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ON ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THIRD ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

Ivan Kiguradze

Abstract. The asymptotic properties of solutions of the equation $u'''(t) = p_1(t)u(\tau_1(t))+p_2(t)u'(\tau_2(t))$, are investigated where $p_i : [a, +\infty[\to R \quad (i = 1, 2)$ are locally summable functions, $\tau_i : [a, +\infty[\to R \quad (i = 1, 2)$ measurable ones and $\tau_i(t) \geq t \quad (i = 1, 2)$. In particular, it is proved that if $p_1(t) \leq 0, p_2^2(t) \leq \alpha(t)|p_1(t)|$,

$$\int_a^{+\infty} [\tau_1(t) - t]^2 p_1(t)dt < +\infty \quad \text{and} \quad \int_a^{+\infty} \alpha(t)dt < +\infty,$$

then each solution with the first derivative vanishing at infinity is of the Kneser type and a set of all such solutions forms a one-dimensional linear space.

Let us consider the differential equation

$$u'''(t) = p_1(t)u(\tau_1(t)) + p_2(t)u'(\tau_2(t)) , \tag{1}$$

where the functions $p_i : [a, +\infty[\to R \quad (i = 1, 2)$ are locally integrable and the functions $\tau_i : [a, +\infty[\to R \quad (i = 1, 2)$ are measurable and

$$\tau_i(t) \geq t \quad \text{for} \quad t \geq a \quad (i = 1, 2) . \tag{2}$$

The solution u of the equation (1) will be called of the **Kneser type** if it satisfies the inequalities

$$u'(t)u(t) \leq 0, \quad u''(t)u(t) \geq 0 \quad \text{for} \quad t \geq a_0,$$

for some $a_0 \in [a, +\infty[$, and will be called **vanishing at infinity** if

$$\lim_{t\to+\infty} u(t) = 0 .$$

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Let K be a set of all Kneser type solutions of (1), W be a set of all solutions of (1) satisfying the condition

$$\int_a^{+\infty} u'^2(t)dt < +\infty\,,$$

and Z be a set of all solutions of the same equation satisfying the condition

$$\lim_{t\to+\infty} u'(t) = 0\,.$$

The results of [1,2] imply that if either of the two conditions

$$\text{(i)} \qquad \tau_1(t) \equiv t, \quad p_2(t) \equiv 0, \quad p_1(t) \leq 0\,;$$

$$\text{(ii)} \qquad \begin{cases} p_1(t) \leq 0, & \int_a^{+\infty} s^2|p_1(s)|ds < +\infty\,, \\ p_2(t) \geq 0, & \int_a^{+\infty} \frac{s^2}{\tau_2(s)} p_2(s)ds < +\infty\,, \end{cases}$$

is fulfilled, then $W \supset K$, $Z \supset K$ and K is a one-dimensional linear space. Questions as to the dimension of K, W and Z and the interconnection of these spaces have virtually remained univestigated in the case when the conditions (i) and (ii) are violated. This paper is devoted exactly to the investigation of these questions.

Theorem 1. *Let $\tau_i(t) \geq t$ $(i = 1, 2)$, $p_1(t) \leq 0$ for $t \geq a$,*

$$\int_a^{+\infty} \big[\tau_1(t) - t\big]^2 \big|p_1(t)\big| dt < +\infty \tag{3}$$

and

$$p_2^2(t) \leq \alpha(t)\big|p_1(t)\big| \quad \text{for } t \geq a, \tag{4}$$

where $\alpha : [a, +\infty[\to [0, +\infty[$ is a summable function. Then

$$K = Z, \quad \dim Z = 1 \tag{5}$$

and for each solution $u \in Z$ to vanish at infinity it is necessary and sufficient that

$$\int_a^{+\infty} t^2 \big|p_1(t)\big| dt = +\infty\,. \tag{6}$$

Before proceeding to the proof of the theorem we shall give two auxiliary statements, using the notation

$$\tau(x) = \underset{a\leq t\leq x}{ess\ sup}\big[\max_{1\leq i\leq 2} \tau_i(t)\big]\,.$$

Lemma 1. *Let the conditions of Theorem 1 be fulfilled and $a_0 \in [a, +\infty[$ be so large that*

$$\int_{a_0}^{+\infty} \big[\tau_1(t) - t\big]^2 \big|p_1(t)\big| dt < \frac{1}{4}, \quad \int_{a_0}^{+\infty} \alpha(t) dt < \frac{1}{8}\,. \tag{7}$$

Then an arbitrary solution u of the equation (1) satisfies the condition

$$\begin{gathered} 2u'^2(t) + \int_t^x \big|p_1(s)\big| u^2(s) ds \le \\ \le 4u(t)u''(t) - 4u(x)u''(x) + 2u'^2(x) + \rho(t,x) \quad \textit{for } \ x > t \ge a_0, \end{gathered} \tag{8}$$

where

$$\rho(t,x) = \sup_{t \le s < \tau(x)} u'^2(s)\,.$$

If however $u \in Z$, then $u \in K$,

$$u'^2(t) + \int_t^{+\infty} \big|p_1(s)\big| u^2(s) ds \le 4u(t)u''(t) \quad \textit{for } \ t \ge a_0, \tag{9}$$

and

$$\int_t^{+\infty} \big[10(s-t)u'^2(s) + (s-t)^2 \big|p_1(s)\big| u^2(s)\big] ds \le 4u^2(t) \quad \textit{for } \ t \ge a_0. \tag{10}$$

Proof. Let u be an arbitrary solution of the equation (1). Then in view of the non-positivity of p_1 and the inequality (4) we have

$$\begin{aligned} & u'''(t)u(t) + |p_1(t)|u^2(t) \\ &= p_1(t)u(t) \int_t^{\tau_1(t)} u'(\xi) d\xi + p_2(t)u(t)u'(\tau_2(t)) \\ &\le \big[\tau_1(t) - t\big] \big|p_1(t)\big| \big|u(t)\big| \big[\rho(t,t)\big]^{1/2} + \big[\alpha(t)\big|p_1(t)\big|\rho(t,t)\big]^{1/2} \big|u(t)\big| \\ &\le \frac{1}{2}\big|p_1(t)\big|u^2(t) + \frac{1}{2}\big[\tau_1(t) - t\big]^2 \big|p_1(t)\big|\rho(t,t) + \frac{1}{4}\big|p_1(t)\big|u^2(t) + \alpha(t)\rho(t,t) \\ &= \frac{3}{4}\big|p_1(t)\big|u^2(t) + \Big(\frac{1}{2}\big[\tau_1(t) - t\big]^2\big|p_1(t)\big| + \alpha(t)\Big)\rho(t,t) \end{aligned}$$

for $t \ge a_0$. Integrating this inequality from t to x and taking into account (7), we find

$$\begin{aligned} & u''(x)u(x) - u''(t)u(t) + \frac{1}{2}\big[u'^2(t) - u'^2(x)\big] + \int_t^x \big|p_1(s)\big| u^2(s) ds \\ & \qquad \le \frac{3}{4} \int_t^x \big|p_1(s)\big| u^2(s) ds + \frac{1}{4}\rho(t,x) \quad \text{for } \ x \ge t \ge a_0. \end{aligned}$$

Thus the inequality (8) is valid.

Let us assume now that $u \in Z$. Then

$$\liminf_{x\to+\infty} \big|u''(x)u(x)\big| = 0.$$

Therefore (8) implies

$$2u'^2(t) + \int_t^{+\infty} \big|p_1(s)\big|u^2(s)ds \le 4u''(t)u(t) + \rho_0(t) \tag{11}$$

for $t \ge a_0$, where

$$\rho_0(t) = \max_{t\le s<+\infty} u'^2(s)\,.$$

We shall show that

$$u''(t)u'(t) \le 0 \quad \text{for} \quad t \ge a_0. \tag{12}$$

Let us assume the opposite: we have

$$u''(t_0)u'(t_0) > 0\,.$$

for some $t_0 \in [a_0, +\infty[$. Then, since u' vanishes at infinity, there is $t_1 \in]t_0, +\infty[$ such that

$$u''(t_1) = 0, \quad u'^2(t_1) = \rho_0(t_1) > 0\,.$$

Therefore from (11) we find

$$2\rho_0(t_1) < \rho_0(t_1)\,.$$

The obtained contradiction proves the validity of the inequality (12), while from (11) and (12) it follows that u satisfies the inequality (9) and $u \in K$.

Integrating twice the inequality (9) we obtain the inequality (10). □

Lemma 2. *Let the conditions of Lemma 1 be fulfilled and there exist a number $b \in]a_0, +\infty[$ such that*

$$p_i(t) = 0 \quad \textit{for} \quad t \ge b \quad (i = 1, 2)\,. \tag{13}$$

Then for any $c \in R$ there exists one and only one solution of the equation (1) satisfying the conditions

$$u(a_0) = c, \quad u'(t) = 0 \quad \textit{for} \;\; t \ge b. \tag{14}$$

Proof. Due to (2) and (13), for any $\gamma \in R$ the equation (1) has the unique solution $v(\cdot, \gamma)$ satisfying the condition

$$v(t;\gamma) = \gamma$$

for $b \le t < +\infty$, and

$$v(t;\gamma) = \gamma v(t;1)\,.$$

Since $v(\cdot;1) \in Z$, by virtue of Lemma 1 we have

$$v(a_0;1) \ge 1\,.$$

From the above reasoning it is clear that

$$u(t) = \gamma_c v(t;1)\,,$$

where

$$\gamma_c = c/v(a_0;1)\,,$$

is the unique solution of the problem (1), (14). □

Proof of Theorem 1. Let a_0 be so large that the inequalities (7) are fulfilled. First of all we shall show that for any $c \in R$ the equation (1) has at least one solution satisfying the conditions

$$u(a_0) = c, \quad \lim_{t\to+\infty} u'(t) = 0\,. \tag{15}$$

Let k be an arbitrary natural number and

$$p_{ik}(t) = \begin{cases} p_i(t) & \text{for } a_0 \le t \le a_0 + k \\ 0 & \text{for } t > a_0 + k \end{cases} \qquad (i = 1, 2)\,. \tag{16}$$

On account of Lemma 2 the equation

$$u'''(t) = p_{1k}(t)u(\tau_1(t)) + p_{2k}(t)u'(\tau_2(t)) \tag{17}$$

has the unique solution u_k satisfying the conditions

$$u_k(a_0) = c, \quad u_k'(t) = 0 \quad \text{for } t \ge a + k. \tag{18}$$

On the other hand, by Lemma 1 $u_k \in K$, i.e.,

$$u_k(t)u_k'(t) \le 0, \quad u_k(t)u_k''(t) \ge 0 \tag{19}$$

for $t \ge a_0$. If alongside with this we take into account the conditions (2) and (16), then we can easily ascertain that the sequences $\left(u_k^{(i)}\right)_{k=1}^{+\infty}$ $(i = 0, 1, 2)$ are uniformly bounded and equicontinuous on each segment contained in $[a, +\infty[$. Therefore, according to the Arzela-Ascoli lemma, from $\left(u_k\right)_{k=1}^{+\infty}$ we can obtain the subsequence $\left(u_{k_m}\right)_{m=1}^{+\infty}$ converging uniformly together with $\left(u_{k_m}^{(i)}\right)_{m=1}^{+\infty}$ $(i = 1, 2)$ on each segment contained in $[a, +\infty[$. By virtue of (16), (18) and (19) the function

$$u(t) = \lim_{m\to+\infty} u_{k_m}(t)$$

is the solution of (1) satisfying the conditions

$$u(a_0) = c, \quad u(t)u'(t) \le 0, \quad u(t)u''(t) \ge 0$$

for $t \ge a_0$. But

$$u \in K \Rightarrow \lim_{t\to+\infty} u'(t) = 0.$$

Therefore u is the solution of the problem (1), (15). We have thereby proved that

$$\dim Z \ge 1\,.$$

Due to Lemma 1 $Z = K$. Let us show that $\dim Z = 1$. For this it is sufficient to establish that for an arbitrary $c \in R$ the problem (1), (15) has at most one solution. Let u_1 and u_2 be arbitrary solutions of this problem and

$$u_0(t) = u_2(t) - u_1(t)\,.$$

Since $u_0 \in Z$ and $u_0(a_0) = 0$, by Lemma 1 we have

$$\int_{a_0}^{+\infty} (s - a_0){u'_0}^2(s)ds \le 0,$$

i.e. $u'_0(t) = 0$ for $t \ge a_0$. Hence it follows that $u_0(t) \equiv 0$, i.e., $u_1(t) \equiv u_2(t)$.

Let us proceed to the proof of the second part of the theorem. Let $u \in Z$. Then by virtue of Lemma 1 $u \in K$ and the inequality (10) is fulfilled. Hence it is clear that if the condition (6) is fulfilled, then u is a vanishing solution at infinity.

To complete the proof of the theorem it remains for us to establish that if

$$\int_a^{+\infty} s^2\big|p_1(s)\big|ds < +\infty\,, \tag{20}$$

then each nontrivial solution $u \in Z$ tends to a limit differing from zero as $t \to +\infty$. Let us assume the opposite: there exists a nontrivial solution $u \in Z$ vanishing at infinity. Then by Lemma 1

$$\begin{gathered}\big|u(t)\big|' \le 0, \quad \big|u'(t)\big|' \le 0, \quad \lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} u'(t) = 0; \\ \liminf_{t\to+\infty} \big|u''(t)\big| = 0\end{gathered} \tag{21}$$

and

$$v(t) \le 2\big|u(t)\big| \tag{22}$$

for $t \ge a_0$, where

$$v(t) = \bigg(\int_t^{+\infty} \big[10(s-t){u'}^2(s) + (s-t)^2\big|p_1(s)\big|u^2(s)\big]\,ds\bigg)^{1/2}.$$

By the conditions (2), (4), (7) and (20)-(22) we have

$$\begin{aligned}
\big|u''(t)\big| &= \bigg|\int_t^{+\infty} \big[p_1(s)u(\tau_1(s)) + p_2(s)u'(\tau_2(s))\big]\,ds\bigg| \\
&\le \int_t^{+\infty} \big|p_1(s)\big|\big|u(s)\big|ds + \bigg[\int_t^{+\infty} \big(\alpha(s)\big|p_1(s)\big|\big)^{1/2} ds\bigg]\big|u'(t)\big| \\
&\le \int_t^{+\infty} \big|p_1(s)\big|\big|u(s)\big|ds + \bigg(\int_t^{+\infty} \alpha(s)ds\bigg)^{1/2}\bigg(\int_t^{+\infty} \big|p_1(s)\big|ds\bigg)^{1/2}\big|u'(t)\big| \\
&\le \int_t^{+\infty} \big|p_1(s)\big|\big|u(s)\big|ds + \bigg(\int_t^{+\infty} \big|p_1(s)\big|ds\bigg)^{1/2}\big|u'(t)\big|
\end{aligned}$$

for $t \ge a_0$, and

$$\begin{aligned}
\big|u(t)\big| &= \int_t^{+\infty} (s-t)\big|u''(s)\big|ds \\
&\le \frac{1}{2}\int_t^{+\infty} (s-t)^2\big|p_1(s)\big|\big|u(s)\big|ds + \int_t^{+\infty} (s-t)\bigg(\int_s^{+\infty} \big|p_1(\xi)\big|d\xi\bigg)^{1/2}\big|u'(s)\big|ds \\
&\le \frac{1}{2}\bigg(\int_t^{+\infty} (s-t)^2\big|p_1(s)\big|ds\bigg)^{1/2}\bigg(\int_t^{+\infty} (s-t)^2\big|p_1(s)\big|u^2(s)ds\bigg)^{1/2} \\
&\quad + \bigg[\int_t^{+\infty} (s-t)\bigg(\int_s^{+\infty} \big|p_1(\xi)\big|d\xi\bigg)ds\bigg]^{1/2}\bigg(\int_t^{+\infty} (s-t)u'^2(s)ds\bigg)^{1/2} \\
&\le \bigg(\int_t^{+\infty} (s-t)^2\big|p_1(s)\big|ds\bigg)^{1/2} v(t) \le 2\bigg(\int_t^{+\infty} (s-t)^2\big|p_1(s)\big|ds\bigg)^{1/2}\big|u(t)\big|
\end{aligned}$$

for $t \ge a_0$. From the latter inequality it is clear that for some sufficiently large a_1

$$u(t) = 0$$

for $t \ge a_1$. Hence, in view of (2), we have $u(t) = 0$ for $t \ge a$. But this contradicts our assumption about the nontriviality of u. The obtained contradiction proves the theorem. □

Theorem 2. *Let $\tau_i(t) \ge t \quad (i = 1,2)$, $p_1(t) \le 0$ for $t \ge a$, and the function τ_2 be locally absolutely continuous and nondecreasing. Let, besides,*

$$\int_a^{+\infty} t\big[\tau_1(t) - t\big]\big|p_1(t)\big|dt < +\infty \tag{23}$$

and

$$p_2^2(t) \le (6-\varepsilon)\frac{\tau_2'(t)}{t-a}\big|p_1(t)\big| \quad \text{for } t > a, \tag{24}$$

where ε is a arbitrary small positive number. Then

$$\dim W = 1$$

and for each solution $u \in W$ to vanish at infinity it is necessary and sufficient that

$$\int_a^{+\infty} t^2 \big|p_1(t)\big| dt = +\infty \,. \tag{25}$$

To prove the theorem we shall need

Lemma 3. *Let the conditions of Theorem 1 be fulfilled and $a_0 \in [a, +\infty[$ be so large that*

$$\int_{a_0}^{+\infty} (s-a_0)[\tau_1(s)-s]\big|p_1(s)\big| ds \le 4\delta^2 \,, \tag{26}$$

where $\delta = \frac{\varepsilon}{20}$. Then an arbitrary solution $u \in W$ satisfies the conditions

$$\delta \int_t^{+\infty} \big[{u'}^2(s) + (s-t)\big|p_1(s)\big|u^2(s)\big]\, ds \le -u'(t)u(t) \tag{27}$$

for $t \ge a_0$, and

$$2\delta \int_t^{+\infty} \big[(s-t){u'}^2(s) + (s-t)^2\big|p_1(s)\big|u^2(s)\big]\, ds \le u^2(t) \tag{28}$$

for $t \ge a_0$.

Proof. Let u be an arbitrary solution of the equation (1). Then in view of the nonpositiveness of p_1

$$\begin{gathered}(s-t)u'''(s)u(s) + (s-t)\big|p_1(s)\big|u^2(s) \\ = (s-t)p_1(s)u(s)\int_s^{\tau_1(s)} u'(\xi)d\xi + (s-t)p_2(s)u(s)u'(\tau_2(s))\,.\end{gathered}$$

The integration of this identity from t to x gives

$$\begin{gathered}u'(t)u(t) + (x-t)u''(x)u(x) - (x-t)\frac{{u'}^2(x)}{2} - u'(x)u(x) \\ + \int_t^x \big[\frac{3}{2}{u'}^2(s) + (s-t)\big|p_1(s)\big|u^2(s)\big]\, ds \\ = \int_t^x \big[(s-t)p_1(s)u(s)\int_s^{\tau_1(s)} u'(\xi)d\xi + (s-t)p_2(s)u(s)u'(\tau_2(s))\big]\, ds\,.\end{gathered} \tag{29}$$

But according to the Schwartz inequality and the conditions (24) and (26)

$$\int_t^x \big[(s-t)p_1(s)u(s)\int_s^{\tau_1(s)} u'(\xi)d\xi\big]\,ds$$
$$\leq \delta\int_t^x (s-t)\big|p_1(s)\big|u^2(s)ds+\frac{1}{4\delta}\int_t^x (s-t)\big|p_1(s)\big|\Big[\int_s^{\tau_1(s)} u'(\xi)d\xi\Big]^2 ds$$
$$\leq \delta\int_t^x (s-t)\big|p_1(s)\big|u^2(s)ds+\frac{1}{4\delta}\int_t^x (s-t)\big[\tau_1(s)-s\big]\,\big|p_1(s)\big|\Big[\int_s^{\tau_1(s)} {u'}^2(\xi)d\xi\Big]ds$$
$$\leq \delta\int_t^x (s-t)\big|p_1(s)\big|u^2(s)ds+\delta\int_t^{\tau(x)} {u'}^2(s)ds$$

for $x\geq t\geq a_0$, and

$$\int_t^x (s-t)p_2(s)u(s)u'(\tau_2(s))ds$$
$$\leq \big(6-\varepsilon\big)^{1/2}\int_t^x \Big[(s-t)\tau_2'(s)\big|p_1(s)\big|\Big]^{1/2}\big|u(s)\big|\big|u'(\tau_2(s))\big|ds$$
$$\leq (1-2\delta)\int_t^x (s-t)\big|p_1(s)\big|u^2(s)ds+\frac{6-\varepsilon}{4(1-2\delta)}\int_t^x \tau_2'(s){u'}^2(\tau_2(s))ds$$
$$\leq (1-2\delta)\int_t^x (s-t)\big|p_1(s)\big|u^2(s)ds+\Big(\frac{3}{2}-2\delta\Big)\int_t^{\tau(x)} {u'}^2(s)ds$$

for $x\geq t\geq a_0$, where

$$\tau(x)=\underset{a\leq t\leq x}{ess\ sup}\Big[\max_{1\leq i\leq 2}\tau_i(x)\Big]\,.$$

Therefore (29) implies

$$\begin{aligned}\delta\int_t^x \Big[{u'}^2(s)+(s-t)p_1(s)u^2(s)\Big]ds &\leq -u'(t)u(t)+(x-t)\frac{{u'}^2(x)}{2}\\ &+u'(x)u(x)-(x-t)u''(x)u(x)+\big(\frac{3}{2}-\delta\big)\int_x^{\tau(x)} {u'}^2(s)ds\end{aligned}\tag{30}$$

for $x\geq t\geq a_0$.

From the condition $u\in W$ it immediately follows that

$$\lim_{x\to+\infty}\frac{1}{x}\int_a^x (s-a){u'}^2(s)ds=0$$

and

$$\lim_{x\to+\infty}\frac{u^2(x)}{x}=0\,.\tag{31}$$

We shall show that for any $t \in [a, +\infty[$

$$\liminf_{x\to+\infty} \Big| (x-t)\frac{u'^2(x)}{2} + u'(x)u(x) - (x-t)u''(x)u(x) \Big| = 0\,. \tag{32}$$

Let us assume the opposite. Then there are numbers $\sigma \in \{-1, 1\}$, $t_0 \in [a, +\infty[$, $t_1 \in]t_0, +\infty[$ and $\eta > 0$ such that

$$\sigma\Big[(x-t_0)\frac{u'^2(x)}{2} + u'(x)u(x) - (x-t_0)u''(x)u(x)\Big] > \eta \tag{33}$$

for $x \geq t_1$,

$$\int_a^x (s-a)u'^2(s)ds \leq \frac{\eta}{6}(x-t_0)\,, \quad u^2(x) < \frac{\eta}{4}(x-t_0). \tag{34}$$

for $x \geq t_1$. Integrating the inequality (33) from t_1 to x, we find

$$\sigma\Big[\frac{3}{2}\int_{t_1}^x (s-t_0)u'^2(s)ds + u^2(x) - (x-t_0)u'(x)u(x)\Big] \geq \eta(x-t_1) - c_1$$

for $x \geq t_1$, where

$$c_1 = u^2(t_1) - (t_1-t_0)u'(t_1)u(t_1)\,.$$

Hence due to (34) we obtain

$$-\sigma(x-t_0)u'(x)u(x) \geq \frac{\eta}{2}(x-t_1) - \frac{\eta}{2}(t_1-t_0) - c_1$$

for $x > t_1$, and

$$-\sigma u'(x)u(x) \geq \frac{\eta}{4}$$

for $x \geq t_2$, where t_2 is some suffieciently large number. Therefore

$$\sigma\Big[u^2(t_2) - u^2(x)\Big] \geq \frac{\eta}{4}(x-t_2)$$

for $x \geq t_0$, which contradicts the condition (31). The obtained contradiction proves the validity of the equality (32).

By virtue of (32) the inequality (30) implies the inequality (27), while by integrating (27) from t to $+\infty$, we obtain the inequality (28). □

The proof of the next lemma repeats that of Lemma 2.

Lemma 4. *Let the conditions of Lemma 3 and the identities (13), where $b \in]a_0, +\infty[$, be fullfilled. Then for any $c \in R$ the problem (1), (14) has one and only one solution.*

Proof of Theorem 2. Let a_0 be so large that the inequality (26) is fulfilled, and $c \in R$ be an arbitrarily fixed number.

According to Lemma 4 for any natural k the differential equation (17), where p_{ik} $(i = 1, 2)$ are the functions given by the equality (16), has the unique solution u_k satisfying the conditions (18).

By virtue of Lemma 3

$$\delta \int_t^{+\infty} \Big[{u'}_k^2(s) + (s-t)\big|p_1(s)\big|u_k^2(s)\Big]\, ds \le -u_k'(t)u_k(t)$$

for $t \ge a_0$, and

$$2\delta \int_t^{+\infty} \Big[(s-t){u'}_k^2(s) + (s-t)^2\big|p_1(s)\big|u_k^2(s)\Big]\, ds \le u_k^2(t)$$

for $t \ge a_0$.

Hence, on account of the Arzela-Ascoli lemma, we readily conclude that the sequence $\big(u_k\big)_{k=1}^{+\infty}$ contains the subsequence $\big(u_{k_m}\big)_{m=1}^{+\infty}$ converging uniformly together with $\big(u_{k_m}^{(i)}\big)_{m=1}^{+\infty}$ $(i = 1, 2)$ on each finite segment from $[a, +\infty[$ and

$$u(t) = \lim_{m \to +\infty} u_{k_m}(t)$$

is the solution of the equation (1) satisfying the conditions

$$u(a_0) = c, \quad \int_{a_0}^{+\infty} {u'}^2(s) ds < +\infty\,.$$

We have thereby proved that $\dim W \ge 1$. Thus to prove the equality $\dim W = 1$ it is sufficient to establish that given the conditions

$$u(a_0) = 0, \quad \int_{a_0}^{+\infty} {u'}^2(s) ds < +\infty \tag{35}$$

the equation (1) has only the trivial solution. Indeed, let u be an arbitrary solution of the problem (1), (35). Then by virtues of Lemma 3

$$\delta \int_{a_0}^{+\infty} \Big[{u'}^2(s) + (s-a_0)\big|p_1(s)\big|u^2(s)\Big]\, ds \le 0\,.$$

Therefore

$$u(t) = 0 \quad \text{for} \;\; t \ge a_0.$$

In view of (2) it follows from the last equality that $u(t) = 0$ for $t \geq a$.

Let us prove the second part of the theorem. Let $u \in W$ be an arbitrary solution. By virtue of Lemma 3 the function $|u(\cdot)|$ does not decrease on $[a_0, +\infty[$ and

$$\int_{a_0}^{+\infty} (s-a_0)^2 |p_1(s)| u^2(s) ds < +\infty .$$

Hence it is clear that if the condition (25) is fulfilled, then u vanishes at infinity.

To complete the proof of the theorem it remains for us to establish that if the condition (20) is fulfilled, then each nontrivial solution $u \in W$ tends to a limit differing from zero as $t \to +\infty$. Let us assume the opposite: there exists a nontrivial solution $u \in W$ vanishing at infinity. Then by Lemma 3

$$u(t)u'(t) \leq 0, \quad v(t) \leq \eta |u(t)| \tag{36}$$

for $t \geq a_0$, where $\eta = (2\delta)^{-1/2}$ and

$$v(t) = \left(\int_t^{+\infty} \Big[(s-t){u'}^2(s) + (s-t)^2 |p_1(s)| u^2(s) \Big] ds \right)^{1/2} .$$

On the other hand, due to (24) and (20) we have

$$\begin{aligned}
|u(t)| &= \frac{1}{2} \left| \int_t^{+\infty} (s-t)^2 \big[p_1(s) u(\tau_1(s)) + p_2(s) u'(\tau_2(s)) \big] ds \right| \\
&\leq \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| u^2(\tau_1(s)) ds \right]^{1/2} \\
&\qquad + 2 \int_t^{+\infty} (s-t)^2 \Big[\frac{|p_1(s)| \tau_2'(s)}{s-a} \Big]^{1/2} |u'(\tau_2(s))| ds \\
&\leq \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| u^2(\tau_1(s)) ds \right]^{1/2} \\
&\quad + 2 \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} \left[\int_t^{+\infty} \frac{(s-t)^2}{s-a} \tau_2'(s) {u'}^2(\tau_2(s)) ds \right]^{1/2}
\end{aligned}$$

for $t \geq a_0$. Hence, taking into account (2) and (36), we find

$$\begin{aligned}
|u(t)| &\leq \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| u^2(s) ds \right]^{1/2} \\
&\quad + 2 \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} \left[\int_t^{+\infty} (s-t) {u'}^2(s) ds \right]^{1/2} \\
&\leq 3 \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} v(t) \\
&\leq 3\eta \left[\int_t^{+\infty} (s-t)^2 |p_1(s)| ds \right]^{1/2} |u(t)|
\end{aligned}$$

for $t \geq a_0$, and therefore

$$u(t) = 0$$

for $t \geq a_1$, where a_1 is some sufficiently large number. In view of (2) the last identity implies that $u(t) = 0$ for $t \geq a$. But this contradicts our assumption about the nontriviality of u. □

Corollary. *Let the conditions of Theorem 2 be fulfilled and*

$$p_2(t) \geq 0$$

for $t \geq a$. Then

$$K = W, \quad \dim K = 1\,. \tag{37}$$

Proof. Let $u \in W$ be an arbitrary solution. Then by virtue of Lemma 3

$$u'(t)u(t) \leq 0$$

for $t \geq a_0$. If, alongside with this we take into account the nonpositivity of p_1 and the nonnegativity of p_2 from (1), we find

$$u'''(t)u(t) \leq 0, \quad u''(t)u(t) \geq 0$$

for $t \geq a_0$. Therefore $u \in K$. We have thereby proved that $W \subset K$. But, according to Theorem 2 and the definition of K and W, we have $\dim W = 1$ and $K \subset W$. Therefore it is clear that the equalities (37) are fulfilled. □

As an example, on the interval $[a, +\infty[$ let us consider the equation

$$u'''(t) = -\frac{3}{8t^3}u(t) + \frac{r}{t^2}u'(t)\,, \tag{38}$$

where a and r are positive numbers. This equation has a solution

$$u_r(t) = t^{\lambda_r}\,,$$

where

$$\frac{1}{2} \leq \lambda_r < 1$$

for $\frac{3}{8} < r \leq \frac{3}{2}$, and

$$0 < \lambda_r < \frac{1}{2}$$

for $r > \frac{3}{2}$. Therefore

$$Z \neq K$$

for $r > \frac{3}{8}$, and

$$W \neq K$$

for $r > \frac{3}{2}$. On the other hand, for the equation (38) all conditions of Theorem 1 except the summability of α (since $\alpha(t) = \frac{8r^2}{3}t^{-1}$) are fulfilled in the case $r > \frac{3}{8}$, and all conditions of Theorem 2 except (24) which is replaced by the inequality

$$p_2^2(t) < (6+\varepsilon)\frac{\tau_2'(t)}{t-a}\big|p_1(t)\big|$$

for $t > a$ are fulfilled in the case $r = \frac{3}{2}\big(1+\varepsilon\big)^{1/2}$. This example shows that in Theorem 1 (in the corollary of Theorem 2) the condition of the summability of α (the condition (24)) is optimal in the definite sense and cannot be weakened.

In conclusion we note that results similar to Theorem 3 for second order differential equations are contained in [3].

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Ivan Kiguradze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
Z. Rukhadze St.1, Tbilisi 380093
Republic of Georgia

E-mail: kig@imath.kheta.georgia.su