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CHARACTERIZATION OF DISTRIBUTIVE SETS BY GENERALIZED ANNIHILATORS

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Abstract. Distributive ordered sets are characterized by so called generalized annihilators.

Let L be a lattice. For $a,\ b \in L$ the annihilator $\langle a, b\rangle$ and the dual annihilator $\langle a, b\rangle_d$ of a relative to b are given by $\langle a, b\rangle := \{x \in L : x \wedge a \leq b\}$ and $\langle a, b\rangle_d := \{x \in L : x \vee a \geq b\}$.

Several authors have studied annihilators in distributive lattices: Mandelker [1], Davey [2]; in modular lattices Davey and Nieminen [3]. In particular Mandelker proved that L is distributive iff $\langle a, b\rangle$ is an ideal for all $a,\ b \in L$.

The aim of this paper is to characterize distributive ordered sets by the so called generalized annihilators.

Let S be an ordered set, $X \subseteq S$. An upper (lower) cone of X in S is the set $U(X) = \{x \in S : x \geq a$ for each $a \in S\}$, $(L(X) = \{x \in S : x \geq a \quad \text{for each} \quad a \in X\})$.

J. Rachůnek in [4] introduced and studied distributive and ordered sets: an ordered set S is
distributive if $\forall\ a, b, c \in S : L(U(a, b), c) = L(U(L(a, c), L(b, c)))$.

Definition 1. Let S be an ordered set, $A \subseteq S$, $B \subseteq S$. A *double generalized annihilator (d-annihilator)* in S is the set defined by

$$\langle A, B\rangle = \{x \in S : UL(A, x) \supseteq U(B)\}\,, \quad \text{and, dually, a double generalized}$$

dual annihilator (dual d-annihilator) in S is:

$$\langle A, B\rangle_d = \{x \in S : LU(A, x) \supseteq L(B)\}\,.$$

If A is a one element set, then the (dual) d-annihilator is called the *(dual) annihilator*.

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Definition 2. Let S be an ordered set. The subset $I \subseteq S$ is called an *ideal (filter)* in S if it holds:

$$x, y \in I \Rightarrow LU(x, y) \subseteq I \quad (x, y \in I \Rightarrow UL(x, y) \subseteq I)\,.$$

Remark. If S is a lattice, then I is an ideal (filter) in S iff I is a lattice ideal (filter).

Theorem 1. *An ordered set S is distributive if and only if each annihilator in S is an ideal in S.*

Proof. (i) Let S be a distributive set, and $\langle a, B\rangle$ be an annihilator in S. Let $x, y \in \langle a, B\rangle$. Then $UL(a, x) \supseteq (B)$,

$$UL(a, y) \supseteq U(B)\,.$$

Let $z \in LU(x, y)$. Then $L(z) \subseteq LU(x, y)$, $U(z) \supseteq U(x, y)$ and henceforth $UL(a, z) = UL(a, U(z)) \supseteq UL(a, U(x, y))$. By the distributive law the right side of the last inclusion is equal to

$$ULU(L(a, x), L(a, y)) = U(L(a, x), L(a, y)) = UL(a, x) \cap UL(a, y) \supseteq U(B)\,,$$

hence $UL(a, z) \supseteq U(B)$, and $z \in \langle a, B\rangle$. Thus $LU(x, y) \subseteq \langle a, B\rangle$ and $\langle a, B\rangle$ is an ideal.

(ii) Let every annihilator in S be an ideal, $a, b, x \in S$. Then $UL(a, x) \supseteq UL(a, x) \cap UL(b, x) = U(L(a, x), L(b, x))$, and, analogously $UL(b, x) \supseteq U(L(a, x), L(b, x))$. Hence for $B = L(a, x) \cup L(b, x)$ it holds $a \in \langle x, B\rangle$, $b \in \langle x, B\rangle$. But $\langle x, B\rangle$ is an ideal, we have

$$LU(a, b) \subseteq \langle x, B\rangle \tag{*}$$

Let $z \in L(U(a, b), x)$; then $z \in LU(a, b) \cap L(x)$ and by (*) $z \in \langle x, B\rangle$. Therefore $UL(z, x) \supseteq U(L(a, x), L(b, x))$. Moreover, $x \in L(x)$ implies $L(z, x) = L(z)$, thus we obtain

$$U(z) \supseteq U(L(a, x), L(b, x)),\ L(z) \subseteq LU(L(a, x), L(b, x))\,,$$

i.e.

$$L(U(a, b), x) \subseteq LU(L(a, x), L(b, x))\,.$$

But the converse inclusion is valid for all element from S (see [4]), proving distributivity of S. $\square$

Corollary. *An ordered set S is distributive iff each dual annihilator in S is the filter in S.*

Example 1. Ordered sets in Fig. 1 and Fig. 2 are not distributive (see [5]), the annihilator $\langle a, \{c\}\rangle = \{b, c\}$ is not an ideal.

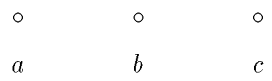

Fig. 1

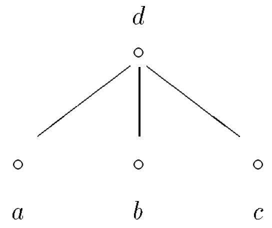

Fig. 2

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