ARCHIVUM MATHEMATICUM (BRNO)
Tomus 30 (1994), 9 – 16

CLOSURE CONDITIONS OF COMMUTATIVITY

V. J. Havel, J. Klouda

Abstract. There are investigated some closure conditions of Thomsen type in 3-webs which gurantee that at least one of coordinatizing quasigroups of a given 3-web is commutative.

We will pose the following question: Which is a necessary and sufficient condition (in form of a conditional identity with constants) for a given quasigroup $\mathbb{Q}$ to be isotopic with a commutative quasigroup? We shall show that such a condition is the fulfilling of a closure condition of Thomsen type (with constants) in the 3-web over $\mathbb{Q}$.

§1 Thomsen closure condition with respect to two constant lines

Let $\mathbb{Q} = (Q, \cdot)$ be a quasigroup of order > 1 and $\mathcal{W}_{\mathbb{Q}}$ the 3-web over $\mathbb{Q}$. The set of all points is $Q \times Q$ and the three line pencils are $\{\{(a, y)|y \in Q\}|a \in Q\}$ (*horizontal* lines), $\{\{(x, b)|x \in Q\}|b \in Q\}$ (*vertical* lines) and $\{\{(x, y)|x \cdot y = c\}|c \in Q\}$ (*skew* lines). We designate these lines briefly by $l_a^{(1)}$, $l_b^{(2)}$, $l_c^{(3)}$, respectively. Let in $\mathbb{Q}$ the commutativity $x \cdot q = q \cdot x$ for all $x, q \in Q$ be valid. This quasigroup identity can be expressed geometrically by special labelings of both pencils of vertical lines and horizontal lines: for every $x \in Q$, the points $l_q^{(2)} \sqcap l_x^{(1)}$, $l_q^{(1)} \sqcap l_x^{(2)}$ must lie on the same skew line. We use the symbol $\sqcap$ for denotation of the intersection point of two lines from different pencils.

1991 *Mathematics Subject Classification*: Primary 05B30, Secondary 20N05.

Key words and phrases: coordinatizing quasigroups of a 3-web, 3-web closure conditions with respect to constant lines, commutative quasigroups.

Received June 3, 1992.

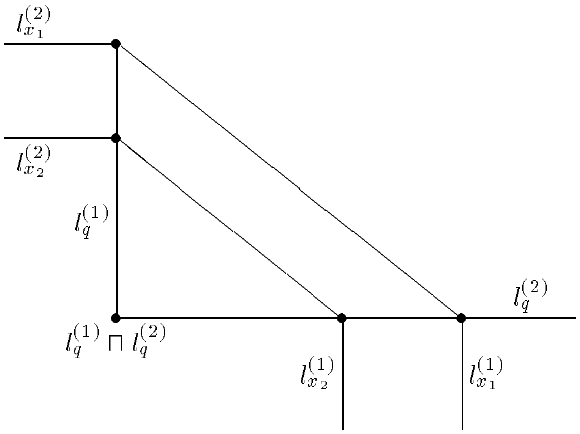

Fig. 1

Further denotations: AB for the line containing distinct points A, B (if it exists), Ai for the line containing the point A and belonging to the i-th pencil.

1. Choose a constant element $q \in Q$. Thus $l_q^{(1)}$, $l_q^{(2)}$ are constant lines and the validity of $x \cdot y = y \cdot x$ for all $x, y \in Q$ implies the validity of the following closure condition in $\mathcal{W}_{\mathbb{Q}}$ (cf. Fig. 2)

$$X2 = Y2 \Rightarrow \Big((X3 \sqcap l_q^{(1)})2 \sqcap Y1\Big)3 = \Big((Y3 \sqcap l_q^{(1)})2 \sqcap X1\Big) .$$

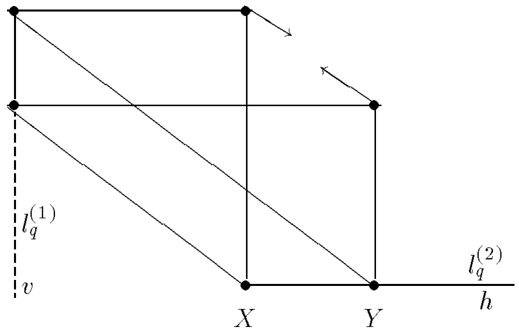

Fig. 2

2. Let in a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}; \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ there hold the closure condition

$$X2 = Y2 \Rightarrow \Big((X3 \sqcap v)2 \sqcap Y2\Big)3 = \Big((Y3 \sqcap h)2 \sqcap X1\Big)3$$

with constant lines $v \in \mathcal{L}_1$, $h \in \mathcal{L}_2$ (the *Thomsen condition with respect to constant lines* v, h ; denotation: $\mathcal{T}_{v,h}$). We assert that, consequently, there is a commutative coordinatizing quasigroup of $\mathcal{W}$.

Single coordinatizing quasigroups of $\mathcal{W}$ are determined if we choose three bijections $\pi_i : \mathcal{L}_i \to Q$, $i \in \{1,2,3\}$, where Q is a set such that $\#\mathcal{Q}$ is the order of $\mathcal{W}$. Then the corresponding quasigroup operation is derived from the concurrency of lines as follows: $x \cdot y = z \Leftrightarrow \pi_1^{-1}(x)$, $\pi_2^{-1}(y)$, $\pi_3^{-1}(z)$ go through the same point. Put $Q = h$ (recall that h is a point set), $\pi_1 : \mathcal{L}_1 \to Q$, $l \mapsto l \sqcap h$, $\pi_2 = \mathcal{L}_2 \to Q$, $l \mapsto (l \sqcap v)3 \sqcap h$ whereas π_3 rests arbitrary. Thus $x \cdot y = z \Leftrightarrow \pi_3(z) = \Big(x1 \sqcap (y3 \sqcap v)2\Big)3$.

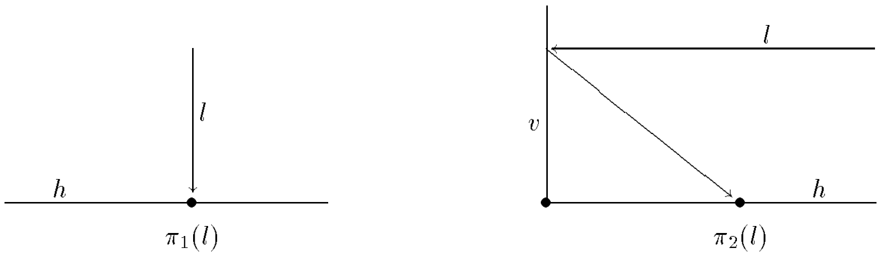

Fig. 3

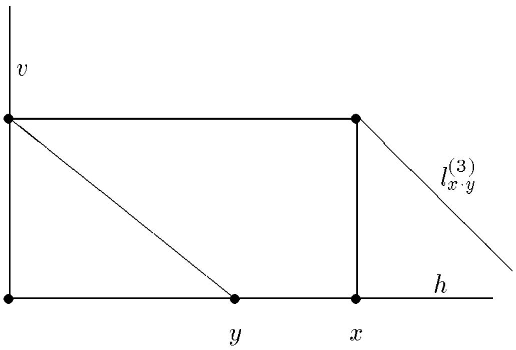

Fig. 4

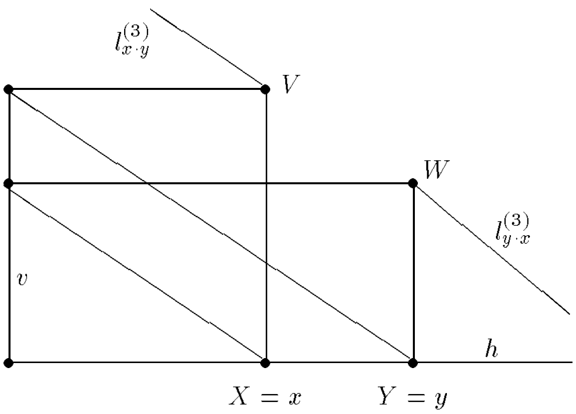

Fig. 5

We assert that $x \cdot y = y \cdot x$ holds for all $x, y \in Q$: in fact, if we put in the closure condition $X = x$, $Y = y$, then we construct the points $V = (Y3 \sqcap v)2 \sqcap X1$, $W = (X3 \sqcap v)2 \sqcap Y1$ so that, as the conclusion of the closure condition, V3 = W3 and consequently $x \cdot y = \pi^{-1}(W3) = y \cdot x$. Thus we obtained

Theorem 1. *Among coordinatizing quasigroups of a given 3-web $\mathcal{W}$ there exists a commutative quasigroup, if and only if there is a prominent vertical line v and a prominent horizontal line h such that, in $\mathcal{W}$, the Thomsen closure condition $\mathcal{T}_{v,h}$ is valid.*

Remark 1 (on universal Thomsen closure condition, cf. [3], pp. 199-200). Let in a 3-web $\mathcal{W}$ the universal Thomsen closure condition $\mathcal{T}$ hold (i.e., the above Thomsen closure condition $\mathcal{T}_{v,h}$ for all vertical lines v and all horizontal lines h). Choose π_1, π_2, π_3 such that the corresponding coordinatizing quasigroup will be a loop, with neutral element 1. Then (cf. Fig. 6) $1 \cdot w = u \cdot y$, $1 \cdot v = u \cdot x$, $x \cdot w = y \cdot v$ so that $x \cdot (u \cdot y) = y \cdot (u \cdot x)$. Putting $u = 1$ we obtain $x \cdot y = y \cdot x$. Thus the equality $x \cdot (u \cdot y) = y \cdot (u \cdot x)$ can be rewritten as $(y \cdot u) \cdot x = y \cdot (u \cdot x)$ and the loop under consideration is a commutative group.

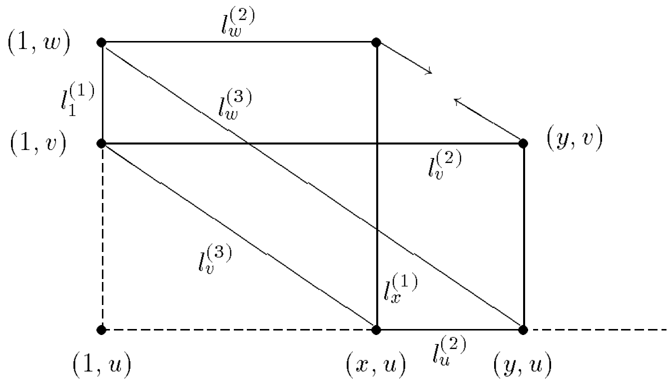

Fig. 6

Now let one of coordinatizing loops of a given 3-web $\mathcal{W}$ be a commutative group. Write the assumptions of the universal Thomsen closure condition $\mathcal{T}$ as $x_1 \cdot y_2 = x_2 \cdot y_1$, $x_1 \cdot y_3 = y_3 \cdot y_1$ (cf. Fig. 7).

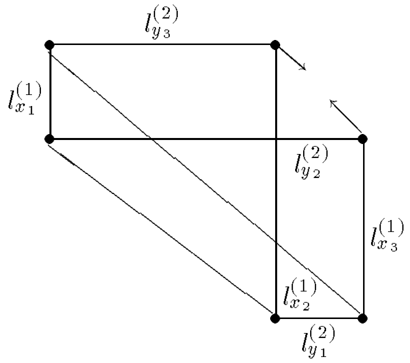

Fig. 7

From this it follows that $x_2 \cdot y_3 = x_2 \cdot (x_1^{-1} \cdot x_3 \cdot y_1) = (x_1^{-1} \cdot x_3) \cdot (x_2 \cdot y_1) = (x_1^{-1} \cdot x_3) \cdot (y_1 \cdot x_2) = x_3 \cdot y_2$, i.e. the conclusion of $\mathcal{T}$.

Remark 2 (on preserving of the commutativity by all loop isotopies):

All loop isotopies of a given commutative loop $\mathbb{L}$ preserve commutativity if $\mathbb{L}$ is a commutative group (so that, consequently, every loop isotopic to a commutative group is also a commutative group).

Examples of commutative loops distinct to groups: central nilpotent loops of class 2 (they are found firstly by Geritt Bol in 1937, cf. [6]) or totally symmetric loops.

Now we start with a given commutative quasigroup $\mathbb{Q} = (Q, \cdot)$, choose arbitrary permutations α, β, γ of Q and form the isotopic quasigroup $\mathbb{Q}' = (Q, \cdot')$ such that $\gamma(x \cdot' y) = \alpha(x) \cdot \beta(y)$ for all $x, y \in Q$. Thus the commutativity of $\mathbb{Q}'$, $x \cdot' y = y \cdot' x$, can be written as $\gamma^{-1}\big(\alpha(x) \cdot \beta(y)\big) = \gamma^{-1}\big(\alpha(y) \cdot \beta(x)\big)$ or, more simply, $x \cdot \varphi(y) = y \cdot \varphi(x)$ with $\varphi = \alpha^{-1}\beta$. As α, β were arbitrary, also φ is arbitrary. Let a be a fixed element of Q. If we put $y = a$ we get $x \cdot \varphi(a) = a \cdot \varphi(x)$ so that $\varphi(x) = L_1^{-1} R_{\varphi(a)}(x)$, $\varphi = L_1^{-1} R_{\varphi(a)}$ and φ is not arbitrary. It results that in a general case not all quasigroup isotopies preserve the commutativity (cf. [5], p. 17).

§2 Thomsen closure condition with respect to four constant lines

Here we start with a 3-web $\mathcal{W}$ in which two vertical lines v_1, v_2 and two horizontal lines h_1, h_2 are fixed. By a *Thomsen closure condition with respect to constant lines* v_1, v_2, h_1, h_2 (denoted by $\mathcal{T}_{v_1,v_2,h_1,h_2}$) we shall mean the assertion

$$\Big(((P1 \sqcap h_1)3 \sqcap v_1)2\Big) \sqcap \Big(((P2 \sqcap v_2)3 \sqcap h_2)1\Big) \in P3$$
$$\text{for all points } P \text{ of } \mathcal{W} \quad (\text{cf. Fig.8}).$$

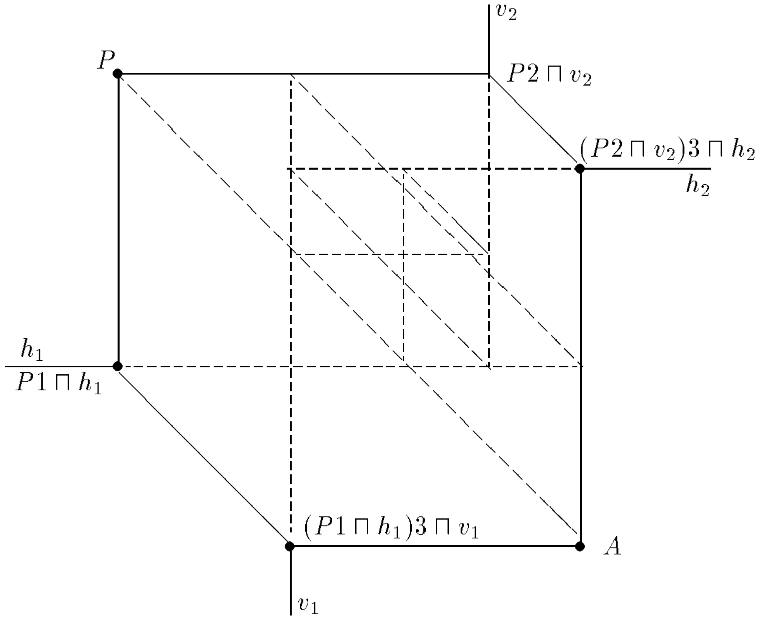

Fig. 8

where $A = \big((P1 \sqcap h_1)3 \sqcap v_1\big)2 \sqcap \big((P2 \sqcap v_2)3 \sqcap h_2\big)1$.

Let $\mathcal{W}$ be a 3-web satisfying $\mathcal{T}_{v_1,v_2,h_1,h_2}$. If we choose $P = v_1 \sqcap h_1$ then the conclusion of $\mathcal{T}_{v_1,v_2,h_1,h_2}$ claims $\Big(((P1 \sqcap h_1)3 \sqcap v_1)2 \sqcap \Big(((P2 \sqcap v_2)3 \sqcap h_2)1\Big) = v_2 \sqcap h_1\Big)$ i.e. $(v_1 \sqcap h_2)3 = (v_2 \sqcap h_1)3$ (assertion (a)). If we take as a new position of P the point $\tilde{P} = P3 \sqcap v_1$ then $\Big(((\tilde{P}1 \sqcap h_1)3 \sqcap v_1)2\Big) \sqcap \Big(((\tilde{P}2 \sqcap v_2)3 \sqcap h_2)1\Big) = h_1 \sqcap \Big((\tilde{P}2 \sqcap v_2)3 \sqcap h_2)1\Big)$, so that $(\tilde{P}2 \sqcap v_2)3 = \big((\tilde{P}3 \sqcap h_1)1 \sqcap h_2\big)3$ (assertion (b)). If we take as a new position of P the point $\overset{\approx}{P} = P2 \sqcap v_1$, then $\overset{\approx}{P}3 = \Big(((= \overset{\approx}{P}2 \sqcap v_2)3 \sqcap h_2)1 \sqcap h_1\Big)3$ (assertion (c)).

We see that $\mathcal{T}_{v_1,v_2,h_1,h_2} \Rightarrow \mathcal{T}_{v_1,h_1}$ and, by Theorem 1, there exists a commutative coordinatizing quasigroup of $\mathcal{W}$.

The direct proof that $\mathcal{T}_{v_1,v_2,h_1,h_2}$ implies the existence of a commutative coordinatizing quasigroup of $\mathcal{W}$ uses the following labeling of lines of $\mathcal{W}$: Let Q be chosen as (the point set) v_1. Let $a = v_1 \sqcap h_1$ be the label of both v_1, h_1 and $b = v_1 \sqcap h_2$ the label of v_2 and h_2. A vertical line v and a horizontal line h have the same label $x \in v_1$ if and only if $x \in h$, $x3 \sqcap v_1 \in v$.

Assertion (a) can be written as $a \cdot b = b \cdot a$, assertion (b) as $x \cdot a = a \cdot x$ together with $x \cdot b = b \cdot x$ for all $x \in Q$ (so that a, b lie in the centre $\mathcal{C}$ of $(Q, \cdot)$). It is easily seen that the label of the skew line through $\Big(((P1 \sqcap h_1)3 \sqcap v_1)2\Big) \sqcap \Big(((P2 \sqcap v_2)3 \sqcap h_2)1$ is $\big((b \cdot y)/b\big) \cdot \big(a\backslash(x \cdot a)\big)$ (cf. Fig. 9). As $a, b \in \mathcal{C}$, this label is equal to $y \cdot x$ and the commutativity of the quasigroup operation is verified.

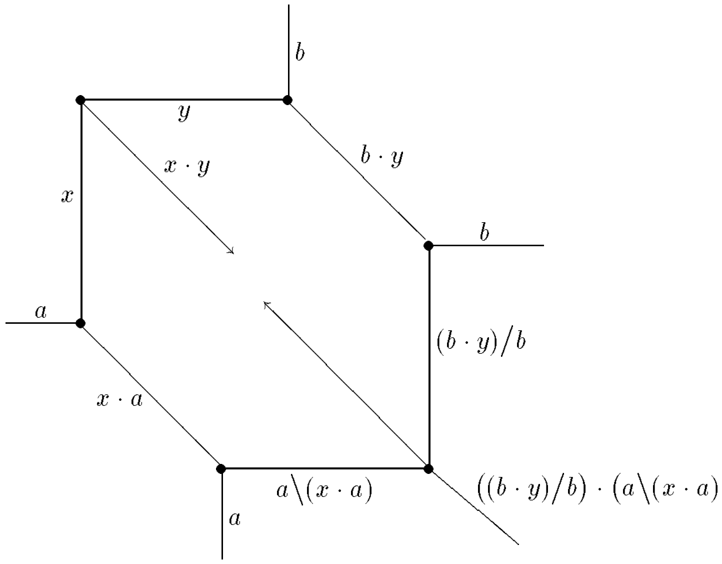

Fig. 9

Let there be given a commutative quasigroup $\mathbb{Q} = (Q, \cdot)$ of order > 1. Choose (mutually distinct) elements $a, b \in Q$. In the 3-web $\mathcal{W}$ over $\mathbb{Q}$ investigate the lines $\{(x,y)|x = a\}$, $\{(x,y)|x = b\}$, $\{(x,y)|y = a\}$, $\{(x,y)|y = b\}$ and denote them by v_1, v_2, h_1, h_2.

For every $x, y \in Q$ start with the point $P = (x,y)$ and construct the points (x,a), $\big(a, a\backslash(x \cdot a)\big)$, (b,y), $\big((b \cdot y)/b, b\big)$, $\big((b \cdot y)/b, a\backslash(x \cdot a)\big)$. As $a\backslash(x \cdot a) = x$, $(b \cdot y)/b = y$, the final point is (y,x). Both points (x, y), (y,x) must lie on the same skew lines because of commutativity of the quasigroup operation. Thus $\mathcal{T}_{v_1,v_2,h_1,h_2}$ holds in $\mathcal{W}$. We shall express the result in the following theorem.

Theorem 2. *There exists a commutative coordinatizing quasigroup of a given 3-web $\mathcal{W}$ if and only if there are (mutually distinct) vertical lines v_1, v_2 and (mutually distinct) horizontal lines h_1, h_2 such that $\mathcal{W}$ satisfies the closure condition $\mathcal{T}_{v_1,v_2,h_1,h_2}$.*

References

[1] Pickert, G., *Projektive Ebenen*, Berlin - Heidelberg - New York, 1975.
[2] Bruck, R. H., *What is a loop?*, Studies in Modern Algebra, Prentice - Hall (1963), 59-99.

[3] Belousov, V. D., *Elements of the theory of quasigroups and loops* (in Russian), Kishinev, 1967.
[4] Havel, V., *3-webs* (in Czech), Knižnice Vys. uč. techn. (Brno) **A-17** (1978), 209-242.
[5] Golovko, I. A., *Closure conditions in quasigroups* (in Russian), Izv. Akad. Nauk Mold. SSR (1971/3), 172.
[6] Bol, G., *Gewebe und Gruppen*, Math. Ann. **114** (1937), 414-431.
[7] Ilse, D., Lehmann, I., Schulz, W., *Gruppoide und Funktionalgleichungen*, Berlin, 1984.

V. J. Havel and J. Klouda
Department of Mathematics
Technical University
Kraví hora 21
602 00 Brno, CZECH REPUBLIC