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ON A FOURTH ORDER PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. Existence and uniqueness of the solution to a fourth order nonlinear vector periodic boundary value problem is proved by using the estimates for derivatives of the Green function for the corresponding homogenous scalar problem

The aim of this paper is to prove the existence and the uniqueness of a solution for the nonlinear vector periodic boundary value problem

$$(1)\quad (L_1 = L_0(y) + K \cdot y \equiv)\, y^{(4)} + (m^2+n^2)\, y'' + (m^2n^2+K)\, y = g(t, y, y', y'', y''')\,,$$

$$(2)\qquad y^{(i)}(0) = y^{(i)}(2\pi)\,, \qquad i = 0, 1, 2, 3\,.$$

where $0 < m < n$, m, $n \in N$ and $K > \frac{1}{4}(n^2 - m^2)^2$, the function $g \in C(D, R^d)$, $D = [0, 2\pi] \times R^d \times R^d \times R^d \times R^d$, $d \geq 1$.

The Green function for the corresponding homogeneous boundary value problem ($d = 1$) has also been constructed. The method of the construction of this function is published for instance in [3], where a nonlinear differential equation of the third order is investigated. Similarly as in that paper where the scalar case is considered we shall prove the existence and uniqueness to (1), (2) in the vector case.

We consider the scalar differential equation of the fourth order

$$(3)\qquad (L_0(x) \equiv)\; x^{(4)} + (m^2 + n^2)\, x'' + m^2n^2\, x = 0$$

with periodic boundary conditions (2). Let the space $X = L^2([0, 2\pi])$ be provided with the usual norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. Consider the differential operator L_0 defined on the subspace

$$D(L_0) = \{x \in C^3([0, 2\pi]) : x^{(4)} \in L^2([0, 2\pi]), x^{(i)}(0) = x^{(i)}(2\pi)\,, \quad i = 0, 1, 2, 3\}\,.$$

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Therefore the operator L_0 maps $D(L_0) \subset X$ into X. The functions $z_1(t) = \cos mt$, $z_2(t) = \sin mt$, $z_3(t) = \cos nt$, $z_4(t) = \sin nt$ form a fundamental system of solutions of the equation (3) and satisfy the boundary conditions (2). Considering the problem

$$L_0(x) = \lambda\, x \tag{4}$$

it is obvious that $\lambda = 0$ is the eigenvalue of the operator L_0. In this case the Green function does not exist. Let us take the equation

$$(L_1 = L_0(x) + K\cdot x \equiv)\ \ x^{(4)} + (m^2+n^2)\, x'' + (m^2n^2 + K) \quad x = 0 \tag{5}$$

instead of the equation (3) and state a condition for the constant $K \in \mathbb{R}$, in order that the operator $L_0 + K\cdot I$ have not the eigenvalue $\lambda = 0$. I is the identical mapping in the space X.

Lemma 1. *Let $K > \frac{1}{4}(n^2 - m^2)^2$. Then 0 is not the eigenvalue of the operator $L_0 + K\cdot I$.*

Proof. λ is the eigenvalue of the problem (4) if and only if there exists such a $k \in Z$ that $i\cdot k$ is the root of characteristic equation

$$r^4 + (m^2+n^2)\, r^2 + m^2n^2 - \lambda = 0\,.$$

This happens, iff k satisfies the equation

$$k^4 - (m^2-n^2)\, k^2 + m^2n^2 - \lambda = 0\,.$$

Denote $g : \mathbb{R} \to \mathbb{R}$ the function

$$g(k) = k^4 - (m^2-n^2)\, k^2 + m^2n^2$$

The eigenvalues of the problem (4) are the values of the function g at $k \in Z$. The function g is an even function and $\min g(k) = -\frac{1}{4}(n^2-m^2)^2$, $k \in Z$ and hence all eigenvalues $\lambda_j \geq -\frac{1}{4}(n^2-m^2)^2$. From the form of the function g it follows that all eigenvalues of the problem (4) form a sequence $\{\lambda_j\}$ which approaches to ∞ to $j \to \infty$. If we add to the function g a constant $K > \frac{1}{4}(n^2-m^2)^2$ to that function, then $g + K$ will be positive for all k. The corresponding characteristic equation will be

$$r^4 + (m^2+n^2)\, r^2 + m^2n^2 + K = 0$$

and the differential operator will be $L_0(x) + K\cdot x$. □

From Lemma 1 it follows that the equation $(L_0 + K\cdot I)(x) = 0$ has only the trivial solution for K. By Lemma 4.3, [1], p.145 it follows that the operator $L_0 + K\cdot I$ is one-to-one and onto X.

Lemma 2. *The operator $L_0 + K\cdot I$, $K \in \mathbb{R}$ is symetric in $D(L_0)$ (i. e. for every $x, y \in D(L_0)$ the equality $\langle (L_0 + K\cdot I)(x), y\rangle = \langle x, (L_0 + K\cdot I)(y)\rangle$ is true.*

Proof. Let $x, y \in D(L_0) = D(L_0 + K\cdot I)$. The assertion of the lemma follows by twice integration by parts and by using boundary condition (2). □

Define a linear operator M in space X by

$$M(g)(t) = \int_0^{2\pi} G(t,s)\, g(s)\, ds\,, \quad 0 \le t \le 2\pi\,,$$

where G is the Green function for the problem (5),(2). By Lemma 2 and Lemma 4.5 [1] p. 147 it follows that M is the self-adjoint operator on X and

$$G(t,s) = G(s,t)\,, \quad \text{for every } (t,s) \in [0,2\pi]\times[0,2\pi] \tag{6}$$

Hence the operator $L_0 + K\cdot I$ is self-adjoint too and $D(L_0 + K\cdot I) = D(L_0)$.

Lemma 1 assurs the existence of the Green function $G(t,s)$ for the operator $L_0 + K\cdot I$. Determine its form. The characteristic equation $L_0(x) + K\cdot x = 0$ is

$$r^4 + (m^2 + n^2)\, r^2 + m^2 n^2 + K = 0\,.$$

Its roots are

$$r_1 = a + i\cdot b\,, \quad r_2 = a - i\cdot b\,, \quad r_3 = -a + i\cdot b\,, \quad r_4 = -a - i\cdot b\,,$$

where

$$a = \sqrt{\frac{1}{2}\left(a_1 + \sqrt{a_1^2 + b_1^2}\right)}\,, \qquad b = \sqrt{\frac{1}{2}\left(-a_1 + \sqrt{a_1^2 + b_1^2}\right)}\,,$$

$$a_1 = -\frac{m^2 + n^2}{2}\,, \qquad b_1 = \sqrt{4\cdot K - (n^2 - m^2)^2} > 0\,,$$

and $0 < a < b$ is true. The Green function will be found in the form

$$G(t,s) = \begin{cases} c_1 e^{at}\cos bt + c_2 e^{at}\sin bt + c_3 e^{-at}\cos bt + c_4 e^{-at}\sin bt\,, \\ \qquad\qquad 0 \le t < s \le 2\pi\,, \\ \\ c_5 e^{at}\cos bt + c_6 e^{at}\sin bt + c_7 e^{-at}\cos bt + c_8 e^{-at}\sin bt\,, \\ \qquad\qquad 0 \le s < t \le 2\pi\,. \end{cases}$$

From (6) follows that it is sufficient to determine the coeficients c_i, $i = 5,6,7,8$. These coeficients are calculated using the standart properties of the Green function. The Green function of the problem (5), (2) is

$$\begin{aligned} G(t,s) = {} & \left[\, \gamma_1\, e^{a(t-s+2\pi)} - \gamma_2 e^{-a(t-s+2\pi)} \right] \\ & \times [\, a\, \sin b(s-t+2\pi) + b\, \cos b(s-t+2\pi)] \\ & - \left[\, \gamma_1\, e^{a(t-s)} - \gamma_2 e^{-a(t-s)} \right] \\ & \times [\, a\, \sin b(s-t) + b\, \cos b(s-t)]\,, \qquad 0 \le s < t \le 2\pi\,, \end{aligned}$$

where

$$\gamma_1 = \left[ab(a^2+b^2)(e^{4a\pi} - 2\,e^{2a\pi}\cos 2b\pi + 1)\right]^{-1},$$
$$\gamma_2 = \left[ab(a^2+b^2)(e^{-4a\pi} - 2\,e^{-2a\pi}\cos 2b\pi + 1)\right]^{-1}.$$

and

$$e^{4a\pi} - 2\,e^{2a\pi}\cos 2b\pi + 1 \geq (e^{2a\pi}-1)^2 > 0\,,$$
$$e^{-4a\pi} - 2\,e^{-2a\pi}\cos 2b\pi + 1 \geq (e^{-2a\pi}-1)^2 > 0\,.$$

Hence $0 < \gamma_1$, $0 < \gamma_2$. Let us introduce the notacion

$$M = \gamma_1(e^{2a\pi}+1)\,, \qquad N = \gamma_2(e^{-2a\pi}+1)\,.$$

Lemma 3. *The following estimates are vatid*

$$|G(t,s)| \leq (M\,e^{at}+N)\sqrt{a^2+b^2}\,,$$
$$|G_t(t,s)| \leq (M\,e^{at}+N)(a+b)\sqrt{a^2+b^2}\,,$$
$$|G_{tt}(t,s)| \leq (M\,e^{at}+N)(a+b)^2\sqrt{a^2+b^2}\,,$$
$$|G_{ttt}(t,s)| \leq (M\,e^{at}+N)(a+b)^3\sqrt{a^2+b^2}\,.$$

Proof. We consider the function $f(u) = A\sin u + B\cos u$ for $[0,2\pi]$, where $0 < \|A\| < \|B\|$, $A, B \in \mathbb{R}$. Look for the maximum of the function f. From the equality $f'(u) = A\cos u - B\sin u = 0$ it follows that $\sin u = \frac{A}{B}\cos u$. Let $A > 0$, $B > 0$. Then $0 < \operatorname{tg} u = \frac{A}{B} < 1$ and therefore there exists such $u_1 \in \left(0, \frac{\pi}{2}\right)$, $u_2 \in \left(\pi, \frac{3}{2}\pi\right)$, that $f'(u_i) = 0$, $i = 1,2$ and $\cos u_1 = B(A^2+B^2)^{-\frac12}$ a $\cos u_2 = -B(A^2+B^2)^{-\frac12}$. The extremal values of f in u_1, u_2 are $f(u_1) = (A^2+B^2)^{\frac12}$, $f(u_2) = -(A^2+B^2)^{\frac12}$. We shall get the same values in the casees when $A>0$, $B<0$, $A<0$, $B>0$, and $A<0$, $B<0$. Therefore $\max|f(u)| = (A^2+B^2)^{\frac12}$ in $[0,2\pi]$ for all $A, B \in \mathbb{R}$.

We use these relations in the following estimations. The function $e^{a(t-s+2\pi)}$ attains its maximum on the set $0 \leq s \leq t$ at $s = 0$ and the function $e^{-a(t-s+2\pi)}$ at $s = t$. Similar results hold for the function $e^{a(t-s)}$, $e^{-a(t-s)}$. Having calculated $\dfrac{\partial^k G(t,s)}{\partial t^k}$, $k = 0,1,2,3$, we get these estimations

$$\begin{aligned}
|G(t,s)| &\leq \left[\gamma_1 e^{at}(e^{2a\pi}+1) + \gamma_2(e^{-2a\pi}+1)\right](a^2+b^2)^{\frac12}\\
&= (Me^{at}+N)(a^2+b^2)^{\frac12}\,,\\
|G_t(t,s)| &\leq \left[\gamma_1 e^{at}(e^{2a\pi}+1) + \gamma_2(e^{-2a\pi}+1)\right](a+b)(a^2+b^2)^{\frac12}\\
&= (Me^{at}+N)(a+b)(a^2+b^2)^{\frac12}\,,\\
|G_{tt}(t,s)| &\leq \left[\gamma_1 e^{at}(e^{2a\pi}+1) + \gamma_2(e^{-2a\pi}+1)\right](a^2+2ab+b^2)\\
&\quad\times(a^2+b^2)^{\frac12} = (Me^{at}+N)(a+b)^2(a^2+b^2)^{\frac12}\,,\\
|G_{ttt}(t,s)| &\leq \left[\gamma_1 e^{at}(e^{2a\pi}+1) + \gamma_2(e^{-2a\pi}+1)\right](a^3+3a^2b+3ab^2+b^3)\\
&\quad\times(a^2+b^2)^{\frac12} = (Me^{at}+N)(a+b)^3(a^2+b^2)^{\frac12}\,. \qquad \square
\end{aligned}$$

Lemma 4.

$$\max_{0\le t\le 2\pi}\int_0^{2\pi} |G(t,s)|\,ds = K_0 \le 2\pi(Me^{2a\pi}+N)(a^2+b^2)^{\frac{1}{2}}\,,$$

$$\max_{0\le t\le 2\pi}\int_0^{2\pi} |G_t(t,s)|\,ds = K_1 \le 2\pi(Me^{2a\pi}+N)(a+b)(a^2+b^2)^{\frac{1}{2}}\,,$$

$$\max_{0\le t\le 2\pi}\int_0^{2\pi} |G_{tt}(t,s)|\,ds = K_2 \le 2\pi(Me^{2a\pi}+N)(a+b)^2(a^2+b^2)^{\frac{1}{2}}\,,$$

$$\max_{0\le t\le 2\pi}\int_0^{2\pi} |G_{ttt}(t,s)|\,ds = K_3 \le 2\pi(Me^{2a\pi}+N)(a+b)^3(a^2+b^2)^{\frac{1}{2}}\,.$$

Proof. The integrals $\int\limits_0^{2\pi}\left|\frac{\partial^j G(t,s)}{\partial t^j}\right| ds$, $j=0,1,2,3$ are continuous functions of the variable t in the compact interval $[0,2\pi]$ and in this interval they attain their maximum K_j, $j=0,1,2,3$. From Lemma 3 we obtain the estimations above. $\square$

Let us consider the nonlinear vector periodic boundary value problem (1), (2). Firstly we introduce the following notations : $x=(x_1,\dots,x_d)^T$ is a column vector, $|x|=(|x_1|,\dots,|x_d|)^T$, $M_{d\times d}$ is the set of all real $d\times d$ matrices, $u_d=(1,\dots,1)\in\mathbb{R}$, $\rho(N)$ is the spectral radius of the matrix $N\in M_{d\times d}$, $\rho(N)=\max|\lambda_i|$, where λ_i are all eigenvalues of N.

For the scalar boundary value problem (5), (2) the estimations

$$\begin{aligned}\left|y^{(j)}(t)\right| \le& \int_0^{2\pi}\left|\frac{\partial^j G(t,s)}{\partial t^j}\right|\cdot\max_{0\le s\le 2\pi}|L_1(y)(s)|\,ds\\ =&K_j\cdot\max_{0\le t\le 2\pi}|L_1(y)(t)|\,,\qquad j=0,1,2,3,\end{aligned}\tag{7}$$

are valid, where the constants K_j are determined in Lemma 4.

Futher we shall use a generalized norm. If E is a real vector space, then the generalized norm in E is a mapping $||\cdot||_G : E\to\mathbb{R}^d$ denoted by

$$||x||_G=(\alpha_1(x),\dots,\alpha_d(x))^T\,,\tag{8}$$

such that

(1) $||x||_G\ge 0$ that is $\alpha_j(x)\ge 0$ for $j=1,\dots,d\,,\ x\in E$,
(2) $||x||_G=0$ iff $x=0$;
(3) $||c\,x||_G=|c|\cdot||x||_G$, $c\in\mathbb{R}\,,\ x\in E$;
(4) $||x+y||_G\le||x||_G+||y||_G\,,\ x,y\in E$.

The couple $(E\,,\,||\cdot||_G)$ is then called a generalized linear normed space. The topology in this space is given in the following way. For each $x \in E$, and $\varepsilon > 0$ let $B_\varepsilon = \{y \in E : ||x-y||_G < \varepsilon \cdot u_d\}$. The same topology can be inducted by the norm which is defined in this way. Let $||x||_G$ is given by (8), then

$$||x|| = \max(\alpha_1(x), \dots, \alpha_d(x))\,, \qquad x \in E\,. \tag{9}$$

The mapping $||\cdot||$ has all properties of the norm. The topology of the normed space $(E\,,\,||\cdot||)$ is given by the basis of neighbourhoods $V_\varepsilon(x) = \{y \in E \;:\; ||y-x|| < \varepsilon\}$, $x \in E\,,\ \varepsilon > 0$ and $V_\varepsilon(x) = B_\varepsilon(x)$. Therefore the norms (8) and (9) define the same topology on E and in this sense are equivalent. We may use the norm (9) instead of the generalized norm (8). The following lemma is true and it is introduced in [2], p. 78.

Lemma 5. *Let $(E\,,\,||\cdot||_G)$ be a generalized Banach space and let $T : E \to E$ be such that for all $x, y \in E$ and for some positive integer p*

$$||T^p(x) - T^p(y)||_G \le M \cdot ||x-y||_G\,,$$

where $M \in M_{d\times d}$ is a nonnegative matrix with $\rho(M) < 1$ and T^p is p-th iterate of T. Then T has a unique fixed point. □

Theorem 1. *Let for all $(t, u_0, u_1, u_2, u_3)\,, (t, v_0, v_1, v_2, v_3) \in D$ the function g satisfy the Lipschitz condition*

$$|g(t, u_0, u_1, u_2, u_3) - g(t, v_0, v_1, v_2, v_3,)| \le \sum_{l=0}^{3} N_l |u_l - v_l|\,, \tag{10}$$

where $N_l \in M_{d\times d}$ are nonnegative matrices. Let $\rho\left(\sum\limits_{l=0}^{3} N_l \cdot K_l\right) < 1$, where K_l are the constant in Lemma 4. Then there exists a unique solution to (1), (2).

Proof. Let us itreduce the notation for each $x \in C([0, 2\pi], \mathbb{R}^d)$, $x(t) = (x_1(t), \dots,$ $x_d(t))^T \max\limits_{0\le t\le 2\pi} |x(t)| = (\max\limits_{0\le t\le 2\pi} |x_1(t)|, \dots, \max\limits_{0\le t\le 2\pi} |x_d(t)|)^T$. Let

$$S_1 = \{x \in C^4([0, 2\pi]\,,\,\mathbb{R}^d) \;:\; x^{(i)}(0) = x^{(i)}(2\pi)\,,\ i = 0, 1, 2, 3\}\,.$$

Then S_1 is a real vector space and the generalized norm is defined on S_1 by

$$||x||_1 = \max_{0\le t\le 2\pi} |L_1(x)(t)|\,, \quad \text{pre v"setky } x \in S_1\,.$$

The properties of the generalized norm can be easily checked. $(S_1\,,\,||\cdot||_1)$ is a generalized Banach space. In fact, if $\{x_n\}_{n=1}^{\infty} \subset S_1$ a Cauchy sequence, then the sequence $\{L_1(x_n)(t)\}_{n=1}^{\infty}$ converge uniformly on $[0, 2\pi]$ to the function $y \in C([0, 2\pi])\,, \mathbb{R}^d)$. The problem $L_1(x)(t) = y(t)$, (2) has a unique solution $x \in S_1$ a $\lim\limits_{n\to\infty} ||x_n - x||_1 = 0$.

Define the mapping $T : S_1 \to S_1$ by $T(y) = x$, where x is a solution of the equation
$$L_1(x)(t) = g(t, y(t), y'(t), y''(t), y'''(t)),$$
which fulfils the boundary conditions (2). By (10) for any of two functions $y, z \in S_1$ we have

$$|L_1(T(y))(t) - L_1(T(z))(t)| \le \sum_{l=0}^{3} N_l |y^{(l)}(t) - z^{(l)}(t)|, \tag{11}$$

for all $t \in [0, 2\pi]$.

Denote the j-th coordinate of the functions y and z by y_j, z_j respectively. Then from (7) we have

$$\begin{aligned} &|y_j^{(l)}(t) - z_j^{(l)}(t)| \le \\ &\le \int_0^{2\pi} \left| \frac{\partial^l G(t,s)}{\partial t^l} \right| \cdot \max_{0 \le t \le 2\pi} |L_1(y_j)(t) - L_1(z_j)(t)| ds \\ &\le K_l \max_{0 \le t \le 2\pi} |L_1(y_j)(t) - L_1(z_j)(t)|, \quad l = 0, 1, 2, 3. \end{aligned}$$

Therefore

$$|y^{(l)}(t) - z^{(l)}(t)| \le K_l \, ||y - z||_1, \quad l = 0, 1, 2, 3. \tag{12}$$

From (11) and (12) it follows

$$|L_1(T(y))(t) - L_1(T(z))(t)| \le \sum_{l=0}^{3} N_l \cdot K_l ||y - z||_1,$$

for all $y, z \in S_1$, $t \in [0, 2\pi]$ and

$$||T(y) - T(z)||_1 \le \sum_{l=0}^{3} N_l \cdot K_l ||y - z||_1.$$

As the assumption of Lemma 5 is fulfiled, there exists a unique fixed point of T in S_1. This means that the problem (1), (2) has a unique solution. $\square$

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