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CURVATURE TENSORS IN DIMENSION FOUR WHICH DO NOT BELONG TO ANY CURVATURE HOMOGENEOUS SPACE

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Abstract. A six-parameter family is constructed of (algebraic) Riemannian curvature tensors in dimension four which do not belong to any curvature homogeneous space. Also a general method is given for a possible extension of this result.

1. Introduction

According to I.M. Singer [SI], a Riemannian manifold is said to be *curvature homogeneous* if, for every two points $p, q \in M$, there is a linear isometry $F : T_pM \to T_qM$ between the corresponding tangent spaces such that $F^*R_q = R_p$ (where R denotes the curvature tensor of type (0,4)). Note that a (locally) homogeneous Riemannian manifold is automatically curvature homogeneous. Explicit *locally non-homogeneous* examples have been constructed by many authors ([SE],[T],[YA],[K-T-V1] – [K-T-V3],[K1] – [K3]; see especially [K-T-V2] and [K-T-V3] for more complete references).

Let K be a curvature-like tensor of Riemannian type (shortly: curvature tensor) on a vector space $\mathbf{V}$ with a positive scalar product $<,>$. I. e., we assume that

$$(1)\qquad \begin{aligned} &K(X,Y,U,V) = -K(Y,X,U,V) = K(U,V,X,Y),\\ &K(X,Y,Z,U) + K(Y,Z,X,U) + K(Z,X,Y,U) = 0. \end{aligned}$$

A natural problem arises whether there is always a curvature homogeneous space (M,g) such that, under a linear isometry between $\mathbf{V}$ and a tangent space T_oM, K coincides with the curvature tensor R_o of (M,g) at the point $o \in M$. In such a case we say that the tensor K belongs to the curvature homogeneous space (M,g).

In dimension $n = 3$, the answer to our problem is always positive, as it was proved quite recently (see [K2],[S-T],[K-P]): *every curvature tensor R in dimension*

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three belongs to some curvature homogeneous space. Moreover, if R is not of the type of constant curvature, one can always find an example which is not locally homogeneous. (On the other hand, it is not always possible to construct a *homogeneous* Riemannian space with a prescribed curvature tensor (see [M],[S-T],[K3]).)

In dimension $n = 4$, the situation is essentially different. In the paper [S-T] the following example is given: let R_{S^4} and R_{CP^2} denote the typical curvature tensors of a four-dimensional sphere and a complex projective plane, respectively and let a, b be nonzero real numbers. Then the curvature tensor $aR_{S^4} + bR_{CP^2}$ does not belong to any curvature homogeneous space. The proof requires nontrivial results from the almost Hermitian geometry (see [T-V]). In this paper we study the problem more systematically, by elementary methods, and we give a new and broader family of curvature tensors with the above property. For our purposes we introduce the notion of σ -rank (cyclic rank) of a curvature tensor, which may be of some interest by itself.

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2. Generic curvature tensors and the Chern bases

Let $\mathbf{V}$ be a vector space with a (positive) scalar product $<,>$ and let $\mathcal{R}$ denote the space of all curvature tensors on $\mathbf{V}$ (satisfying the identities (1)). The orthogonal group $O(\mathbf{V})$ acts on $\mathcal{R}$ in a natural way. A tensor $R \in \mathcal{R}$ is said to be *generic* if the subgroup $H = \{A \in O(\mathbf{V}) | A(R) = R\}$ is finite. Another characterization of a generic tensor $R \in \mathcal{R}$ is that its orbit in $\mathcal{R}$ under $O(\mathbf{V})$ has the maximal dimension (equal to $n(n-1)/2$). Finally, $R \in \mathcal{R}$ is generic if and only if there is no nonzero skew-symmetric endomorhpism P of $\mathbf{V}$ (acting as a derivation on the tensor algebra of $\mathbf{V}$) such that $P \cdot R = 0$. Indeed, the last identity means $(\exp tP)(R) = R,\ t \in (-\infty, +\infty)$.

Let us consider now the four-dimensional case.

According to S.S. Chern [C], or R. Klinger [KL], $\dim \mathcal{R} = 20$, and for every $R \in \mathcal{R}$ there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbf{V}$ (called a Chern basis) such that the components R_{ijkl} satisfy

$$R_{1213} = R_{1214} = R_{1223} = R_{1224} = 0,\ R_{1314} = R_{1323} = 0. \tag{2}$$

We have also the following observations:

a) If $R \in \mathcal{R}$ is generic, then its Chern basis is uniquely determined up to a finite group of reflections.
b) If $R \in \mathcal{R}$ is an Einsteinian curvature tensor, then its Singer-Thorpe basis $\{f_1, f_2, f_3, f_4\}$ (see [SI-TH]) is always a Chern basis.
c) For the curvature tensor $R = cg \wedge g$ of a space form, each orthonormal basis is a Chern basis.

For the purpose of this paper, we shall introduce the following notation w.r. to a

Chern basis:

$$
\begin{aligned}
&A_1 = R_{1212}, \quad A_2 = R_{1234},\\
&B_1 = R_{1313}, \quad B_2 = R_{1324}, \quad B_3 = R_{1334},\\
&C_1 = R_{1414}, \quad C_2 = R_{1424}, \quad C_3 = R_{1434},\\
&D_1 = R_{2323}, \quad D_2 = R_{2324}, \quad D_3 = R_{2334},\\
&E_1 = R_{2424}, \quad E_2 = R_{2434},\\
&F_1 = R_{3434}.
\end{aligned} \tag{3}
$$

All curvature components in (3) are independent, in general, and they uniquely determine the tensor R w.r. to a given basis.

Let us recall that an Einstein curvature tensor is characterized, w.r. to a Singer-Thorpe basis, by the equalities (2) and the equalities

$$A_1 = F_1, \ B_1 = E_1, \ C_1 = D_1, \ C_2 = D_2 = E_2 = 0, \ B_3 = C_3 = D_3 = 0.$$

We give now some more examples of generic curvature tensors:

d) A curvature tensor $R \in \mathcal{R}$ with four distinct Ricci eigenvalues is always generic. In particular, assume that, w.r. to a Chern basis, the components $A_1, B_1, C_1, D_1, E_1, F_1$ are the only (possible) nonzero components. It can be easily seen that the Ricci eigenvalues are all distinct if and only if the numbers $|A_1 - F_1|, \ |B_1 - E_1|, \ |C_1 - D_1|$ are all distinct.
In particular, if $D_1 = E_1 = F_1 = 0$ holds but $|A_1|, |B_1|, |C_1|$ are all distinct, the tensor R is still generic.

e) If $A_1 = B_1 = E_1 \neq C_1 = D_1 = F_1$, and all other components are zero, then there are only two different Ricci eigenvalues but the curvature tensor R is still generic.

To see the last result, one can use the following criterion of genericity w.r. to a Chern basis $\{e_1, \ldots, e_4\}$:

Let E_{ij} denote the elementary skew-symetric endomorphism of $\mathbf{V}$, i.e. those defined by

$$E_{ij}(e_i) = -e_j, \ E_{ij}(e_j) = e_i, \ E_{ij}(e_k) = 0 \text{ for } k \neq i, j, \ 1 \leq i < j \leq 4. \tag{4}$$

Denote

$$P = \sum_{1 \leq i < j \leq 4} \alpha_{ij} E_{ij}, \tag{5}$$

and let P act as a derivation on the tensor algebra $\mathcal{T}(\mathbf{V})$. Then a tensor $R \in \mathcal{R}$ is generic if and only if $P \cdot R = 0$ implies $\alpha_{ij} = 0$ for all $i, j, \ 1 \leq i < j \leq 4$.

We conclude this Section with the following example:

f) Suppose that $B_1 = D_1, \ C_1 = E_1, \ A_2 = 2B_2, \ A_1, \ F_1$ are arbitrary, $C_2 = D_2 = E_2 = 0, \ B_3 = C_3 = D_3 = 0$. Then $A_{12} \cdot R = 0$, and hence the corresponding tensor R is not generic.

3. General properties of curvature homogeneous spaces

According to [SI] (see also [N-T] and [S-T]) we have the following

Theorem A. *Any curvature homogeneous space (M, g) admits an open covering $\{V_\alpha\}_{\alpha\in J}$ such that, in each V_α, an orthonormal moving frame $\{E_1^\alpha, \dots, E_n^\alpha\}$ exists for which all the components of the Riemannian curvature tensor field R are constant.*

Hence we obtain the first of the following corollaries:

Corollary 3.1. *In each V_α there exists a flat connection $\tilde{\nabla}^\alpha$ (with torsion) such that $\tilde{\nabla}^\alpha g = \tilde{\nabla}^\alpha R = 0$. Here $\tilde{\nabla}^\alpha$ is defined as the (unique) connection for which the vector fields $E_1^\alpha, \dots, E_n^\alpha$ define an absolute parallelism.*

Corollary 3.2. *Suppose that (M, g) is curvature homogeneous and the typical curvature tensor R is generic. Then there is a unique global affine connection $\tilde{\nabla}$ on (M, g) such that $\tilde{\nabla} g = \tilde{\nabla} R = 0$. The connection $\tilde{\nabla}$ is locally flat.*

Proof. In each neighborhood V_α we choose a point p_α and a "generalized Chern basis" $\{e_1^\alpha, \dots, e_n^\alpha\}$ at p_α in the sense of R. Klinger [KL]. Then we can extend this basis to a moving frame $\{E_1^\alpha, \dots, E_n^\alpha\}$ in V_α for which all the components R_{ijkl} are constant and each basis is a generalized Chern basis. Because R is generic, such a moving frame is uniquely determined up to a finite group of reflections. Hence we see $\tilde{\nabla}^\alpha = \tilde{\nabla}^\beta$ on each intersection $V_\alpha \cap V_\beta$, and we get a globally defined connection $\tilde{\nabla}$ on (M, g), which is locally flat. For the uniqueness part, consider two connections $\tilde{\nabla}_1$, $\tilde{\nabla}_2$ on (M, g) satisfying $\tilde{\nabla}_i g = \tilde{\nabla}_i R = 0$. Then their difference tensor field D satisfies $D_X \cdot g = D_X \cdot R = 0$ for each tangent vector $X \in TM$. Suppose that $D_X \neq 0$ for some vector $X \in T_m M$. Because D_X is a skew-symmetric endomorphism of $T_m M$ such that $D_X \cdot R = 0$, we obtain a contradiction to the genericity of R. Hence $D = 0$ and $\tilde{\nabla}_1 = \tilde{\nabla}_2$ on M. □

The following simple result is basic for our further investigations:

Proposition 3.3. *Let (M, g) be a curvature homogeneous space. Then, in a neighborhood U_p of each point $p \in M$, there exists a tensor field S of type $(1, 2)$ such that*

$$S_X \cdot g = 0 \quad \text{for every } X \in T_m M,\ m \in U_p, \tag{6}$$

$$\underset{X,Y,Z}{\mathfrak{S}} (S_X \cdot R)(Y, Z, U, V) = 0 \text{ for every } X, Y, Z, U, V \in T_m M, m \in U_p, \tag{7}$$

where $\mathfrak{S}$ denotes the cyclic sum.

Proof. Put $S_X = \nabla_X - \tilde{\nabla}_X$, where $\tilde{\nabla}$ is a flat connection as in Corollary 2.1 and ∇ is the Levi-Civita connection. Then (6) is trivial and (7) follows from the second Bianchi identity for ∇R. □

4. The cyclic rank of a curvature tensor in dimension four

We shall go back to the four dimensional case. Consider a curvature homogeneous space (M, g), $\dim M = 4$, and an orthonormal moving frame $\{E_1, \dots, E_4\}$ on some open subset $U \subset M$ for which all the curvature components are constant. We can again assume that $\{E_1, \dots, E_4\}$ consists of Chern bases: we choose any orthonormal moving frame $\{F_1, \dots, F_4\}$ for which all components R_{ijkl} are constant and then transform $\{F_1, \dots, F_4\}$ in $\{E_1, \dots, E_4\}$ by a *constant* orthogonal matrix. (If the curvature tensor R is not generic, then the choice of the Chern adapted frame $\{E_1, \dots, E_4\}$ is far from being unique.)

Let S be the tensor field defined on U by means of the moving frame $\{E_1, \dots, E_4\}$. Here the vector fields E_i are parallel with respect to the corresponding flat connection $\widetilde{\nabla}$ and we have

$$\nabla_{E_i} E_j = S_{E_i} E_j \quad (i, j = 1, \dots, 4). \tag{8}$$

Let us introduce the notation

$$S_{E_i} E_j = \sum_{k=1}^{4} S_{ij}^k E_k. \tag{9}$$

Obviously, $S_X \cdot g = 0$ implies

$$S_{ij}^k + S_{ik}^j = 0 \quad (i, j, k = 1, \dots, 4). \tag{10}$$

Thus we have 24 independent functions S_{ij}^k, $1 \le i \le 4$, $1 \le j < k \le 4$, which have to satisfy (7), i.e., in the classical notation

$$\mathop{\mathfrak{S}}_{i,j,k} (S_i \cdot R)_{jklu} = 0 \tag{11}$$

In detail, we obtain

$$\begin{aligned}(S_{ij}^p - S_{ji}^p)R_{pklu} + (S_{ki}^p - S_{ik}^p)R_{pjlu} + (S_{jk}^p - S_{kj}^p)R_{pilu} \\ +S_{il}^p R_{jkpu} + S_{jl}^p R_{kipu} + S_{kl}^p R_{ijpu} + S_{iu}^p R_{jklp} \\ +S_{ju}^p R_{kilp} + S_{ku}^p R_{ijlp} \quad = \quad 0\end{aligned} \tag{12}$$

($1 \le i < j < k \le 4,\ 1 \le l < u \le 4$ arbitrary, p is a summation index).

This gives a system of *24 linear equations* for the *24 unknown functions* S_{ij}^k.

We shall write down all these equations explicitly using our Chern basis and thus the conditions (2) and the notation (3). We shall first distribute these equations in four groups putting $(i, j, k) = (1,2,3), (1,2,4), (1,3,4), (2,3,4)$, respectively. Then, in each group separately, we put $(l, u) = (1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$, in this order. We obtain finally the following system:

$$
\begin{aligned}
&\text{(E1)} && (A_1-D_1)S^3_{11}-D_2S^4_{11}+(B_2-2A_2)S^4_{12}\\
&&& +(A_2+B_2)S^4_{21}+(A_1-B_1)S^3_{22} = 0\\
&\text{(E2)} && (D_1-B_1)S^2_{11}-D_3S^4_{11}-B_3S^4_{12}+(2B_2-A_2)S^4_{13}\\
&&& +2B_3S^4_{21}-(A_2+B_2)S^4_{31}+(B_1-A_1)S^3_{32} = 0\\
&\text{(E3)} && D_2S^2_{11}+D_3S^3_{11}-C_3S^4_{12}+C_2S^4_{13}\\
&&& +(A_2-2B_2)S^2_{21}-B_3S^3_{21}+C_3S^4_{21}+(B_1-C_1)S^4_{23}\\
&&& +(2A_2-B_2)S^3_{31}-C_2S^4_{31}+(C_1-A_1)S^4_{32} = 0\\
&\text{(E4)} && -2D_3S^4_{12}+2D_2S^4_{13}+(B_1-D_1)S^2_{21}+D_3S^4_{21}\\
&&& +B_3S^4_{22}+(A_2-2B_2)S^4_{23}+(A_1-D_1)S^3_{31}\\
&&& -D_2S^4_{31}+(B_2-2A_2)S^4_{32} = 0\\
&\text{(E5)} && (A_2-2B_2)S^2_{11}+D_3S^3_{12}-E_2S^4_{12}+(E_1-D_1)S^4_{13}\\
&&& -D_2S^2_{21}+E_2S^4_{21}-B_3S^3_{22}-C_2S^4_{23}-D_2S^3_{31}\\
&&& +(A_1-E_1)S^4_{31}+(A_2+B_2)S^3_{32}+C_2S^4_{32} = 0\\
&\text{(E6)} && -B_3S^2_{11}+(D_1-F_1)S^4_{12}+(2A_2-B_2)S^3_{11}-D_2S^3_{12}\\
&&& +E_2S^4_{13}-D_3S^2_{21}+(F_1-B_1)S^4_{21}+(A_2+B_2)S^3_{22}\\
&&& -C_3S^4_{23}-D_3S^3_{31}-E_2S^4_{31}+B_3S^3_{32}+C_3S^4_{32} = 0\\
&\text{(E7)} && -D_2S^3_{11}+(A_1-E_1)S^4_{11}+(A_2+B_2)S^3_{12}+C_2S^4_{12}\\
&&& +(B_2-2A_2)S^3_{21}+C_2S^4_{21}+(A_1-C_1)S^4_{22} = 0\\
&\text{(E8)} && D_2S^2_{11}-E_2S^4_{11}+B_3S^3_{12}+C_2S^4_{13}\\
&&& +(A_2-2B_2)S^2_{21}-B_3S^3_{21}+C_3S^4_{21}+(B_1-C_1)S^4_{23}\\
&&& -(A_2+B_2)S^4_{41}+(B_1-A_1)S^3_{42} = 0\\
&\text{(E9)} && (E_1-C_1)S^2_{11}+E_2S^3_{11}+C_3S^3_{12}+(A_2-2B_2)S^4_{13}\\
&&& -2C_2S^2_{21}-2C_3S^3_{21}+(2A_2-B_2)S^3_{41}\\
&&& -C_2S^4_{41}+(C_1-A_1)S^4_{42} = 0\\
&\text{(E10)} && (A_2-2B_2)S^2_{11}+D_3S^3_{12}-E_2S^4_{12}\\
&&& +(E_1-D_1)S^4_{13}-D_2S^2_{21}-D_3S^3_{21}+C_3S^4_{22}\\
&&& -C_2S^4_{23}+(A_1-D_1)S^3_{41}-D_2S^4_{41}+(B_2-2A_2)S^4_{42} = 0\\
&\text{(E11)} && -2C_2S^2_{11}+2E_2S^3_{12}-2D_2S^4_{13}+(C_1-E_1)S^2_{21}\\
&&& -E_2S^3_{21}-C_3S^3_{22}+(2B_2-A_2)S^4_{23}-D_2S^3_{41}\\
&&& +(A_1-E_1)S^4_{41}+(A_2+B_2)S^3_{42}+C_2S^4_{42} = 0\\
&\text{(E12)} && -C_3S^2_{11}-C_2S^3_{11}+(F_1-E_1)S^3_{12}+(A_2+B_2)S^4_{11}\\
&&& +D_2S^4_{12}-D_3S^4_{13}-E_2S^2_{21}+(C_1-F_1)S^3_{21}\\
&&& +C_2S^3_{22}+(2A_2-B_2)S^4_{22}+B_3S^4_{23}\\
&&& -D_3S^3_{41}-E_2S^4_{41}+B_3S^3_{42}+C_3S^4_{42} = 0
\end{aligned}
$$

$$\begin{aligned}
\text{(E13)}\quad & -D_3S^3_{11} - E_2S^4_{11} + B_3S^3_{12} + C_3S^4_{12} \\
& +(B_2 - 2A_2)S^3_{31} + C_2S^4_{31} + (A_1 - C_1)S^4_{32} \\
& - (A_2 + B_2)S^4_{41} + (B_1 - A_1)S^3_{42} \quad = \quad 0 \\
\text{(E14)}\quad & D_3S^2_{11} + (B_1 - F_1)S^4_{11} - (A_2 + B_2)S^3_{12} \\
& +C_3S^4_{13} + (A_2 - 2B_2)S^2_{31} - B_3S^3_{31} \\
& + C_3S^4_{31} + (B_1 - C_1)S^4_{33} - 2B_3S^4_{41} \quad = \quad 0 \\
\text{(E15)}\quad & E_2S^2_{11} + (F_1 - C_1)S^3_{11} - C_2S^3_{12} + (B_2 - 2A_2)S^4_{12} \\
& -B_3S^4_{13} - 2C_2S^2_{31} - 2C_3S^3_{31} \\
& + (2B_2 - A_2)S^2_{41} + B_3S^3_{41} - C_3S^4_{41} + (C_1 - B_1)S^4_{43} \quad = \quad 0 \\
\text{(E16)}\quad & -B_3S^2_{11} + (2A_2 - B_2)S^3_{11} - D_2S^3_{12} + (D_1 - F_1)S^4_{12} \\
& +E_2S^4_{13} - D_2S^2_{31} - D_3S^3_{31} + C_3S^4_{32} - C_2S^4_{33} \\
& +(D_1 - B_1)S^2_{41} - D_3S^4_{41} - B_3S^4_{42} + (2B_2 - A_2)S^4_{43} \quad = \quad 0 \\
\text{(E17)}\quad & -C_3S^2_{11} - C_2S^3_{11} + (A_2 + B_2)S^4_{11} \\
& +(F_1 - E_1)S^3_{12} + D_2S^4_{12} - D_3S^4_{13} + (C_1 - E_1)S^2_{31} \\
& - E_2S^3_{31} - C_3S^3_{32} + (2B_2 - A_2)S^4_{33} \\
& +D_2S^2_{41} - E_2S^4_{41} + B_3S^3_{42} + C_2S^4_{43} \quad = \quad 0 \\
\text{(E18)}\quad & -2C_3S^3_{11} + 2B_3S^4_{11} - 2E_2S^3_{12} + 2D_3S^4_{12} - E_2S^2_{31} \\
& +(C_1 - F_1)S^3_{31} + C_2S^3_{32} + (2A_2 - B_2)S^4_{32} + B_3S^4_{33} \\
& +D_3S^2_{41} + (B_1 - F_1)S^4_{41} - (A_2 + B_2)S^3_{42} + C_3S^4_{43} \quad = \quad 0 \\
\text{(E19)}\quad & -D_3S^3_{21} - E_2S^4_{21} + B_3S^3_{22} + C_3S^4_{22} + D_2S^3_{31} \\
& +(E_1 - A_1)S^4_{31} - (A_2 + B_2)S^3_{32} - C_2S^4_{32} \\
& + (A_1 - D_1)S^3_{41} - D_2S^4_{41} + (B_2 - 2A_2)S^4_{42} \quad = \quad 0 \\
\text{(E20)}\quad & D_3S^2_{21} + (B_1 - F_1)S^4_{21} - (A_2 + B_2)S^3_{22} \\
& +C_3S^4_{23} - D_2S^2_{31} + E_2S^4_{31} - B_3S^3_{32} - C_2S^4_{33} \\
& + (D_1 - B_1)S^2_{41} - D_3S^4_{41} - B_3S^4_{42} + (2B_2 - A_2)S^4_{43} \quad = \quad 0 \\
\text{(E21)}\quad & E_2S^2_{21} + (F_1 - C_1)S^3_{21} - C_2S^3_{22} + (B_2 - 2A_2)S^4_{22} \\
& -B_3S^4_{23} + (C_1 - E_1)S^2_{31} - E_2S^3_{31} - C_3S^3_{32} \\
& +(2B_2 - A_2)S^4_{33} + D_2S^2_{41} + D_3S^3_{41} - C_3S^4_{42} + C_2S^4_{43} \quad = \quad 0 \\
\text{(E22)}\quad & -B_3S^2_{21} + (2A_2 - B_2)S^3_{21} - D_2S^3_{22} + (D_1 - F_1)S^4_{22} \\
& +E_2S^4_{23} + (2B_2 - A_2)S^2_{31} - D_3S^3_{32} + E_2S^4_{32} \\
& + (D_1 - E_1)S^4_{33} - 2D_3S^4_{42} + 2D_2S^4_{43} \quad = \quad 0 \\
\text{(E23)}\quad & -C_3S^2_{21} - C_2S^3_{21} + (A_2 + B_2)S^4_{21} + (F_1 - E_1)S^3_{22} \\
& +D_2S^4_{22} - D_3S^4_{23} + 2C_2S^2_{31} - 2E_2S^3_{32} + 2D_2S^4_{33} \\
& +(A_2 - 2B_2)S^2_{41} + D_3S^3_{42} - E_2S^4_{42} + (E_1 - D_1)S^4_{43} \quad = \quad 0
\end{aligned}$$

$$\text{(E24)} \quad \begin{aligned} &-2C_3S^3_{21} + 2B_3S^4_{21} - 2E_2S^3_{22} + 2D_3S^4_{22} + C_3S^2_{31} + C_2S^3_{31} \\ &-(A_2 + B_2)S^4_{31} + (E_1 - F_1)S^3_{32} - D_2S^4_{32} + D_3S^4_{33} - B_3S^2_{41} \\ &+(2A_2 - B_2)S^3_{41} - D_2S^3_{42} + (D_1 - F_1)S^4_{42} + E_2S^4_{43} = 0 \end{aligned}$$

We see immediately that the following linear relations hold: $(E5) + (E19) = (E10)$, $(E6) + (E20) = (E16)$, $(E12) + (E21) = (E17)$, $(E3) + (E13) = (E8)$.

Hence we obtain

Proposition 4.1. *The rank of the system (12) is not greater than twenty.*

We shall now give the following purely algebraic definition:

Definition 4.2. Let $\mathcal{R}$ denote the space of all 4-dimensional curvature tensors on $(\mathbf{V}, <, >)$ and let $A(\mathbf{V}, \mathbf{V})$ denote the space of all skew-symmetric endomorphismus of $(\mathbf{V}, <, >)$. *The σ-rank of a curvature tensor* $R \in \mathcal{R}$ is defined as the difference $24 - k$ where k is the dimension of the subspace of all elements $S \in \mathbf{V}^* \otimes A(\mathbf{V}, \mathbf{V})$ satisfying the identity (7).

Hence we see that, for any orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbf{V}$, and any fixed $R \in \mathcal{R}$, the corresponding σ-rank can be defined as the rank of the corresponding system (12) of linear algebraic equations for the 24 unknowns S^k_{ij}. Especially, we can use the Chern bases for the calculation of the σ-rank.

Now, let $\widetilde{\mathcal{R}}$ denote the orbit space of $\mathcal{R}$ (w.r. to the action of the orthogonal group $O(\mathbf{V})$), provided with the factor topology. Every curvature tensor $R \in \mathcal{R}$ can be represented as a point in $\mathbf{R}^{14}$ via some Chern basis. Such representation is not unique but a generic curvature tensor has only a finite number of representatives (because we have only a finite number of Chern bases). Obviously, the corresponding orbits $[R] \in \widetilde{\mathcal{R}}$ can be also represented as points in $\mathbf{R}^{14}$ and maximal orbits have only finite number of representatives. It is also obvious that all curvature tensors belonging to the same orbit have the same σ-rank and thus the σ-rank can be considered as a function on $\widetilde{\mathcal{R}}$. Now we have

Theorem 4.3. *The σ-rank is equal to 20 on a dense open subset of $\widetilde{\mathcal{R}}$.*

Proof. It is sufficient to give an example of a curvature tensor whose σ-rank is equal to twenty. Then using the equations (E1) - (E24) written w.r. to a Chern basis we see that the condition for the σ-rank to be less than twenty would mean a system of algebraic equations for the components (3), which is not satisfied identically. This gives a subset of zero measure in $\mathbf{R}^{14}$ and our Theorem will follow.

Let us suppose that a curvature tensor R is represented w.r. to a Chern basis and

$$C_2 = D_2 = E_2 = 0, \quad B_3 = C_3 = D_3 = 0. \tag{13}$$

We shall omitt the equations (E13),(E19),E20),(E21) which are linearly dependent on the others. After re-aranging the order of our unknowns S^k_{ij} in a convenient way and after a permutation of our equations we find that the resulting coefficient matrix decomposes into four blocks which are (6,5)-matrices. The corresponding subsystems of equations with separated variables are given by the following tables:

Eq.	S^3_{11}	S^4_{12}	S^4_{21}	S^3_{22}	S^2_{41}	S^4_{43}
(E1)	$D_1 - A_1$	$2A_2 - B_2$	$-(A_2 + B_2)$	$B_1 - A_1$	0	0
(E6)	$2A_2 - B_2$	$D_1 - F_1$	$F_1 - B_1$	$A_2 + B_2$	0	0
(E15)	$F_1 - C_1$	$B_2 - 2A_2$	0	0	$2B_2 - A_2$	$C_1 - B_1$
(E16)	$B_2 - 2A_2$	$F_1 - D_1$	0	0	$B_1 - D_1$	$A_2 - 2B_2$
(E23)	0	0	$A_2 + B_2$	$F_1 - E_1$	$A_2 - 2B_2$	$E_1 - D_1$

Eq.	S^2_{11}	S^4_{13}	S^4_{31}	S^3_{32}	S^3_{41}	S^4_{42}
(E2)	$D_1 - B_1$	$2B_2 - A_2$	$-(A_2 + B_2)$	$B_1 - A_1$	0	0
(E5)	$A_2 - 2B_2$	$E_1 - D_1$	$A_1 - E_1$	$A_2 + B_2$	0	0
(E9)	$E_1 - C_1$	$A_2 - 2B_2$	0	0	$2A_2 - B_2$	$C_1 - A_1$
(E10)	$A_2 - 2B_2$	$E_1 - D_1$	0	0	$A_1 - D_1$	$B_2 - 2A_2$
(E24)	0	0	$-(A_2 + B_2)$	$E_1 - F_1$	$2A_2 - B_2$	$D_1 - F_1$

Eq.	S^4_{11}	S^3_{12}	S^3_{21}	S^4_{22}	S^2_{31}	S^4_{33}
(E7)	$A_1 - E_1$	$A_2 + B_2$	$B_2 - 2A_2$	$A_1 - C_1$	0	0
(E12)	$A_2 + B_2$	$F_1 - E_1$	$C_1 - F_1$	$2A_2 - B_2$	0	0
(E14)	$B_1 - F_1$	$-(A_2 + B_2)$	0	0	$A_2 - 2B_2$	$B_1 - C_1$
(E17)	$A_2 + B_2$	$F_1 - E_1$	0	0	$C_1 - E_1$	$2B_2 - A_2$
(E22)	0	0	$2A_2 - B_2$	$D_1 - F_1$	$2B_2 - A_2$	$D_1 - E_1$

Eq.	S^2_{21}	S^4_{23}	S^3_{31}	S^4_{32}	S^4_{41}	S^3_{42}
(E3)	$A_2 - 2B_2$	$B_1 - C_1$	$2A_2 - B_2$	$C_1 - A_1$	0	0
(E4)	$B_1 - D_1$	$A_2 - 2B_2$	$A_1 - D_1$	$B_2 - 2A_2$	0	0
(E8)	$A_2 - 2B_2$	$B_1 - C_1$	0	0	$-(A_2 + B_2)$	$B_1 - A_1$
(E11)	$C_1 - E_1$	$2B_2 - A_2$	0	0	$A_1 - E_1$	$A_2 + B_2$
(E18)	0	0	$C_1 - F_1$	$2A_2 - B_2$	$B_1 - F_1$	$-(A_2 + B_2)$

The rank of each of these matrices is five, in general, and hence the σ-rank is equal to twenty for a general R satisfying (13). This concludes the proof of Theorem 4.3. $\square$

The previous tables will be used in the next section for the proof of our main Theorem. Here we only make the following remark:

It is natural to call a curvature tensor $R \in \mathcal{R}$ σ-generic if its σ-rank is equal to twenty. One can ask, what is the relation between the notion "generic" and "σ-generic". The following examples show that there is no direct relation.

Example 1. Suppose, in addition to (13), that $A_2 = B_2 = 0$, $D_1 = E_1 = F_1 = 0$ and the numbers $|A_1|, |B_1|, |C_1|$ are all distinct. Then the corresponding tensor R is generic (see Section 2,d)) but the σ-rank is ≤ 17, as we see from our tables.

Example 2. Suppose, in addition to (13), that $B_1 = D_1$, $C_1 = E_1$, $A_2 = 2B_2$, A_1, D_1,E_1, F_1 are arbitrary and $B_2 \neq 0$. Then we see that, in general, the σ-rank is maximal. But due to f) of Section 2, the corresponding curvature tensor is not generic.

It is not known to the authors if the σ-genericity may have some geometrical meaning.

5. The main existence theorem

Theorem 5.1. *Let R be a curvature tensor on $(\mathbf{V}, <,>)$, $\dim \mathbf{V} = 4$. Suppose that, w.r. to some orthonormal basis $\{e_1, \dots, e_4\}$, all the components A_1, B_1 ,C_1, D_1, E_1, F_1 (written shortly as $A, B, \dots, F$) are distinct and all the other curvature components are zero. Further, suppose that the following two sets of inequalities hold for $A, B, \dots, F$:*

$$\text{(A)} \quad \begin{vmatrix} D-A & B-A & 0 \\ F-C & 0 & C-B \\ 0 & F-E & E-D \end{vmatrix} \neq 0, \quad \begin{vmatrix} D-B & B-A & 0 \\ E-C & 0 & C-A \\ 0 & E-F & D-F \end{vmatrix} \neq 0,$$

$$\begin{vmatrix} A-E & A-C & 0 \\ B-F & 0 & B-C \\ 0 & D-F & D-E \end{vmatrix} \neq 0, \quad \begin{vmatrix} B-D & A-D & 0 \\ C-E & 0 & A-E \\ 0 & C-F & B-F \end{vmatrix} \neq 0.$$

(B) Under the notations

$$\alpha_1 = \frac{B-C}{A-C}, \ \beta_1 = \frac{B-C}{A-B}, \ \alpha_2 = \frac{E-D}{E-A}, \ \beta_2 = \frac{E-D}{D-A},$$
$$\alpha_3 = \frac{D-F}{B-F}, \ \beta_3 = \frac{D-F}{B-D}, \ \alpha_4 = \frac{F-E}{F-C}, \ \beta_4 = \frac{F-E}{E-C},$$

the rank of the matrix

$$\begin{pmatrix} 0 & 0 & 2\alpha_3 & 2\alpha_4 & A \\ 0 & 2\alpha_2 & 0 & -2\beta_4 & B \\ 0 & -2\beta_2 & -2\beta_3 & 0 & C \\ 2\alpha_1 & 0 & 0 & 2\alpha_4\beta_4 & D \\ -2\beta_1 & 0 & 2\alpha_3\beta_3 & 0 & E \\ 2\alpha_1\beta_1 & 2\alpha_2\beta_2 & 0 & 0 & F \end{pmatrix}$$

is equal to five.

Then the curvature tensor R does not belong to any curvature homogeneous space.

Remark. a) The inequalities (A) are all independent. E.g., if we take $A = B = C = u$, $D = E = F = v$, $u \neq v$, then the first three determinants are zero and the last one is non zero.
b) The coefficients α_i, β_i from (B) are not independent but they satisfy the equalities

$$\alpha_i\beta_i = \beta_i - \alpha_i \quad (i = 1, \dots, 4). \tag{14}$$

Proof of the Theorem. The four subsystems of linear equations with separated variables coming from (E1) - (E24) are now given by the tables

S^3_{11}	S^4_{12}	S^4_{21}	S^3_{22}	S^2_{41}	S^4_{43}
D - A	0	0	B - A	0	0
0	D - F	F - B	0	0	0
F - C	0	0	0	0	C - B
0	F - D	0	0	B - D	0
0	0	0	F - E	0	E - D

S^2_{11}	S^4_{13}	S^4_{31}	S^3_{32}	S^3_{41}	S^4_{42}
D - B	0	0	B - A	0	0
0	E - D	A - E	0	0	0
E - C	0	0	0	0	C - A
0	E - D	0	0	A - D	0
0	0	0	E - F	0	D - F

S^4_{11}	S^3_{12}	S^3_{21}	S^4_{22}	S^2_{31}	S^4_{33}
A - E	0	0	A - C	0	0
0	F - E	C - F	0	0	0
B - F	0	0	0	0	B - C
0	F - E	0	0	C - E	0
0	0	0	D - F	0	D - E

S^2_{21}	S^4_{23}	S^3_{31}	S^4_{32}	S^4_{41}	S^3_{42}
0	B - C	0	C - A	0	0
B - D	0	A - D	0	0	0
0	B - C	0	0	0	B - A
C - E	0	0	0	A - E	0
0	0	C - F	0	B - F	0

We put

$$U_1 = S^4_{23},\ U_2 = S^4_{13},\ U_3 = S^4_{12},\ U_4 = S^3_{12} \tag{15}$$

and we shall try to express the other unknown functions S^k_{ij} through U_1, U_2, U_3, U_4 from the previous systems of equations. Now the inequalities (A) from our Theorem guarantee that the Cramer's rule can be used in all cases. We obtain easily

$$S^j_{ii} = 0,\ S^j_{ji} = 0 \text{ for all } i, j,\ (1 \le i < j \le 4), \tag{16}$$

and

$$\begin{aligned} S^4_{32} &= \alpha_1 U_1, \quad S^3_{42} = \beta_1 U_1, \qquad S^4_{31} = \alpha_2 U_2, \quad S^3_{41} = \beta_2 U_2, \\ S^4_{21} &= \alpha_3 U_3, \quad S^2_{41} = \beta_3 U_3, \qquad S^3_{21} = \alpha_4 U_4, \quad S^2_{31} = \beta_4 U_4. \end{aligned} \tag{17}$$

Next, we shall use the known identities

$$[\nabla_{E_i}, \nabla_{E_j}]E_k - \nabla_{[E_i,E_j]}E_k = R(E_i, E_j)E_k. \tag{18}$$

Due to (8),(9) we have

$$[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = \sum_l (S^l_{ij} - S^l_{ji}) E_l \tag{19}$$

and (18) can be rewritten in the form

$$E_i(S^l_{jk}) - E_j(S^l_{ik}) + S^u_{jk} S^l_{iu} - S^u_{ik} S^l_{ju} - S^u_{ij} s^l_{uk} + S^u_{ji} S^l_{uk} = -R_{ijkl}. \tag{20}$$

(36 equations, in which u is a summation index.)

Now, let us consider the six equations of (20) for which $1 \leq i < j \leq 4$, $(k,l) = (i,j)$. According to (16), all these equations are purely algebraic. Substituting from (15)-(17) we get easily

$$\begin{aligned} 2\alpha_3(U_3)^2 + 2\alpha_4(U_4)^2 &= A, \\ 2\alpha_2(U_2)^2 - 2\beta_4(U_4)^2 &= B, \\ -2\beta_2(U_2)^2 - 2\beta_3(U_3)^2 &= C, \\ 2\alpha_1(U_1)^2 + 2\alpha_4\beta_4(U_4)^2 &= D, \\ -2\beta_1(U_1)^2 + 2\alpha_3\beta_3(U_3)^2 &= E, \\ 2\alpha_1\beta_1(U_1)^2 + 2\alpha_2\beta_2(U_4)^2 &= F. \end{aligned} \tag{21}$$

This is a system of six linear algebraic equations for four unknown functions $(U_1)^2$, $(U_2)^2$, $(U_3)^2$, $(U_4)^2$. The matrix of the corresponding homogeneous system is of rank ≤ 4. Hence, if the condition (B) of Theorem 5.1 is satisfied, then the system (21) has no solution according to the Frobenius Theorem. But this is a contradiction to Proposition 3.3, and a curvature homogeneous space with such a type of curvature tensor cannot exist. This concludes the proof of Theorem 5.1.$\square$

Remark 1. The condition (B) is a bit ackward to check. But we can obtain very simple sufficient conditions which already imply contradictions. E.g., if we require $\operatorname{sgn} \alpha_3 = \operatorname{sgn} \alpha_4 = -\operatorname{sgn} A$, then the first equation (21) is contradictory, and we get similar conditions for the other equations.

Remark 2. For the curvature tensor of the form $R = aR_{S^4} + R_{CP^2}$ from the Introduction one can check easily that, with respect to a convenient orthonormal basis, $A_1 = F_1 = 4\lambda + c$, $B_1 = C_1 = D_1 = E_1 = \lambda + c, A_2 = 2\lambda, B_2 = \lambda, \lambda c \neq 0$, and the other components are zero. Hence the σ-rank of R is equal to eight, and our method cannot be used in any way.

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