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WEIGHTED ESTIMATES FOR THE HANKEL–, K– AND Y–TRANSFORMATIONS

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Abstract. We give conditions on pairs of non-negative functions u and v which are sufficient that, for $0 < q < p$, $p > 1$

$$\left[\int_0^\infty |u(x)(Tf)(x)|^q \, dx\right]^{\frac{1}{q}} \le C \left[\int_0^\infty |v(x)f(x)|^p \, dx\right]^{\frac{1}{p}} ,$$

where T is the Hankel-, K-, or the Y-transformations.

1. Introduction

The weighted Lebesgue spaces $L^p_w(R^+)$ consist of those functions f for which

$$\|f\|_{L^p_W(R^+)} = \left(\int_{R^+} |f(x)w(x)|^p \, dx\right)^{\frac{1}{p}} < \infty .$$

A continuous non-decreasing function $w : R^+ \to R^+$ belongs to the class B_K [12] if

$$\int_0^\infty \min(1, 1/t)\tilde{w}(t)t^{-1}dt < \infty ,$$

where $\tilde{w}(s) = \sup\limits_{y>0} \dfrac{w(s)}{w(y)}$ and $\tilde{w}(s) < \infty$ for $s > 0$.

Clearly, if $0 < \theta < 1$ then $w(t) = t^\theta \in B_K$. Also Gustavsson [6] has shown, the function $t^\beta / \log(1 + t^\alpha) \in B_K$ if $0 < \alpha < \beta < 1$.

Let $w \in B_K$, then the weighted Lorentz spaces $L^{p,w}$, $0 < p \le \infty$ consist of all measurable functions f on R^+ such that

$$\|f\|_{w,p} = \left[\int_0^\infty [tf^*(t)/w(t)]^p t^{-1} \, dt\right]^{\frac{1}{p}} , \quad 0 < p < \infty$$
$$\operatorname*{ess\,sup}_{t>0} tf^*(t)/w(t) , \quad p = \infty .$$

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Here f^* is the equimeasurable decreasing rearrangement of f (with respect to Lebesgue measure).

Note that if $w(t) = t^{1-(\frac{1}{q})}$, $0 < q \leq \infty$ these spaces reduce to Lorentz spaces $L(p,q)$.

The weight class B_ψ consists of all non-negative continuously differentiable functions w on R^+ such that

$$\sup_{t>0} tw'(t)/w(t) = \beta < 1 \quad \text{and} \quad \inf_{t>0} tw'(t)/w(t) = \alpha > 0 .$$

It is not difficult to see that $B_\psi \subset B_K$.

Again $w(t) = t^\theta$, $0 < \theta < 1$ is in B_ψ and also $w(t) = t^\alpha(\log(1+t^\gamma))^\theta$, $0 < \alpha < 1$, θ real and γ is a sufficiently small neighbourhood of zero (Gustavsson [6]).

Let $[X, Y]$ denote the collections of bounded operators from the Banach space X to the Banach space Y. An operator T is bounded from $L_v^p(R^+)$ to $L_u^q(R^+)$, written $T\varepsilon[L_v^p(R^+), L_u^q(R^+)]$, provided there exists a constant C such that

$$\|Tf\|_{L_u^q(R^+)} \leq C\,\|f\|_{L_v^p(R^+)}\,, \quad \text{for all} \quad f \geq 0 .$$

Throughout, constants are denoted by C and may be different at different apperances but are always independent of the function f in question. The indices p', q' and r are defined by $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{q} - \frac{1}{p} = \frac{1}{r}$.

The purpose of this paper is to establish a norm inequality for the Hankel transformation

$$(H_\alpha f)(x) = \int_0^\infty (xt)^{\frac{1}{2}} J_\alpha(xt) f(t)\, dt, \quad x > 0, \quad \alpha \geq -\frac{1}{2}\,,$$

where J_α is the Bessel function of order α, which for $0 < q < p$, $p > 1$ is new. If $1 < p$, $q < \infty$, the result was proved in [2], in fact, for $1 < p \leq q < \infty$ the result was similar to a weighted estimate of Heywood and Rooney [10], but with different weight conditions. We also prove a corresponding weighted norm inequalities for the <u>K</u>- and Y-transformations to the case $0 < q < p$, $p > 1$. Our weight conditions of these transformations are described in the following definitions:

Definition 1. Let u and v be non-negative functions defined on R^+ and let u^* and $(1/v)^*$ be the equimeasurable rearrangements of u and $1/v$. We write $(u,v) \in F^*_{p,q}$, $0 < q < p$, $p > 1$, if

$$\left\{\int_0^\infty \left[\left(\int_0^{\frac{1}{x}} u^*(t)^q\, dt\right)^{\frac{1}{q}} \left(\int_0^x (1/v)^*(t)^{p'}\, dt\right)^{\frac{1}{q'}}\right]^r (1/v)^*(x)^{p'}\, dx\right\}^{\frac{1}{r}} < \infty\,,$$

$$\left\{\int_0^\infty \left[\left(\int_{1/x}^\infty [t^{-\frac{1}{2}} u^*(t)]^q\, dt\right)^{\frac{1}{q}} \left(\int_x^\infty [t^{-\frac{1}{2}}(1/v)^*(t)]^{p'}\, dt\right)^{\frac{1}{q'}}\right]^r \right.$$
$$\left. \times\, [x^{-\frac{1}{2}}(1/v)^*(x)]^{p'}\, dx\right\}^{\frac{1}{r}} < \infty$$

hold, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

2. Weighted inequalities

We require the following lemmas:

Lemma 2 ([11]). *If f is a non-negative non-increasing function defined on R^+ then for α real and $0 < p \le q \le \infty$*

$$\left\{ \int_0^\infty [t^\alpha f(t)]^q t^{-1}\, dt \right\}^{\frac{1}{q}} \le C \left\{ \int_0^\infty [t^\alpha f(t)]^p t^{-1}\, dt \right\}^{\frac{1}{p}}$$

holds.

Lemma 3 ([13]), ([14]). *Suppose $0 < q < p$, $p > 1$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then*

$$\text{(a)} \qquad \left[\int_0^\infty |u(x) \int_0^x f(y)\, dy|^q\, dx \right]^{\frac{1}{q}} \le C \left[\int_0^\infty |f(x)v(x)|^p\, dx \right]^{\frac{1}{p}}$$

holds for all f, if and only if

$$\left[\int_0^\infty \left[\left(\int_x^\infty u(y)^q\, dy \right)^{\frac{1}{q}} \left(\int_0^x v(y)^{-p'}\, dy \right)^{\frac{1}{q'}} \right]^r v(x)^{-p'}\, dx \right]^{\frac{1}{r}} < \infty\,.$$

For the dual operator

$$\text{(b)} \qquad \left[\int_0^\infty |u(x) \int_x^\infty f(y)\, dy|^q\, dx \right]^{\frac{1}{q}} \le C \left[\int_0^\infty |f(x)v(x)|^p\, dx \right]^{\frac{1}{p}}$$

holds for all f, if and only if

$$\left[\int_0^\infty \left[\left(\int_0^x u(y)^q\, dy \right)^{\frac{1}{q}} \left(\int_x^\infty v(y)^{-p'}\, dy \right)^{\frac{1}{q'}} \right]^r v(x)^{-p'}\, dx \right]^{\frac{1}{r}} < \infty\,.$$

Lemma 4 ([6]). *If $w \in B_K$ then*

(i) $\underset{\sim}{w}(s)\tilde{w}(1/s) = 1$, *where* $\underset{\sim}{w}(s) = \inf_{t>0} \frac{w(st)}{w(t)}$.

(ii) $0 < \underset{\sim}{w}(s)w(t) \le w(st) \le \tilde{w}(s)w(t)$.

(iii) $\tilde{w}$ *and* $\underset{\sim}{w}$ *are non-decreasing and* $\tilde{w}(1) = \underset{\sim}{w}(1) = 1$.

(iv) *For any* $p > 0$, $\{\int_0^\infty [\min(1, 1/t)\tilde{w}(t)]^p t^{-1}\, dt\}^{\frac{1}{p}} < \infty$, *with the usual modification if* $p = \infty$.

(v) *There exist constants* $A, B > 0$ *such that for* $A \le s^{-1}w(s)\{\int_0^s [t/w(t)]^p t^{-1}\, dt\}^{\frac{1}{p}} \le B$, $p > 0$. *In fact,* $A = p^{-\frac{1}{p}}$ *and* $B = \{\int_0^\infty [\tilde{w}(t)/t]^p t^{-1}\, dt\}^{\frac{1}{p}}$ *if* $p < \infty$.

(vi) *There are positive constants* C, D *such that*

$C \leq w(s)\{\int_s^\infty [1/w(t)]^p t^{-1}\, dt\}^{\frac{1}{p}} \leq D,\ p > 0\,.$
Here,
$C = \{\int_1^\infty [1/\tilde{w}(t)]^p t^{-1}\, dt\}^{\frac{1}{p}}$ and $D = \{\int_0^1 \tilde{w}(t)^p t^{-1}\, dt\}^{\frac{1}{p}}$ *if* $p < \infty\,.$

Remark 5 ([3]). If $w_i \in B_\psi$, $i = 0, 1$ and $\tau(t) = w_1(t)/w_0(t)$ satisfying $t\tau'(t)/\tau(t) \geq \alpha > 0$ for all $t > 0$, then τ has an inverse and $\lim_{t\to 0} \tau(t) = 0$, $\lim_{t\to\infty} \tau(t) = \infty$.

Holmstedt's K-functional estimate in the context takes the form:

Lemma 6. *Suppose $w_i \in B_\psi$, $i = 0, 1$ with $\tau(t) = w_1(t)/w_0(t)$ and $\eta(t) = \tau^{-1}(t)$ such that*

$$t\tau'(t)/\tau(t) \geq \alpha > 0\,, \tag{1}$$

holds. Then for $1 \leq q_0, q_1 \leq \infty$.

$$K(t, f : L^{q_0,w_0},\, L^{q_1,w_1}) \leq C\left\{ \int_0^{\eta(t)} [f^*(s)/w_0(s)]\, ds + t\int_{\eta(t)}^\infty [f^*(s)/w_1(s)]\, ds \right\}. \tag{2}$$

Proof. In [8] Heinig showed that

$$K(t, f; L^{q_0,w_0}, L^{q_1,w_1}) \sim \left\{ \int_0^{\eta(t)} [sf^*(s)/w_0(s)]^{q_0} s^{-1} ds \right\}^{\frac{1}{q_0}} + t\left\{ \int_{\eta(t)}^\infty [sf^*(s)/w_1(s)]^{q_1} s^{-1} ds \right\}^{\frac{1}{q_1}}. \tag{3}$$

We complete the proof by showing that both summands in (3) are bounded by the right side of (2). Since $f^*(s)/w_0(s)$ is non-increasing it follows directly from Lemma 2 that

$$\left\{ \int_0^t [sf^*(s)/w_0(s)]^{q_0} s^{-1} ds \right\}^{\frac{1}{q_0}} \leq C\int_0^t [f^*(s)/w_0(s)]\, ds\,, \quad 1 \leq q_0 \leq \infty\,. \tag{4}$$

Now we define g by

$$g(s) = \begin{cases} f^*(t)\left\{ \int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy \right\}^{\frac{1}{q_1}}, & \text{if} \quad 0 < s \leq t \\ f^*(s)\left\{ \int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy \right\}^{\frac{1}{q_1}}, & \text{if} \quad s > t, \end{cases}$$

where $t > 0$ is fixed. Then g is non-increasing and from (vi) of Lemma 4 and Lemma 2, we obtain for fixed $t > 0$, $1 \le q_1 \le \infty$,

$$\begin{aligned}
&\left\{ \int_t^\infty [sf^*(s)/w_1(s)]^{q_1} s^{-1} ds \right\}^{\frac{1}{q_1}} \\
&\quad \le C\left\{ \int_t^\infty s^{q_1} f^*(s)^{q_1} \Big(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big) s^{-1} ds \right\}^{\frac{1}{q_1}} \\
&\quad \le C\left\{ \int_0^t s^{q_1} f^*(t)^{q_1} \Big(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big) s^{-1} ds \right. \\
&\qquad \left. + \int_t^\infty s^{q_1} f^*(s)^{q_1} \Big(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big) s^{-1} ds \right\}^{\frac{1}{q_1}} \\
&\quad = C\left\{ \int_0^\infty [sg(s)]^{q_1} s^{-1} ds \right\}^{\frac{1}{q_1}} \le C \int_0^\infty g(s)\, ds \\
&\quad \le C\left\{ \int_0^t f^*(t) \Big(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big)^{\frac{1}{q_1}} ds \right. \\
&\qquad \left. + \int_t^\infty f^*(s) \Big(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big)^{\frac{1}{q_1}} ds \right\} \\
&\quad \le C\left\{ \int_0^t f^*(s) \Big(\int_t^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big)^{\frac{1}{q_1}} ds \right. \\
&\qquad \left. + \int_t^\infty f^*(s) \Big(\int_s^\infty [1/w_1(y)]^{q_1} y^{-1} dy \Big)^{\frac{1}{q_1}} ds \right\}.
\end{aligned}$$

Here the second inequality is obtained by adding the first integral term. On applying (vi) of Lemma 4, then the right side of the previous inequality is dominated by

$$\begin{aligned}
&C\left\{ w_1(t)^{-1} \int_0^t f^*(s)\, ds + \int_t^\infty [f^*(s)/w_1(s)]\, ds \right\} \\
&\qquad \le C\left\{ [1/\tau(t)] \int_0^t [f^*(s)/w_0(s)]\, ds + \int_t^\infty [f^*(s)/w_1(s)]\, ds \right\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(5) \qquad &\left\{ \int_t^\infty [sf^*(s)/w_1(s)]^{q_1} s^{-1}\, ds \right\}^{\frac{1}{q_1}} \\
&\qquad \le C\left\{ [1/\tau(t)] \int_0^t [f^*(s)/w_0(s)]\, ds + \int_t^\infty [f^*(s)/w_1(s)]\, ds \right\}.
\end{aligned}$$

From (3), (4), and (5) one gets the desired result, which proves the lemma.

We now prove the following interpolation theorem:

Theorem 7. *Suppose $w_i,\ \bar{w}_i \in B_\psi,\ i = 0,1$ with $\tau = w_1/w_0,\ \bar{\tau} = \bar{w}_1/\bar{w}_0$ and $\eta = \tau^{-1},\ \bar{\eta} = \bar{\tau}^{-1}$ satisfy $t\tau'(t)/\tau(t) \geq \alpha > 0$ and $|t\bar{\tau}'(t)/\bar{\tau}(t)| \geq \bar{\alpha} > 0$. Let $\sigma = \bar{\eta} o \tau$ and $T : L^{q_i,w_i} \to L^{\bar{q}_i,\bar{w}_i}$, $1 \leq q_i,\ \bar{q}_i \leq \infty$, $i = 0,1$ be quasilinear operator.*

If $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$, $0 < q < p$, $p > 1$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$; u and w are non-negative weight functions satisfying

$$\left\{ \int_0^\infty \Big[\Big(\int_{\sigma(s)}^\infty [u^*(t)\bar{w}_0(t)/t]^q \, dt\Big)^{\frac{1}{q}} \times \Big(\int_0^s [1/(1/v)^*(t)w_0(t)]^{-p'} \, dt\Big)^{\frac{1}{q'}}\Big]^r [1/(1/v)^*(s)w_0(s)]^{-p'} \, ds \right\}^{\frac{1}{r}} < \infty \tag{6}$$

and

$$\left\{ \int_0^\infty \Big[\Big(\int_0^{\sigma(s)} [u^*(t)\bar{w}_1(t)/t]^q \, dt\Big)^{\frac{1}{q}} \times \Big(\int_s^\infty [1/(1/v)^*(t)w_1(t)]^{-p'} \, dt\Big)^{\frac{1}{q'}}\Big]^r [1/(1/v)^*(s)w_1(s)]^{-p'} \, ds \right\}^{\frac{1}{r}} < \infty \tag{7}$$

then for all simple functions f

$$T \in [L_v^p(R^+),\ L_u^q(R^+)] \,. \tag{8}$$

If $-t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$, (8) still holds provided the ranges of the fisrt inner integrals in (6) and (7) are interchanged.

Proof. For the case $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$ we apply Lemma 4 (v) with s, w and p replaced by $\bar{\eta}$, $\bar{w}_0$ and $\bar{q}_0$, respectively and use (3) to obtain

$$\begin{aligned}
(Tf)^*(\bar{\eta}(t))\bar{\eta}(t)/\bar{w}_0(\bar{\eta}(t)) &\leq C(Tf)^*(\bar{\eta}(t))\Big\{ \int_0^{\bar{\eta}(t)} [s/\bar{w}_0(s)]^{\bar{q}_0} s^{-1} \, ds\Big\}^{\frac{1}{\bar{q}_0}} \\
&\leq C\Big\{ \int_0^{\bar{\eta}(t)} [s(Tf)^*(s)/w_0(s)]^{\bar{q}_0} s^{-1} \, ds\Big\}^{\frac{1}{\bar{q}_0}} \\
&\leq CK(t, Tf; L^{\bar{q}_0,\bar{w}_0}, L^{\bar{q}_1,\bar{w}_1}) \,.
\end{aligned}$$

Now, by hypothesis

$$(Tf)^*(\bar{\eta}(t))\bar{\eta}(t)/\bar{w}_0(\bar{\eta}(t)) \leq CK(tM_1/M_0, f; L^{q_0,w_0}, L^{q_1,w_1})$$

and since $K(t,f)$ is increasing whereas $t^{-1}K(t,f)$ is decreasing we may take without loss of generality $M_1/M_0 = 1$. Therefore,

$$(Tf)^*(\bar{\eta}(t)) \leq C[\bar{w}_0(\bar{\eta}(t))/\bar{\eta}(t)]K(t,f;L^{q_0,w_0},L^{q_1,w_1})\,.$$

Hence, it follows from (2) of Lemma 6 that

$$(Tf)^*(\bar{\eta}(t)) \leq C[\bar{w}_0(\bar{\eta}(t))/\bar{\eta}(t)]\left\{\int_0^{\eta(t)}[f^*(s)/w_0(s)]\,ds + t\int_{\eta(t)}^{\infty}[f^*(s)/w_1(s)]\,ds\right\}.$$

Now, let $\bar{\eta}(t) = y$, then $t = \bar{\tau}(y)$ and

$$(Tf)^*(y) \leq C[\bar{w}_0(y)/y]\left\{\int_0^{\bar{\sigma}(y)}[f^*(s)w_0(s)]\,ds + \bar{\tau}(y)\int_{\bar{\sigma}(y)}^{\infty}[f^*(s)/w_1(s)]\,ds\right\},$$

where $\bar{\sigma}(y) = \eta(\bar{\tau}(y))$. Utilizing properties of rearrangenent of functions and Minkowski's inequality one obtains from this estimate

$$\begin{aligned}
\left\{\int_0^\infty |u(x)(Tf)(x)|^q dx\right\}^{\frac{1}{q}} &\leq \left\{\int_0^\infty [u^*(y)(Tf)^*(y)]^q\,dy\right\}^{\frac{1}{q}}\\
&\leq C\left\{\int_0^\infty u^*(y)^q\bar{w}_0(y)^q y^{-q}[\int_0^{\bar{\sigma}(y)}[f^*(s)/w_0(s)]\,ds\right.\\
&\quad\left.+\,\bar{\tau}(y)\int_{\bar{\sigma}(y)}^\infty [f^*(s)/w_1(y)]\,ds]^q\,dy\right\}^{\frac{1}{q}}\\
&\leq C\left\{[\int_0^\infty u^*(y)^q\bar{w}_0(y)^q y^{-q}(\int_0^{\bar{\sigma}(y)}[f^*(s)/w_1(s)]\,ds)^q\,dy]^{\frac{1}{q}}\right.\\
&\quad\left.+\,[\int_0^\infty u^*(y)^q\bar{w}_1(y)^q y^{-q}(\int_{\bar{\sigma}(y)}^\infty [f^*(s)/w_1(s)]\,ds)^q\,dy]^{\frac{1}{q}}\right\}
\end{aligned}$$

(9)$\equiv$ $C\{Z_1 + Z_2\}$, respectively. Here we used in the second inequality the assumption that $\bar{\tau}(t)\bar{w}_0(y) = \bar{w}_1(y)$. We complete the proof by showing that both summands Z_1 and Z_2 in (9) are bounded by $C\{\int_0^\infty |v(x)f(x)|^p\,dx\}^{\frac{1}{p}}$.

Now, let $\bar{\sigma}(y) = t$, then by definition of $\bar{\sigma}$, $\eta(\bar{\tau}(y)) = t$ or $y = \bar{\tau}^{-1}(\eta^{-1}(t))$. But since $\bar{\eta}(t) = \bar{\tau}^{-1}(t)$ and $\tau(t) = \eta^{-1}(t)$ one obtains $y = \bar{\eta}(n^{-1}(t)) = \bar{\eta}(\tau(t)) = \sigma(t)$, and similarly $\bar{\tau}(y) = \tau(t)$. Also by Remark 5, $\tau(t)$, $\bar{\tau}(t)$ tend to zero and infinity as $t \to 0$, respectively $t \to \infty$. Also $t\bar{\tau}'(t)/\bar{\tau}(t) \geq \bar{\alpha} > 0$ implies $\bar{\tau}'(t) > 0$ and $\tau(t) = \bar{\tau}(\sigma(t))$ we obtain $\tau'(t) = \bar{\tau}'(\sigma(t))\sigma'(t)$, which implies $\sigma'(t) > 0$. Therefore, the first summand yields

$$\begin{aligned}
Z_1 &= C\left\{\int_0^\infty (u^*(\sigma(t))\bar{w}_0(\sigma(t))\sigma(t)^{-1}\int_0^t [f^*(s)/w_0(s)]\,ds)^q\sigma'(t)\,dt\right\}^{\frac{1}{q}}\\
&\leq C\left\{\int_0^\infty [1/(1/v)^*(s)f^*(s)]^p\,ds\right\}^{\frac{1}{p}},
\end{aligned}$$

where the above inequality holds by Lemma 3 (a) provided (6) holds.

The bound of the second summand of (9) follows from condition (7) in exactly the same way. We omit the details. Therefore,

$$\left\{\int_0^\infty |u(x)(Tf)(x)|^q\,dx\right\}^{\frac{1}{q}} \le C\left\{[1/(1/v)^*(x)f^*(x)]^p\,dx\right\}^{\frac{1}{p}}$$
$$\le C\left\{\int_0^\infty |v(x)f(x)|^p\,dx\right\}^{\frac{1}{p}},$$

where the second inequality follows the integral analogue of [7, Theorem 368] obtained by approximating v by appropriate simple function and using Lebesgue's theorem of monotone convergence.

The rest of the proof is similar to the case $t\bar{\tau}'(t)/\bar{\tau}(t) \ge \bar{\alpha} > 0$ and therefore omitted. This completes the proof of the theorem. □

The following corollary is a consequence of Theorem 7 with $w_i(t) = t^{1-(\frac{1}{p_i})}$ and $\bar{w}_i(t) = t^{1-(1/\bar{p}_i)}$, $i = 0, 1$.

Corollary 8. *Let* $0 < p_i,\ \bar{p}_i \le \infty$, $1 \le q_i,\ \bar{q}_i \le \infty$, $i = 0, 1$ *and* $T : L(p_i, q_i) \to L(\bar{p}_i, \bar{q}_i)$ *be a quasi-linear operator and*

$$\lambda = \frac{1}{p_0} - \frac{1}{p_1} > 0, \quad \bar{\lambda} = 1/\bar{p}_0 - 1/\bar{p}_1 \ne 0\,.$$

If $\bar{\lambda} > 0$, $0 < q < p$, $p > 1$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$; u, v *satisfying*

$$\left\{\int_0^\infty [(\int_{s^{\lambda/\bar{\lambda}}}^\infty [u^*(t)t^{-1/\bar{p}_0}]^q\,dt)^{\frac{1}{q}}(\int_0^s [1/(1/v)^* t^{\frac{1}{p}-\frac{1}{p_0}}]^{-p'} t^{-1}\,dt)^{\frac{1}{q'}}]^r \right.$$
$$\left. \times [1/(1/v)^*(s)s^{\frac{1}{p}-\frac{1}{p_0}}]^{-p'} s^{-1}\,ds\right\}^{\frac{1}{r}} < \infty \tag{10}$$

and

$$\left\{\int_0^\infty [(\int_0^{s^{\lambda/\bar{\lambda}}} [u^*(t)t^{-1/\bar{p}_1}]^q\,dt)^{\frac{1}{q}}(\int_s^\infty [1/(1/v)^*(t) t^{\frac{1}{p}-\frac{1}{p_1}}]^{-p'} t^{-1}\,dt)^{\frac{1}{q'}}]^r \right.$$
$$\left. \times [1/(1/v)^*(s)s^{\frac{1}{p}-\frac{1}{p_1}}]^{-p'} s^{-1}\,ds\right\}^{\frac{1}{r}} < \infty\,, \tag{11}$$

then (8) *holds.*

If $\lambda < 0$, (8) still holds provided the ranges of the first inner integrals in (10) and (11) are interchanged.

Our next corollary extends and complets these results obtained in [1, Theorem 1.1] and [9, Corollary 2.5 and Proposition 2.6] for the range $0 < q < p$, $p > \infty$, when $p_0 = 1$, $\bar{p}_0 = \infty$, $\bar{p}_1 = p_1 = 2$; $\lambda = \frac{1}{2}$, $\bar{\lambda} = -\frac{1}{2}$.

Corollary 9. *If* $T \in [L^1(R^+), L^\infty(R^+)] \cap [L^2(R^+), L^2(R^+)]$ *and* $(u, v) \in F^*_{p,q}$, $0 < q < p$, $p > 1$, *then (8) holds.*

A consequence of this result is the following:

Corollary 10. *Let T be as in Corollary 9 and B defined by*

$$(Bf)(x) = w(x)(Tg)(x), \quad \textit{where } g(x) = w(x)f(x)\,.$$

If $(uw, v/w) \in F^*_{p,q}$, *then* $B \in [L^p_v(R^+), L^q_u(R^+)]$.

Now, we state and prove the weighted inequality for the Hankel-transformation.

Theorem 11. *Let* $\alpha \geq -\dfrac{1}{2}$, $\alpha \neq 0$, *u and v be non-negative functions defined on* R^+ *and* $u_\alpha(x) = x^{\alpha+\frac{1}{2}}u(x)$, $v_\alpha(x) = x^{-\alpha-\frac{1}{2}}v(x)$. *If* $(u_\alpha, v_\alpha) \in F^*_{p,q}$, $0 < q < p$, $p > 1$, *then* $H_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$.

Proof. If $\alpha = \pm\dfrac{1}{2}$, the result reduces to a weighted estimate for the Fourier sine- and cosine transformations.

Let $\alpha \geq -\dfrac{1}{2}$ and suppose f is simple. Since the Bessel function has an integral representation ([5, 952 (4)]),

$$\begin{aligned}
J_\alpha(x) &= \frac{2^{1-\alpha}x^\alpha}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \cos(x\cos y)\sin^{2\alpha} y\,dy\,,\\
(H_\alpha f)(x) &= \frac{2^{1-\alpha}x^{\alpha+\frac{1}{2}}}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^\infty t^{\alpha+\frac{1}{2}}f(t)(\int_0^{\frac{\pi}{2}} \cos(xt\cos y)\sin^{2\alpha} y\,dy)\,dt\\
&= \frac{2^{1-\alpha}x^{\alpha+\frac{1}{2}}}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin^{2\alpha} y(\int_0^\infty \cos(xt\cos y)t^{\alpha+\frac{1}{2}}f(t)\,dt)\,dy\\
&\equiv x^{\alpha+\frac{1}{2}}(T_\alpha g)(x)\,,
\end{aligned}$$

where $g(t) = t^{\alpha+\frac{1}{2}}f(t)$ and

$$(T_\alpha g)(x) = \frac{2^{1-\alpha}}{\pi^{\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})}\int_0^{\frac{\pi}{2}} \sin^{2\alpha} y(\int_0^\infty \cos(xt\cos y)g(t)\,dt)\,dy\,.$$

The interchange of order of integration is justified by Fubini's theorem. It is not hard to check that $T_\alpha \in [L^1(R^+), L^\infty(R^+)] \cap [L^2(R^+), L^2(R^+)]$, [2, Theorem 3].

Now if $(u_\alpha, v_\alpha) \in F^*_{p,q}$, then by Corollary 10 with $(Bf)(x) \equiv (H_\alpha f)(x)$, $w(x) = x^{\alpha+\frac{1}{2}}$ and $(Tg)(x) = (T_\alpha g)(x)$ we obtain $H_\alpha \in [L^p_v(R^+), L^g_u(R^+)]$, which proves the theorem. □

3. The $\underline{K}$ and Y-transformations

Since the $\underline{\mathrm{K}}$-transformation of f of order α is defined formally by

$$(\underline{K}_\alpha f)(x) = \int_0^\infty (xy)^{\frac{1}{2}} K_\alpha(xy) f(y)\,dy, \quad x > 0, \quad \alpha \geq -\frac{1}{2}, \tag{12}$$

where K_α is the modified Bessel function of the third kind ([4, Chapter X]).

If $\alpha = \pm\frac{1}{2}$, the transformation reduces to the Laplace transform

$$(\underline{K}_{\pm\frac{1}{2}})(x) = (\frac{\pi}{2})^{\frac{1}{2}} \int_0^\infty e^{-xy} f(y)\,dy\,.$$

If $\alpha \geq -\frac{1}{2}$ the kernel K_α can be written as:

$$K_\alpha(x) = \frac{2^{-\alpha}\Gamma(\frac{1}{2})x^\alpha}{\Gamma(\alpha+\frac{1}{2})} \int_1^\infty e^{-xt}(t^2-1)^{\alpha-\frac{1}{2}}\,dt, \quad x > 0, \quad [5, \text{p. } 958\ (3)]$$

or

$$K_\alpha(x) = \frac{2^{\alpha}\Gamma(\alpha+\frac{1}{2})x^{-\alpha}}{\Gamma(\frac{1}{2})} \int_0^\infty \frac{\cos xt\,dt}{(1+t^2)^{\alpha+\frac{1}{2}}}, \quad x > 0 \quad [5, \text{p. } 959\ (5)]\,.$$

It follows that (12) has the representations

$$(\underline{K}_\alpha f)(x) = \frac{2^{-\alpha}\Gamma(\frac{1}{2})x^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_0^\infty y^{\alpha+\frac{1}{2}} f(y)\ \Big[\int_1^\infty e^{-xyt}(t^2-1)^{\alpha-\frac{1}{2}}\,dt\Big]\,dy\,, \tag{13}$$

$$(\underline{K}_\alpha f)(x) = \frac{2^{\alpha}\Gamma(\alpha+\frac{1}{2})x^{-\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2})} \int_0^\infty y^{-\alpha+\frac{1}{2}} f(y)\Big[\int_0^\infty \frac{\cos xyt\,dt}{(1+t^2)^{\alpha+\frac{1}{2}}}\Big]\,dy\,, \tag{14}$$

which are needed to prove the weighted norm inequality for $\underline{K}_\alpha$.

Theorem 12. *Let $\alpha \geq -\frac{1}{2}$, $\alpha \neq 0$, u, v be non-negative functions defined on R^+ and $u_\alpha(x) = x^{\frac{1}{2}-|\alpha|}u(x), v_\alpha(x) = x^{-\frac{1}{2}+|\alpha|}v(x)$. If $(u_\alpha, v_\alpha) \in F^*_{p,g}$, $0 < q < p$, $p > 1$, then $\underline{K}_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$.*

Proof. Consider first the case $-\frac{1}{2} < \alpha < 0$, ([2, Theorem 4]), then by (13)

$$\begin{aligned}(\underline{K}_\alpha f)(x) &= \frac{2^{-\alpha+1}\Gamma(\frac{1}{2})x^{\alpha+\frac{1}{2}}}{\Gamma(\alpha+\frac{1}{2})} \int_1^\infty (t^2-1)^{\alpha-\frac{1}{2}}\Big[\int_0^\infty e^{-xyt}y^{\alpha+\frac{1}{2}}f(y)\,dy\Big]\,dt \\ &= x^{\alpha+\frac{1}{2}}(T'_\alpha g)(x)\,,\end{aligned}$$

where $g(y) = y^{\alpha+\frac{1}{2}}f(y)$ and

$$(T'_\alpha g)(x) = \frac{2^{-\alpha}\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}\int_1^\infty (t^2-1)^{\alpha-\frac{1}{2}}[\int_0^\infty e^{-xyt}g(y)\,dy]\,dt\,.$$

Hence $T'_\alpha \in [L^1(R^+), L^\infty(R^+)]\cap[L^2(R^+), L^2(R^+)]$, [2, Theorem 4] and if $(u_\alpha, v_\alpha) \in F^*_{p,q}$, then by Corollary 10 we obtain $\underline{K}_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$, where $-\dfrac{1}{2} < \alpha < 0$.

If $\alpha > 0$, we use the representation (14) so that

$$\begin{aligned}(\underline{K}_\alpha f)(x)&\frac{2^\alpha\Gamma(\alpha+\frac{1}{2})x^{-\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2})}\int_0^\infty (1+t^2)^{-\alpha-\frac{1}{2}}[\int_0^\infty \cos xyt y^{-\alpha+\frac{1}{2}}f(y)\,dy]\,dt\\ &= x^{-\alpha+\frac{1}{2}}(T''_\alpha g)(x)\,,\end{aligned}$$

where $g(y) = y^{-\alpha+\frac{1}{2}}f(y)$ and

$$(T''_\alpha g)(x) = \frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty (1+t^2)^{-\alpha-\frac{1}{2}}[\int_0^\infty \cos xytg(y)\,dy]\,dt\,.$$

Again $T''_\alpha \in [L^1(R^+), L^\infty(R^+)] \cap [L^2(R^+), L^2(R^+)]$, [2, Theorem 4].

By applying Corollary 10, we obtain $\underline{K}_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$, $\alpha > 0$. This proves the theorem. □

The final application involves the Y-transform defined for $0 < |a| \leq \dfrac{1}{2}$ by

$$(Y_\alpha f)(x) = \int_0^\infty y_\alpha(xt)(xt)^{\frac{1}{2}}f(t)\,dt\,,\quad x > 0\,, \tag{15}$$

where y_α is the Bessel function of second kind or Neumann's function. If $0 < |\alpha| < \dfrac{1}{2}$, the kernel y_α can be written as

$$\begin{aligned}y_\alpha(x) = \frac{2(x/2)^\alpha}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2}))}&\left\{\int_0^{\frac{\pi}{2}}\sin(x\sin\theta)\cos^{2\alpha}\theta\,d\theta\right.\\ &\left.-\int_0^\infty [e^{-xy}/(1+y^2)^{-\alpha+\frac{1}{2}}]\,dy\right\},\quad ([5, \text{p. } 955\ (5)])\end{aligned}$$

or

$$y_\alpha(x) = -\frac{2(x/2)^{-\alpha}}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{1}{2})}\int_1^\infty [\cos xt/(t^2-1)^{\alpha+\frac{1}{2}}]\,dt\,,$$

([5, p. 955 (2)]). If $\alpha = \dfrac{1}{2}$, $y_{\frac{1}{2}}(x) = -(2/(\pi x))^{\frac{1}{2}}\cos x$, ([5, p. 967 (1)]) and if $\alpha = -\dfrac{1}{2}$, $y_{-\frac{1}{2}}(x) = (2/(\pi x))^{\frac{1}{2}}\sin x$, ([5, p. 967 (2)]).

It follows that (15) has the two integral representations

$$(Y_\alpha f)(x) = \frac{2^{1-\alpha}x^{\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2})}\int_0^\infty t^{\alpha+\frac{1}{2}}f(t) \times\left\{\int_0^{\frac{\pi}{2}}\sin(xt\sin\theta)\cos^{2\alpha}\theta\,d\theta - \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+\frac{1}{2}}]\,dy\right\}dt \tag{16}$$

and

$$(Y_\alpha f)(x) = -\frac{2^{1+\alpha}x^{\frac{1}{2}-\alpha}}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{1}{2})}\int_0^\infty t^{\frac{1}{2}-\alpha}f(t) \times\left\{\int_0^\infty [\cos xty/(y^2-1)^{\alpha+\frac{1}{2}}]\,dy\right\}dt \tag{17}$$

which are needed to prove the weighted inequality for Y_α. □

Theorem 13. *Let $0 < |\alpha| \le \dfrac{1}{2}$, u, v be non-negative functions defined on R^+ and $u_\alpha(x) = x^{\frac{1}{2}-|\alpha|}u(x)$ and $v_\alpha(x) = x^{-\frac{1}{2}+|\alpha|}v(x)$. If $(u_\alpha, v_\alpha) \in F^*_{p,g}$, $0 < q < p$, $p > 1$, then $Y_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$.*

Proof. First consider the case $-\dfrac{1}{2} < \alpha < 0$ then by (16)

$$\begin{aligned}(Y_\alpha f)(x) &= \frac{2^{1-\alpha}x^{\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2})}\left\{\int_0^{\frac{\pi}{2}}\cos^{2\alpha}\theta\Big[\int_0^\infty \sin(xt\sin\theta)\right. \\ &\qquad \left.\times\, t^{\alpha+\frac{1}{2}}f(t)\,dt\Big]\,d\theta - \int_0^\infty (1+y^2)^{\alpha-\frac{1}{2}}\Big[\int_0^\infty e^{-xty}t^{\alpha+\frac{1}{2}}f(t)\,dt\Big]\,dy\right\} \\ &\equiv x^{\alpha+\frac{1}{2}}(F'_\alpha g)(x), \quad \text{where} \quad g(t) = t^{\alpha+\frac{1}{2}}f(t)\end{aligned}$$

and

$$\begin{aligned}(F'_\alpha g)(x) &= \frac{2^{1-\alpha}}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})}\left\{\int_0^{\frac{\pi}{2}}\cos^{2\alpha}\theta\Big[\int_0^\infty \sin(xt\sin\theta)\right. \\ &\qquad \left.\times\, g(t)\,dt\Big]\,d\theta - \int_0^\infty (1+y^2)^{\alpha-\frac{1}{2}}\Big[\int_0^\infty e^{-xty}g(t)\,dt\Big]\,dy\right\}.\end{aligned}$$

So that the above integchange of order of integrations is justified by Fubini's theorem since f vanishes outside $(0,a)$ for some $a > 0$, Hölder's inequality shows that

$$\int_0^\infty |t^{\alpha+\frac{1}{2}} v_\alpha(t) f(t) v_\alpha(t)^{-1}| \Big\{ \int_0^{\frac{\pi}{2}} |\sin(xtsin\theta)\cos^{2\alpha}\theta|\,d\theta + \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+\frac{1}{2}}]\,dy \Big\}\,dt$$

$$\leq \|f\|_{L_v^p(R^+)} \Big\{ \int_0^a (1/v_\alpha)^*(t)^{p'} [\int_0^{\frac{\pi}{2}} |\sin(xt\sin\theta)\cos^{2\alpha}\theta|\,d\theta + \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+\frac{1}{2}}]\,dy]^{p'}\,dt \Big\}^{\frac{1}{p}'} < \infty\,.$$

Since the sum of the two inner integral is dominated by

$$B(\alpha+\frac{1}{2},\frac{1}{2}) + \int_0^\infty (1-y^2)^{\alpha-\frac{1}{2}}\,dy = B(\alpha+\frac{1}{2},\frac{1}{2}) + B(-\alpha,\frac{1}{2})/2 < \infty$$

where B denotes the beta function, it follows that

$$|(F'_\alpha g)(x)| \leq \frac{2^{1-\alpha}}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2})} \int_0^\infty |g(t)| \Big\{ \int_0^{\frac{\pi}{2}} \sin(xt\sin\theta) \times \cos^{2\alpha}\theta\,d\theta| + \int_0^\infty [e^{-xty}/(1+y^2)^{-\alpha+\frac{1}{2}}]\,dy \Big\}\,dt \leq C\|g\|_{L^1(R^+)}\,,$$

which shows that $F'_\alpha \in [L^1(R^+), L^\infty(R^+)]$. Also, by Minkowski's integral inequality

$$\Big\{ \int_0^\infty |(F'_\alpha g)(x)|^2\,dx\}^{\frac{1}{2}} \leq \frac{2^{1-\alpha}}{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2})} \Big\{ \int_0^{\frac{\pi}{2}} \cos^{2\alpha}\theta \times [\int_0^\infty |\int_0^\infty \sin(xt\sin\theta) g(t)\,dt|^2\,dx]^{\frac{1}{2}}\,d\theta + \int_0^\infty (1+t^2)^{\alpha-\frac{1}{2}} [\int_0^\infty |\int_0^\infty e^{-xty} g(y)\,dy|^2\,dx]^{\frac{1}{2}}\,dt \Big\}\,.$$

If we let $x\sin\theta = z$, $\theta \in (0,\pi/2)$ in the first inner integral and $xt = z$ in the second integral, then the above integrals are dominated by

$$C\Big\{ \int_0^{\frac{\pi}{2}} \cos^{2\alpha}\theta \sin^{-\frac{1}{2}}\theta\,d\theta [\int_0^\infty |\int_0^\infty \sin(tz) g(t)\,dt|^2\,dz]^{\frac{1}{2}} + \int_0^\infty (1+t^2)^{\alpha-\frac{1}{2}} t^{-\frac{1}{2}} [\int_0^\infty |\int_0^\infty e^{-yz} g(y)\,dy|^2\,dz]^{\frac{1}{2}}\,dt \Big\}$$

$$\leq C\{B(\alpha+\frac{1}{2},\frac{1}{2}) + B(-\alpha+\frac{1}{4},\frac{1}{4})/2\}\ \|g\|_{L^2(R^+)}\,,$$

where the last inequality follows from Plancherel's theorem for the Fourier sine-transform and the fact that the Laplace transform maps $L^2(R^+)$ to $L^2(R^+)$. Hence $F'_\alpha \in [L^1(R^+), L^\infty(R^+)] \cap [L^2(R^+), L^2(R^+)]$.

If $0 < \alpha < \dfrac{1}{2}$ we use the integral representation (17) so that

$$\begin{aligned}(Y_\alpha f)(x) &= -\frac{2^{1+\alpha} x^{-\alpha+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{1}{2})} \int_1^\infty (t^2-1)^{-\alpha-\frac{1}{2}} \\ &\quad \times \left\{ \int_0^\infty \cos(xyt) y^{-\alpha+\frac{1}{2}} f(y)\, dy \right\} dt \\ &= x^{-\alpha+\frac{1}{2}} (F''_\alpha)(x), \quad \text{where} \quad g(y) = y^{-\alpha+\frac{1}{2}} f(y)\end{aligned}$$

and

$$(F''_\alpha g)(x) = -\frac{2^{1+\alpha}}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{1}{2})} \int_1^\infty (t^2-1)^{-\alpha-\frac{1}{2}} \{ \int_0^\infty \cos(xyt) g(y)\, dy \}\, dt\,.$$

Again it is seen that $F''_\alpha \in [L^1(R^+), L^\infty(R^+)] \cap [L^2(R^+), L^2(R^+)]$.

Now, if $(u_\alpha, v_\alpha) \in F^*_{p,q}$, then by Corollary 10 we obtain $Y_\alpha \in [L^p_v(R^+), L^q_u(R^+)]$, which proves the theorem. □

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