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REGULATED BUILDUPS OF 3-CONFIGURATIONS

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Abstract. We deal with two types of buildups of 3-configurations: a generating buildup over a given edge set and a regulated one (according to maximal relative degrees of vertices over a penetrable set of vertices). Then we take account to minimal generating edge sets, i.e., to edge bases. We also deduce the fundamental relation between the numbers of all vertices, of all edges from edge basis and of all terminal elements. The topic is parallel to a certain part of Belousov' "Configurations in algebraic nets" edited in 1979. We attempt to find an apparatus, which, beside others, will decode some less readible chapters of the monograph [1], and which can be useful by further study of 3-configurations with simple edge bases and of corresponding quasigroup identities.

A (3)-*configuration* is defined as a triple $(\mathcal{V}, \mathcal{E}, \mathrm{I})$, where $\mathcal{V}$ is a finite non-empty set (of *vertices*), $\mathcal{E}$ is a non-empty set (of *edges*) disjoint to $\mathcal{V}$ and I is a binary relation (of *adjacency*) in $\mathcal{V} \cup \mathcal{E}$ such that $x \ \mathrm{I} \ y$ implies either $x \in \mathcal{V}$, $y \in \mathcal{E}$ or $y \in \mathcal{V}$, $x \in \mathcal{E}$. Moreover, the number of edges adjacent to the same vertex is at most 3 and any two distinct vertices are simultaneously adjacent to one edge at most. If, especially, every edge is adjacent to at least two vertices and every vertex is adjacent to just three edges, then $\mathfrak{C}$ is said to be a 3-*configuration*.

A (3)-configuration is said to be *connected* if to any two distinct vertices v, w there exists a finite sequence $(e_1, \dots, e_l)$ of edges such that $e_1 = v$, $e_l = w$ a that any two consecutive edges are distinct and simultaneously adjacent to the same vertex.

Further notions: Two vertices are *neighboring* if they are different and simultaneously adjacent to the same edge. Two edges are *neighboring* if they are different and simultaneously adjacent to the same vertex. Three mutually distinct edges adjacent to the same vertex are said to be *concurrent*.

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In the sequel every 3-configuration under consideration is assumed to be connected whereas by (3)-configurations this assumption will not be made.

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a (3)-configuration. We shall denote every ordering of the set $\mathcal{V} \cup \mathcal{E}$ as a *buildup* of $\mathfrak{C}$. Under a *relative degree* of an element $x \in \mathcal{V} \cup \mathcal{E}$ we mean the number of all preceding elements adjacent to x. Under a *socle* of a vertex v we shall understand the set of all preceding edges adjacent to v. Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration. A non-void vertex set $\mathcal{X} \subseteq \mathcal{V}$ will be denoted as *penetrable* if for every $x \in \mathcal{X}$ there exists an edge e_x (the *turning edge*) adjacent to x but to no further vertex from $\mathcal{X}$.

We describe the construction of every penetrable set $\mathcal{X} \subseteq \mathcal{V}$. Start with arbitrary couple of adjacent elements (of a vertex v_1) and an edge $e_1 \mathrel{\mathrm{I}} v_1$. Choose a further $\begin{cases} \text{edge} & e_2 \not\mathrel{\mathrm{I}} v_1 \\ \text{vertex} & v_2 \not\mathrel{\mathrm{I}} e_1 \end{cases}$ and, after then, a further $\begin{cases} \text{vertex} & v_2 \mathrel{\mathrm{I}} e_2 \\ \text{edge} & e_2 \mathrel{\mathrm{I}} v_2 \end{cases}$ with $e_2 \not\mathrel{\mathrm{I}} v_1$. The third step consists of the choice of a further $\begin{cases} \text{edge} & e_3 \not\mathrel{\mathrm{I}} v_1, v_2 \\ \text{vertex} & v_3 \not\mathrel{\mathrm{I}} e_1, e_2 \end{cases}$ and, after then, of a further $\begin{cases} \text{vertex} & v_3 \mathrel{\mathrm{I}} e_3 \\ \text{edge} & e_3 \mathrel{\mathrm{I}} v_3 \end{cases}$ with $e_3 \not\mathrel{\mathrm{I}} v_1, v_2$. Further steps are obvious. We proceed as far as possible. After finitely many steps, we finish and find a maximal penetrable set of vertices $v_1, v_2, \dots$ with turning edges $e_1 = e_{v_1}$, $e_2 = e_{v_2}, \dots$. As it is easily seen, this procedure enables to obtain all maximal penetrable vertex sets. Minimal penetrable vertex sets are, of course, just all one-vertex sets (turning edges are at the same time arbitrary adjacent edges).

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration. For every non-empty set $\mathcal{X} \subseteq \mathcal{V}$, or $\mathcal{X} \subseteq \mathcal{E}$, respectively, define the *envelope* $[\mathcal{X}]$ as the set of just all edges such that each of them is adjacent to at least one vertex from $\mathcal{X}$, or as the set of just all vertices such that each of them is adjacent to at least two edges from $\mathcal{X}$.

Now we restrict ourselves onto a starting set $\mathcal{X} \subseteq \mathcal{E}$ and construct successive envelopes $\mathcal{X}_0 := \mathcal{X}$, $\mathcal{X}_1 := [\mathcal{X}_0]$, $\mathcal{X}_2 := [\mathcal{X}_1]$, $\mathcal{X}_3 := [\mathcal{X}_2], \dots$. Then $\bigcup\limits_{i=0}^{\infty} \mathcal{X}_{2i}$ is called the *edge cover* of $\mathcal{X}$ (denotation :$\langle \mathcal{X} \rangle$). If $\langle \mathcal{X} \rangle = \mathcal{E}$ (which implies $\bigcup\limits_{i=0}^{\infty} \mathcal{X}_{2i+1} = \mathcal{V}$) then we shall say that the set $\mathcal{X}$ *generates* the given 3-configuration $\mathfrak{C}$. To the sequence $(\mathcal{X}_i)_{i=0}^{\infty}$ a buildup of $\mathfrak{C}$ can be associated (a *generating buildup* over $\mathcal{X}$) such that instead of each $\mathcal{X}_i$ we insert an arbitrary ordering of all new elements (which are added in the present step). If $\mathcal{X}$ is a minimal generating edge set of $\mathfrak{C}$, then $\mathcal{X}$ is said to be an *edge basis* of $\mathfrak{C}$.

Example 1. Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a Desargues' 3-configuration on Fig. 1. The set $\mathcal{X} = \{e_1, e_2, e_3, e_4\}$ is generating because of $[[\mathcal{X}]] = \mathcal{E}$.

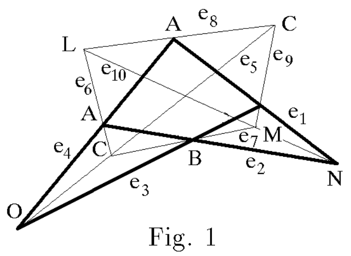


Fig. 1

As $\langle \mathcal{X} \backslash \{e_i\} \rangle \neq \mathcal{E}$ for all $i = \{1, 2, 3, 4\}$, $\mathcal{X}$ is an edge basis of $\mathfrak{C}$.

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration and $\mathcal{A}$ a penetrable vertex set. For every $a \in \mathcal{A}$ denote by e_a the turning edge at a. Vertices from $\mathcal{A}$ and edges from $\mathcal{G}_0 = [\mathcal{A}] \backslash \{e_a | a \in \mathcal{A}\}$ will be denoted as *initial*; all remaining vertices will be called *proper*. Proper vertices will be ordered onto a finite sequence $(w_1, w_2, \ldots, w_e)$ as follows: w_1 is some of proper vertices adjacent to α_1 edges from $[\mathcal{A}]$ with $\alpha_1 \in \{1, 2, 3\}$ to be maximal; w_2 is some of remaining proper vertices adjacent to α_2 edges from $[\mathcal{A}] \cup [w_1]$ with $\alpha_2 \in \{1, 2, 3\}$ to be maximal, w_3 is some of remaining proper vertices adjacent to α_3 edges from $[\mathcal{A}] \cup [w_1] \cup [w_2]$ with $\alpha_3 \in \{1, 2, 3\}$ to be maximal. In this way we proceed so long as all proper vertices will be exhausted.

With this procedure an buildup (called *regulated*) of $\mathfrak{C}$ over $\mathcal{A}$ is associated (cf. [1], Lemma 2.6 on p. 47): We start with initial vertices (in an arbitrary order), on the further place we give the vertex w_1, after then we input new edges adjacent to w_1 (in an arbitrary order) and the same act will be repeated with w_2 and new edges adjacent to w_2 and so forth, till to w_l and to new edges adjacent to w_l. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ are relative degrees of vertices $w_1, w_2, \ldots, w_l$ with respect to the buildup just described.

Under *distinguished* edges we shall first understand all initial edges. Secondly, each proper vertex w with relative degree 1 gives rise to one distinguished edge as one of both subsequent edges adjacent to w. Other edges will be not distinguished.

Theorem 1. Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration with a regulated buildup $\mathscr{L}$ over a penetrating set $\mathcal{A}$. Then the set of all distinguished edges is a generating edge set of $\mathfrak{C}$.

Proof. Let $\mathcal{E}_0$ be the set of all initial edges. As every turning edge forms together with two further edges from $\mathcal{E}_0$ a concurrent triple, it must belong to $\mathcal{E}_1 = [[\mathcal{E}_x 0]]$. Now we shall investigate the edges adjacent to proper vertices $w_1, \ldots, w_l$ (ordered within the framework of $\mathscr{L}$) and proceed by induction according to $i \in \{1, \ldots, l\}$. Let $e_1^{(1)}, e_1^{(2)}, e_1^{(3)}$ be the edges adjacent to w_1. If $\alpha_1 = 1$, then just one of them lies in $\mathcal{E}_1$. One of remaining edges, which we denote by g_1, is distinguished and the last edge belongs to $[[\mathcal{E}_1 \cup \{g_1\}]]$. If $\alpha_2 = 2$, then just two of the edges $e_1^{(1)}, e_1^{(2)}, e_1^{(3)}$ lie in $\mathcal{E}_1$ and the remaining one belongs to $[[\mathcal{E}_1]]$. If $\alpha_1 = 3$, then $e_1^{(1)}, e_1^{(2)}, e_1^{(3)}$ lie already in $\mathcal{E}_1$. An analogous investigation will be made for every $j \in \{2, \ldots, l-1\}$ under assumption that every edge adjacent to some of vertices $w_1, \ldots, w_{j-1}$ belongs to the edge socle $\mathcal{S}_j$ of w_j with respect to $\mathscr{L}$. Let $e_j^{(1)}, e_j^{(2)}, e_j^{(3)}$ be edges adjacent to

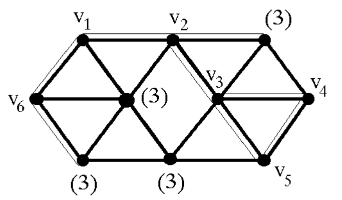
w_j. If $\alpha_j = 1$, then just one of them belongs to $\mathcal{S}_j$, one of two remaining edges (which we denote by g_j) is distinguished and the last one belongs to $[[\mathcal{S}_j \cup \{g_j\}]]$. If $\alpha_j = 2$, then two of edges $e_j^{(1)}, e_j^{(2)}, e_j^{(3)}$ lie in $\mathcal{S}_j$ and the remaining one in $[[\mathcal{S}_j]]$. If $\alpha_j = 3$, then all the $e_j^{(1)}, e_j^{(2)}, e_j^{(3)}$ belong already to $\mathcal{S}_j$. As the sets $\mathcal{V}, \mathcal{E}$ are finite, the socle $\mathcal{S}_l$ of w_l must contain all edges of $\mathfrak{C}$. Let us note that there must exist at least two initial edges and at least one further distinguished edge. $\square$

Theorem 2 (folklore — cf. also [1], Lemma 2.8 on pp. 50-51). *Let $\mathcal{G}$ be the set of all distinguished edges and $\mathcal{T}$ the set of all vertices of relative degree 3 according to a regulated buildup of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$. Then $m + \mu = n + \nu$, where $m = \#\mathcal{V}$, $\nu = \#\mathcal{G}$, $n = \#\mathcal{E}$, $\nu = \#\mathcal{T}$.*

Proof. Denote by $\mathcal{V}_0$, $\mathcal{G}_0$, respectively $\mathcal{V}_i$ $(i \in \{1,2,3\})$ the set of all initial vertices, of all initial edges, respectively of all vertices of relative degree i and put $\#\mathcal{G}_0 = \nu_0$, $\#\mathcal{V}_i = m_i$ $(i \in \{0,1,2,3\})$. Thus $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ with mutually disjoint sets $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, and consequently $m = m_0 + m_1 + m_2 + m_3$. Further, we have $\#[\mathcal{V}_0] = \#\mathcal{G}_0 + \#\mathcal{V}_0 = \mu_0 + m_0$ and $\mu = \#\mathcal{G} = \#\mathcal{G}_0 + \#\mathcal{V}_0 = \mu_0 + m_0$, so that $n = \#\mathcal{E} = \#[\mathcal{V}_0] + 2 \cdot \#\mathcal{V}_1 + 1 \cdot \#\mathcal{V}_2 = (\mu_0 + m_0) + 2 \cdot m_1 + 1 \cdot m_2$. Thus $m + \mu = (m_0 + m_1 + m_2 + m_3) + (\mu_0 + m_1)$, $n + \mu = n + m_3 = (\mu_0 + m_0 + 2m_1 + m_2) + m_3$. We see that both left sides $m + \mu$, $n + \nu$ are equal to the same expression $m_0 + 2m_1 + m_2 + m_3 + \mu_0$ so that $m + \mu = n + \nu$. $\square$

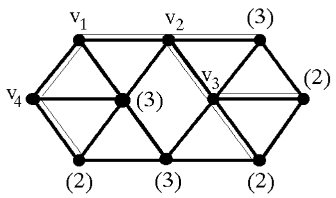
Corollary. *As $\mu \leq 3$, it follows that $n \geq m - \nu + 3$. For $\nu = 1$we have $n \geq m + 2$.*

Example 2a



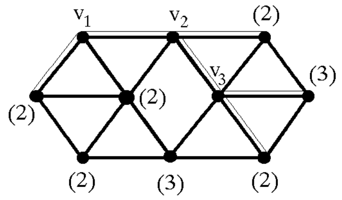
$$\begin{array}{cccc} 10 + & 6 = & 12 + & 4 \\ | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$$

Example 2b



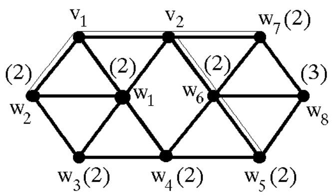
$$\begin{array}{cccc} 10 + & 5 = & 12 + & 3 \\ | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$$

Example 2c



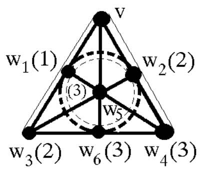
$$\begin{array}{cccc} 10 + & 4 = & 12 + & 2 \\ | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$$

Example 2d



$$\begin{array}{cccc} 10 + & 3 = & 12 + & 1 \\ | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$$

Example 3



$$\mu_0 = 2$$

$$\begin{array}{cccc} 7 & +\,3 = & 7 & +\,3 \\ | & | & | & | \\ \text{m} & \mu & \text{n} & \nu \end{array}$$

where ═══ are initial edges, $====$ are further distinguished edges and $(\cdot)$ are relative degrees.

Theorem 3. *Every edge basis $\mathcal{B}$ of a 3-configuration $\mathfrak{C}= (\mathcal{V},\mathcal{E},\ \mathrm{I})$ is the set of all distinguished edges according to some regulated buildup of $\mathfrak{C}$.*

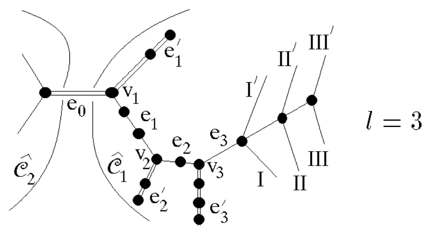
Proof. Let $\mathcal{B}_0$ is the set $\mathcal{B}$ without *isolated* edges (i.e. such which have not an adjacent edge in $\mathcal{B}$). As $\mathcal{A} = [\mathcal{B}_0]$ is a penetrating vertex set, we can use it as the starting set of regulated buildups over $\mathcal{A}$. One of these buildups must be such that its non-initial distinguished edges are just all isolated edges of $\mathcal{B}$. This can be verified if we follow proper vertices in their order under considered buildups. If there are few possibilities for the choice of the consecutive proper vertex of relative degree 1 then at least one from vertices under consideration must be adjacent to an isolated edge from $\mathcal{B}$. In fact, if no of such vertices is adjacent to an isolated edge from $\mathcal{B}$, then we have a contradiction: in this case namely $\mathcal{B}$ cannot be an edge basis. $\square$

Remark. We return to an arbitrary generating set $\mathcal{G}$ of edges of a 3-configuration $\mathfrak{C}= (\mathcal{V},\mathcal{E},\ \mathrm{I})$. If we investigate the corresponding generating buildup $\mathscr{G}$ over $\mathcal{G}$ then we may denote vertices of relative degree 3 and edges of relative degree ≥ 2 as *terminal elements* according to $\mathscr{G}$. The number of terminal elements (if a terminal edge has the relative degree $\alpha > 2$ then we count its "terminality" with multiplicity $\alpha - 1$) agrees with the number of proper vertices of relative degree 3 under each regulated buildup over $\mathcal{G}$. We omit the details.

A *bridge* of a 3-configuration $\mathfrak{C}= (\mathcal{V},\mathcal{E},\ \mathrm{I})$ is defined as an edge e adjacent to just two vertices and having the property that the (3)-configuration $(\mathcal{V},\mathcal{E}\setminus\{e\},\ \mathrm{I})$ is not connected.

Theorem 4. No bridge of a 3-configuration $\mathfrak{C}$ can belong to an edge basis of $\mathfrak{C}$.

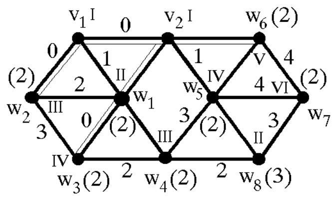
Proof. Let $\mathcal{B}$ be an edge basis of a given 3-configuration $\mathfrak{C}= (\mathcal{V},\mathcal{E},\ \mathrm{I})$ and let there exists a bridge $e_0 \in \mathcal{B}$. Then the (3)-configuration $\hat{\mathfrak{C}} = (\mathcal{V},\mathcal{E} \setminus \{e_0\},\ \mathrm{I})$ splits onto connected (3)-configurations $\hat{\mathfrak{C}}_1 = (\mathcal{V}_1,\mathcal{E}_1,\ \mathrm{I})$, $\hat{\mathfrak{C}}_2 = (\mathcal{V}_2,\mathcal{E}_2,\ \mathrm{I})$. Put $\mathcal{B}_1 = \mathcal{B} \cap \mathcal{E}_1$, $\mathcal{B}_2 = \mathcal{B} \cap \mathcal{E}_2$. Let v_1 be a vertex from $\mathcal{V}_1$, adjacent to e_0 and let e_1, e_1' be both further edges adjacent to v_1. These two edges cannot belong simultaneously to $\langle\mathcal{B}_1\rangle$ (if both are in $\langle\mathcal{B}_1\rangle$ then $e_0 \in \langle\mathcal{B}_1\rangle$, contrary to $e_0 \in \mathcal{B}$) but they cannot simultaneously lie out of $\langle\mathcal{B}_1\rangle$ (if both are out of $\langle\mathcal{B}_1\rangle$, then $\langle\mathcal{B}\rangle \neq \mathcal{V}$, a contradiction). Thus one of both edges must belong to $\langle\mathcal{B}_1\rangle$ and the remaining, for example e_1, must be out of $\langle\mathcal{B}_1\rangle$. Choose one of further vertices, v_2, adjacent to e_1 and denote by e_2, e_2' both further edges adjacent to v_2. For similar reasons as in the preceding, one of them lies in $\langle\mathcal{B}_1\rangle$ and the remaining, for example e_2, is out of $\langle\mathcal{B}_1\rangle$. We proceed similarly as long as possible. After finitely many steps we finish at an edge $e_l \notin \langle\mathcal{B}_1\rangle$ such that for all remaining vertices v adjacent to e_l the further edges adjacent to v either both belong to $\langle\mathcal{B}_1\rangle$ or both are out of $\langle\mathcal{B}_1\rangle$. In any case we get a contradiction. (Cf. Fig. 2.) $\square$



e_0– a bridge of $\mathfrak{C}$
both I,I'$\in \langle\mathcal{B}_1\rangle$ or both $\notin \langle\mathcal{B}_1\rangle$
both II,II'$\in \langle\mathcal{B}_1\rangle$ or both $\notin \langle\mathcal{B}_1\rangle$
both III,III'$\in \langle\mathcal{B}_1\rangle$ or both $\notin \langle\mathcal{B}_1\rangle$
$\vdots$

Fig. 2

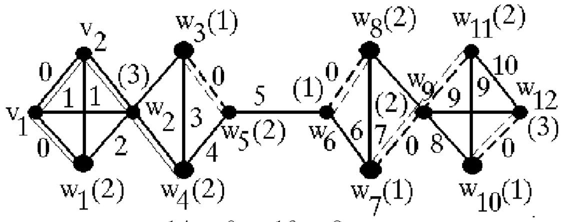
Example 4



$$\begin{array}{ccccccc} 10 & + & 3 & = & 14 & + & 1 \\ | & & | & & | & & | \\ \mathrm{m} & & \mu & & \mathrm{n} & & \nu \end{array}$$

the valuation $i \in \{0,1,2,3,4,5\}$ means that the edges
belongs to every set $\mathcal{X}_{2j}$, $j \geq i$

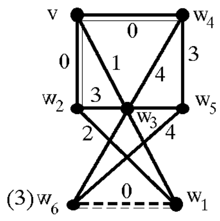
Example 5



$14 + 6 = 18 + 2$

$\begin{array}{cccc} | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$

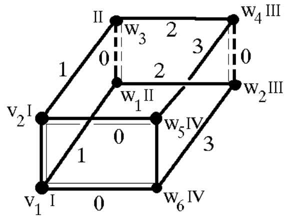
Example 6



$10 + 6 = 12 + 4$

$\begin{array}{cccc} | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$

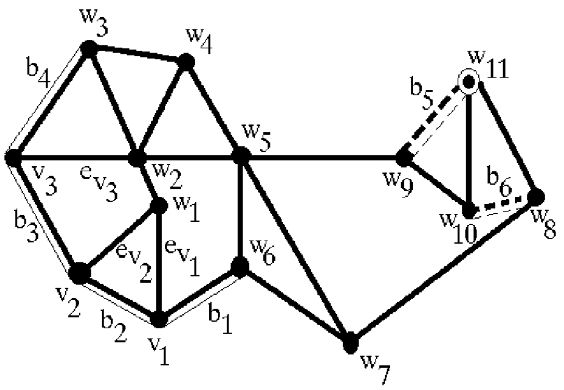
Example 7



$10 + 5 = 12 + 3$

$\begin{array}{cccc} | & | & | & | \\ \mathrm{m} & \mu & \mathrm{n} & \nu \end{array}$

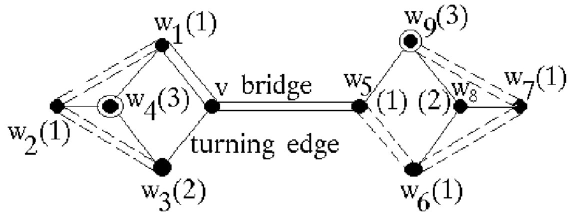
Example 8



$\{b_1, b_2, b_3, b_4, b_5, b_6\}$ is an edge basis

m$= 14$, $\mu_0 = 4$, $\mu = 6$, n$= 19$, $\nu = 1$

Example 9



where ═══ are initial edges, = = = = are further distinguished edges and $(\cdot)$ are relative degrees.

Questions:

(1) If $\mathfrak{C}$ is a 3-configuration without bridges, does every one-element penetrable set $\{v\}$ lead to an edge basis (as the set of all distinguished edges according to a regulated buildup of $\mathfrak{C}$ over $\{v\}$)?

(2) Under which conditions do all edge bases of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ have the same number of edges? (If $\mathfrak{C}$ is a graph then every edge basis is the edge complement to a cycle-free connected factor of $\mathfrak{C}$ so that $\#\mathcal{E} - \#\sqsubseteq + 1$ is the constant number of edges of every edge basis of $\mathfrak{C}$.)

Appendix (folklore):

Let m be the number of all vertices and n the number of all edges of a 3-configuration. Then $\frac{2}{3}n \le \frac{n^2-n}{6}$.

Proof. Let α be the number of all unordered couples of adjacent elements. As every vertex is adjacent to exactly three edges, we have $\alpha = 3m$. As every edge is adjacent to at least two vertices, we get $\alpha \ge 2n$. Thus $3m \ge 2n$.

Further let β be the number of all unordered couples of neighboring edges. Every vertex v determines just three couples of neighboring edges (adjacent to v) so that $\beta = 3m$. The total numbers of unordered couples of distinct edges is $\binom{n}{2} = \frac{n^2-n}{2}$. Thus $\beta \le \frac{n^2-n}{2}$ and consequently $3m \le \frac{n^2-n}{2}$. □

References

[1] Belousov, V. D., *Configurations in algebraic nets* (in Russian), Kishinev (1979).

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