then there exists at most one maximal interval $J \subset (a, b)$ with Property V.

Theorem 5. *Let the assumptions of Theorem 2 be valid and at the same time the validity of (29) is supposed instead of (10). Then y has no Z-point on (a, b) and the statement of Theorem 4, (i) is valid.*

Theorem 6. *Let $\alpha = 1$, $x : (a, b) \to R$ be a non-trivial solution of (4), (5).*

(i) Let the assumptions of Theorem 3(i) be valid. Then x has no Z-point in (a, b) and the statement of Theorem 4, (i) holds.

(ii) Let $\frac{a_2}{a_1} \in C^1(R)$. Then x has at most one Z-interval and one of the two following relations holds:

1° *y is successively of Types {I, IV, VI, VII} and {II, IV, V, VI, VII} in (a, b).*

2° *y is successively of Types {I, IV, VI, VII}, II, VIII and V in (a, b). If y is not of the Type VIII in some right neighbourhood of a, then Types II and VIII are both present.*

Remark 6. Similar conclusions hold as in Remark 5.

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