ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 135 – 143

(L,φ) – REPRESENTATIONS OF ALGEBRAS

Andrzej Walendziak

Abstract. In this paper we introduce the concept of an (L,φ)-representation of an algebra A which is a common generalization of subdirect, full subdirect and weak direct representation of A. Here we characterize such representations in terms of congruence relations.

Let I be a nonvoid set. $P(I)$ and $F(I)$ denote the set of all subsets of I and the set of all finite subsets of I, respectively. We denote by $\mathcal{P}(I)$ the Boolean algebra $\langle P(I), \cap, \cup, \prime, \emptyset, I\rangle$. If f is a function from X into Y, then the kernel of f, written $\ker(f)$, is defined to be the binary relation $\{(a,b) \in X^2 : f(a) = f(b)\}$.

Let $(A_i : i \in I)$ be a system of similar algebras, and let $B = \prod(A_i : i \in I)$ denote the direct product of the A_i, $i \in I$. For each $i \in I$, we denote by p_i the i^{th} projection function from B onto A_i. For two elements $x, y \in B$ we define

$$I(x,y) = \{i \in I : x(i) \neq y(i)\}.$$

Definition 1. Let A be a subalgebra of $\prod(A_i : i \in I)$, L be an ideal of $\mathcal{P}(I)$ and let $\varphi \subseteq A^2$. We say that A is an (L,φ)-product of algebras $A_i (i \in I)$, and write $A = \prod_{(L,\varphi)}(A_i : i \in I)$ iff the following conditions hold:

(A1) A is a subdirect product of the $A_i, i \in I$,
(A2) for every $x, y \in A$, $I(x,y) \in L$,
(A3) for any $i \in I$ and any $x, y \in A$, if $(x,y) \in \varphi$, then there is $z \in A$ such that $z(i) = x(i)$, $z(j) = y(j)$ for each $j \in I - \{i\}$.

If $L = P(I)$, we will write $\prod_{\varphi}(A_i : i \in I)$ for $\prod_{(L,\varphi)}(A_i : i \in I)$.

Let $\mathrm{Con}(A)$ denote the set of all congruence relations on an algebra A. Then $\mathrm{Con}(A)$ forms a complete and algebraic lattice with 0_A and 1_A, the smallest and the largest congruence relation, respectively.

1991 *Mathematics Subject Classification*: 08A05, 08A30.
Key words and phrases: finitely restricted subdirect product, full subdirect product, weak direct product, congruence lattice, distributivity.
Received September 9, 1991.

Proposition 1. *Let A be a subalgebra of $\prod(A_i : i \in I)$ and let L be an ideal of $\mathcal{P}(I)$.*

(i) $A = \prod_{0_A}(A_i : i \in I)$ *iff A is a subdirect product of $A_i, i \in I$.*

(ii) $A = \prod_{(L,0_A)}(A_i : i \in I)$ *iff A is an L-restricted subdirect product of $A_i, i \in I$ (cf. [3], p. 92).*

(iii) $A = \prod_{1_A}(A_i : i \in I)$ *iff A is a full subdirect product of $A_i, i \in I$ (cf. [2] or [4]).*

(iv) $A = \prod_{(F(I),1_A)}(A_i : i \in I)$ *iff A is a weak direct product of $A_i, i \in I$ (cf. [2] or [4]).*

Proof. The first three statements are obvious.

To prove (iv), assume first that A is an $(F(I), 1_A)$-product of algebras $A_i (i \in I)$. We can see that A satisfies the following two conditions:

(B1) if $x, y \in A$, then $I(x, y)$ is finite,

(B2) if $x \in A$, $y \in \prod(A_i : i \in I)$ and if $I(x, y)$ is finite, then $y \in A$.

It is clear that (B1) holds. To prove (B2), let $x \in A$ and $y \in \prod(A_i : i \in I)$. Suppose that the set $I(x, y)$ contains only one element i_1. Since A is a subdirect product of $A_i (i \in I)$, there is $t \in A$ such that $t(i_1) = y(i_1)$. From the condition (A3) of Definition 1 it follows that there exists $z \in A$ satisfying $z(i_1) = t(i_1)$ and $z(i) = x(i)$ for each $i \in I - \{i_1\}$. Clearly $y = z$, thus $y \in A$. From this we get by induction that (B2) holds. Then A is a weak direct product of algebras $A_i, i \in I$. Conversely, assume that A satisfies conditions (B1) and (B2). Then A is a full subdirect product of $A_i (i \in I)$, and obviously, (A2) holds, for $L = F(I)$. Therefore, $A = \prod_{(F(I),1_A)}(A_i : i \in I)$. □

Definition 2. Let A be an algebra of type τ and $\varphi \subseteq A^2$. Let I be a nonvoid set and let L be an ideal of the Boolean algebra $\mathcal{P}(I)$. By an (L, φ)-representation of A we will mean an ordered pair $\langle (A_i : i \in I), f \rangle$, where $(A_i : i \in I)$ is a system of algebras of type τ and f is an embedding from A into $\prod(A_i : i \in I)$ such that $f(A) = \prod_{(L,f(\varphi))}(A_i : i \in I)$.

The mapping $f_i = p_i \circ f$, which is a homomorphism of A onto A_i will be referred to as the i^{th} f-projection.

An (L, φ)-representation of A is called

(i) subdirect, if $L = P(I)$ and $\varphi = 0_A$,

(ii) finitely restricted subdirect, if $L = F(I)$ and $\varphi = 0_A$,

(iii) full subdirect, if $L = P(I)$ and $\varphi = 1_A$,

(iv) weak direct, if $L = F(I)$ and $\varphi = 1_A$.

We shall now correlate (L, φ)-representations of an algebra A with congruence relations on A.

Let $\theta_i (i \in I)$ be congruences on A, and let L be an ideal of $\mathcal{P}(I)$. For any set $M \in L$, we define a congruence relation $\theta(M)$ of A by

$$\theta(M) = \bigwedge(\theta_j : j \notin M).$$

For $i \in I$, we set $\bar{\theta}_i = \bigwedge(\theta_j : j \in I - \{i\})$. For some $\alpha \in \mathrm{Con}(A)$ and $\varphi \subseteq A^2$ we write

$$\alpha = \prod\nolimits_{(L,\varphi)}(\theta_i : i \in I)$$

iff the following conditions hold:

(C1) $\alpha = \bigwedge(\theta_i : i \in I)$,

(C2) $1_A = \bigvee(\theta(M) : M \in L)$,

(C3) for all $i \in I$, $\varphi \subseteq \theta_i \circ \bar{\theta}_i$ ($\theta_i \circ \bar{\theta}_i$ denotes the relational product of congruences θ_i and $\bar{\theta}_i$).

Theorem 1.

(i) *Let A be an algebra and φ be a binary relation on A. Let I be a nonvoid set and L be an ideal of $\mathcal{P}(I)$. If $\langle (A_i : i \in I), f\rangle$ is an (L, φ)-representation of A and if $\theta_i (i \in I)$ is the kernel of the i^{th} f-projection f_i, then $0_A = \prod_{(L,\varphi)}(\theta_i : i \in I)$.*

(ii) *Let $(\theta_i : i \in I)$ be a system of congruences of A such that $0_A = \prod_{(L,\varphi)}(\theta_i : i \in I)$. We put $A_i = A/\theta_i$ for $i \in I$ and define the mapping $f : A \to \prod(A_i : i \in I)$ by setting $f(x) = (x/\theta_i : i \in I)$. ($x/\theta_i$ is the congruence class containing x.) Then $\langle (A_i : i \in I), f\rangle$ is an (L, φ)-representation of A.*

Proof. (i) By assumption the mapping f is one-to-one, and hence $0_A = \bigwedge(\theta_i : i \in I)$.

To prove (C2), let $x, y \in A$. Clearly,

$$M = \{i \in I : f_i(x) \neq f_i(y)\} = I(f(x), f(y)) \in L$$

and $(x, y) \in \theta(M)$. Then $(x, y) \in \bigvee(\theta(M)) : M \in L)$ and hence (C2) holds. Moreover, (C3) immediately follows from (A3). Thus $0_A = \prod_{(L,\varphi)}(\theta_i : i \in I)$.

(ii) The fact that f is an embedding is easy to check. Of course, the mapping f_i is onto for each $i \in I$. Therefore, $\bar{A} = f(A)$ is a subdirect product of algebras $A_i, i \in I$. Let $x, y \in A$. Now we prove that

$$I(f(x), f(y)) \in L\,. \tag{1}$$

By (C2), $(x, y) \in \bigvee(\theta(M) : M \in L)$. Then, there exists a finite number of sets $M_1, \ldots, M_n \in L$ such that $(x, y) \in \theta(M_1) \vee \cdots \vee \theta(M_n)$. Observe that

$$\{i \in I : f_i(x) \neq f_i(y)\} \subseteq M_1 \cup \cdots \cup M_n\,. \tag{2}$$

Indeed, let $f_i(x) \neq f_i(y)$ for some $i \in I$, and suppose on the contrary that $i \notin M_1 \cup \cdots \cup M_n$. Therefore, $\theta(M_1) \vee \cdots \vee \theta(M_n) \leq \theta_i$, and hence $(x, y) \in \theta_i$, i.e. $f_i(x) = f_i(y)$, a contradiction. From (2), by the definition of ideal we conclude that $\{i \in I : f_i(x) \neq f_i(y)\} \in L$. Thus (1) is satisfied. Finally, from (C3) it follows that for any $i \in I$ and any $\bar{x}, \bar{y} \in \bar{A}$, if $(\bar{x}, \bar{y}) \in f(\varphi)$, then there is $\bar{z} \in \bar{A}$ such that $\bar{z}(i) = \bar{x}(i)$ and $\bar{z}(j) = \bar{y}(j)$ for each $j \in I - \{i\}$. Then $f(A) = \prod_{(L,f(\varphi))}(A_i : i \in I)$, which was to be proved. $\square$

Corollary 1. *Let $(\theta_i : i \in I)$ be a system of congruence relations on an algebra A. If $0_A = \bigwedge(\theta_i : i \in I)$, then*

(i) *$(\theta_i : i \in I)$ gives a subdirect representation of A,*
(ii) *$(\theta_i : i \in I)$ constitutes a finitely restricted subdirect representation of A iff $1_A = \bigvee(\theta(M) : M \in F(I))$,*
(iii) *$(\theta_i : i \in I)$ gives a full subdirect representation of A iff $1_A = \theta_i \circ \bar{\theta}_i$ for all $i \in I$,*
(iv) *$(\theta_i : i \in I)$ constitutes a weak direct representation of A iff $1_A = \bigvee(\theta(M) : M \in F(I))$ and $1_A = \theta_i \circ \bar{\theta}_i$ for each $i \in I$.*

Lemma 1. *Let I, J be two sets of indices and L_1, L_2 ideals of the Boolean algebras $\mathcal{P}(I)$, $\mathcal{P}(J)$, respectively. Let A be an algebra with $\mathrm{Con}(A)$ completely distributive and let $\varphi \subseteq A^2$. If*

$$0_A = \prod_{(L_1,\varphi)}(\alpha_i : i \in I) = \prod_{(L_2,\varphi)}(\beta_j : j \in J)$$

for congruences α_i, β_j on A, then there exist congruences $\delta_{ij} (i \in I, j \in J)$ such that, for all i and j,

$$\alpha_i = \prod_{(L_2,\varphi)}(\delta_{ij} : j \in J), \text{and } \beta_j = \prod_{(L_1,\varphi)}(\delta_{ij} : i \in I).$$

Proof. For $i \in I$ and $j \in J$, we put $\delta_{ij} = \alpha_i \vee \beta_j$. Let i be a fixed but arbitrary element of I. Observe that

$$\alpha_i = \bigwedge(\delta_{ij} : j \in J)\,. \tag{3}$$

Indeed, by completely distributivity of $\mathrm{Con}(A)$ we have

$$\alpha_i = \alpha_i \vee \bigwedge(\beta_j : j \in J) = \bigwedge(\alpha_i \vee \beta_j : j \in J) = \bigwedge(\delta_{ij} : j \in J),$$

i.e (3) holds.
For $M \in L_2$, we set $\delta(M) = \bigwedge(\delta_{ij} : j \notin M)$. Now we prove that

$$1_A = \bigvee(\delta(M) : M \in L_2)\,. \tag{4}$$

Let $x, y \in A$. Since $(x,y) \in \bigvee(\beta(M) : M \in L_2)$ we can choose a finite number of sets $M_1, \ldots, M_2 \in L_2$ such that

$$(x,y) \in \beta(M_1) \vee \cdots \vee \beta(M_n)\,.$$

We set $M = \{j \in J : (x,y) \notin \delta_{ij}\}$. Let $j \in M$ and $j \notin M_1 \cup \cdots \cup M_n$. It is obvious that $\beta(M_k) \leq \beta_j$ for each $k = 1, \ldots, n$. Therefore, $\beta(M_1) \vee \cdots \vee \beta(M_n) \leq \delta_{ij}$. Then $(x,y) \in \delta_{ij}$, which gives us a contradiction. Consequently, $M \subseteq M_1 \cup \cdots \cup M_n$, and hence $M \in L_2$. Thus $(x,y) \in \delta(M)$ and (4) is satisfied.
For each $j \in J$, let us write $\bar{\delta}_{ij}$ for $\bigwedge(\delta_{ik} : k \in J - \{j\})$. Clearly, $\delta_{ij} \geq \beta_j$ and $\bar{\delta}_{ij} \geq \bar{\beta}_j$. Since $\varphi \subseteq \beta_j \circ \bar{\beta}_j$, we have

$$\varphi \subseteq \delta_{ij} \circ \bar{\delta}_{ij}\,, \tag{5}$$

for all $j \in J$. From (3), (4) and (5) it follows that $\alpha_i = \prod_{(L_2,\varphi)}(\delta_{ij} : j \in J)$. The proof that $\beta_j = \prod_{(L_1,\varphi)}(\delta_{ij} : i \in I)$ is similar. □

Lemma 2. *Let I, J be two sets of indices and L_1, L_2 ideals of $\mathcal{P}(I)$ and $\mathcal{P}(J)$, respectively. Let A be an algebra whose congruence lattice is distributive. If*

$$0_A = \prod_{(L_1,1_A)}(\alpha_i : i \in J) = \prod_{(L_2,1_A)}(\beta_j : j \in J)$$

for congruences α_i, β_j on A, then
$\alpha_i = \prod_{(L_2,1_A)}(\alpha_i \vee \beta_j : j \in J)$ and $\beta_j = \prod_{(L_1,1_A)}(\alpha_i \vee \beta_j : i \in I)$ for all i and j.

Proof. For $i \in I$ and $j \in J$, we set $\delta_{ij} = \alpha_i \vee \beta_j$. First we show that (3) holds. By distributivity of $\mathrm{Con}(A)$ we have $\bar{\alpha}_i \wedge \delta_{ij} = \bar{\alpha}_i \wedge (\alpha_i \vee \beta_j) = \bar{\alpha}_i \wedge \bar{\beta}_j \leq \beta_j$. Hence $\bar{\alpha}_i \wedge \bigwedge(\delta_{ij} : j \in J) = \bigwedge(\bar{\alpha}_i \wedge \delta_{ij} : j \in J) \leq \bigwedge(\beta_j : j \in J) = 0_A$. Therefore, using distributivity we get

$$\bigwedge(\delta_{ij} : j \in J) = \bigwedge(\delta_{ij} :\ j \in J) \wedge (\alpha_i \vee \bar{\alpha}_i) = \alpha_i \wedge \bigwedge(\delta_{ij} :\ j \in J) = \alpha_i\,,$$

i.e. (3) is satisfied. By the proof of Lemma 1 we conclude that (4) holds. Finally, since $1_A = \beta_j \circ \bar{\beta}_j$ we have

$$1_A = \delta_{ij} \circ \bigwedge(\delta_{ij} : k \in J - \{j\})\,, \tag{6}$$

for all $j \in J$. From (3), (4) and (6) it follows that $\alpha_i = \prod_{(L_2,1_A)}(\delta_{ij} : j \in J)$. The proof that $\beta_j = \prod_{(L_1,1_A)}(\delta_{ij} : i \in I)$ is similar. □

A subset $\Gamma \subseteq \mathrm{Con}(A)$ is called meet irredundant iff for all proper subsets Γ' of Γ we have $\bigwedge \Gamma < \bigwedge \Gamma'$. An (L, φ)-representation $\langle (A_i : i \in I), f\rangle$ of A is said to be irredundant if the set $\{\ker(f_i) : i \in I\}$ is meet irredundant, where f_i is the i^{th} f-projection.

Lemma 3. *Let $(L, 1_A)$-representation $\langle (A_i : i \in I), f\rangle$ of A be given. If $|A_i| > 1$ for each $i \in I$, then this representation of A is irredundant.*

Proof. Let $\theta_i (i \in I)$ be the kernel of the i^{th} f-projection f_i. By Theorem 1,

$$0_A = \prod_{(L,1_A)}(\theta_i : i \in I)\,.$$

We shall prove that the set $\{\theta_i : i \in I\}$ is meet irredundant. Suppose on the contrary that $0_A = \bar{\theta}_i$ for some $i \in I$. Then $1_A = \theta_i \circ \bar{\theta}_i = \theta_i$. Hence $|A/\theta_i| = 1$, and therefore $|A_i| = 1$, since $A_i \cong A/\theta_i$. This is a contrary to the assumption. Consequently, the representation $\langle (A_i : i \in I), f\rangle$ of A is irredundant. □

Let $\varphi \subseteq A^2$. We say that $\alpha \in \mathrm{Con}(A)$ is φ-irreducible if $\alpha \neq 1_A$ and for every system $(\theta_i : i \in I)$ of congruences on A, $\alpha = \prod_\varphi(\theta_i : i \in I)$ implies that there is an element $i \in I$ such that $\alpha = \theta_i$.

Proposition 2. *Let $\alpha \in \mathrm{Con}(A)$.*

(i) *α is 0_A-irreducible iff α is a completely meet irreducible element of $\mathrm{Con}(A)$ (i.e. $\alpha \neq 1_A$ and for all $\Gamma \subseteq \mathrm{Con}(A)$, if $\alpha = \bigwedge \Gamma$, then $\alpha \in \Gamma$).*

(ii) *α is 1_A-irreducible iff α is indecomposable (i.e. $\alpha \neq 1_A$ and for any $\beta, \gamma \in \mathrm{Con}(A)$, if $\alpha = \beta \wedge \gamma$ and $1_A = \beta \circ \gamma$, then $\beta = 1_A$ or $\gamma = 1_A$).*

Proof. The proof of statement (i) is trivial.

To prove the second statement, assume first that α is indecomposable. Let $\alpha = \prod_{1_A}(\theta_i : i \in I)$ and i be an index of I such that $\theta_i \neq 1_A$. Clearly, $\alpha = \theta_i \wedge \bar{\theta}_i$ and $1_A = \theta_i \circ \bar{\theta}_i$. Since α is indecomposable and $\theta_i \neq 1_A$, we have $\bar{\theta}_i = 1_A$. Consequently, $\alpha = \theta_i$, and thus we obtain that α is 1_A-irreducible. The converse is obvious. □

Lemma 4. *Let A be an algebra and $\alpha \in \mathrm{Con}(A)$.*

(i) *A/α is subdirectly irreducible iff α is 0_A-irreducible.*
(ii) *A/α is directly indecomposable iff α is 1_A-irreducible.*

Proof. (i) It is well known that A/α is subdirectly irreducible iff α is completely meet irreducible in $\mathrm{Con}(A)$. Hence in view of Proposition 2 we obtain (i).

(ii) By Lemma 2(§ 5.2) in [5] we deduce that A/α is directly indecomposable iff α is indecomposable. Now, using Proposition 2 we get (ii). □

Theorem 2. *Let the assumptions of Lemma 1 be satisfied. Let $\langle (A_i : i \in I), f\rangle$ be an irredundant (L_1, φ)-representation of A and $\langle (B_j : j \in J), g\rangle$ be an irredudant (L_2, φ)-representation of A. Suppose that each $\alpha_i = \ker(f_i)$and each $\beta_j = \ker(g_j)$ is φ-irreducible. Then there is a bijection $\sigma : I \to J$ for which the following conditions hold:*

(D1) *for each $i \in I$, there exists an isomorphism*

$$h_i : A_i \to B_{\sigma(i)}, \quad \text{such that } h_i \circ f_i = g_{\sigma(i)}$$

(D2) *$\sigma(I(f(x), f(y))) = J(g(x), g(y))$ for all $x, y \in A$.*

Proof. By Theorem 1,
$0_A = \prod_{(L_1,\varphi)}(\alpha_i : i \in I)$ and $0_A = \prod_{(L_2,\varphi)}(\beta_j : j \in J)$.
For each $i \in I$ and each $j \in J$, we set

$$\delta_{ij} = \alpha_i \vee \beta_j \text{ and } D_{ij} = A/\delta_{ij}\,.$$

Using Lemma 1 we obtain

$$\alpha_i = \prod\nolimits_{(L_2,\varphi)}(\delta_{ij} : j \in J) \text{ and } \beta_j = \prod\nolimits_{(L_1,\varphi)}(\delta_{ij} : i \in I)\,.$$

Hence, $\alpha_i = \prod_{\varphi}(\delta_{ij} : j \in J)$ and $\beta_j = \prod_{\varphi}(\delta_{ij} : i \in I)$. Since α_i is φ-irreducible, we infer that there is an index $\sigma(i) = j \in J$ such that $\alpha_i = \delta_{ij}$. But β_j is also φ-irreducible, and therefore, $\beta_j = \delta_{i'j}$ for some $i' = \pi(j) \in I$. Consequently, $\alpha_i = \alpha_i \vee \beta_j$ and $\beta_j = \alpha_{i'} \vee \beta_j$. Then $\alpha_i \geq \beta_j \geq \alpha_{i'}$. Observe that $i = i'$. Indeed, if $i \neq i'$, then $\bar{\alpha}_i \leq \alpha_{i'} \leq \alpha_i$, and hence $0_A = \alpha_i \wedge \bar{\alpha}_i = \bar{\alpha}_i$. This is a contrary to the fact that the representation $\langle (A_i : i \in I), f\rangle$ of A is irredundant. Therefore, $\pi\sigma(i) = i$ for all $i \in I$, and similarly, $\sigma\pi(j) = j$ for all $j \in J$. Then π is a two-sided inverse of σ, and this proves that σ is a bijection. If $\sigma(i) = j$, then we have

$$A_i \cong A/\alpha_i = D_{ij} = A/\beta_j \cong B_j\,.$$

The map

$$f_i(x) \to x/\delta_{ij}(x \in A)$$

defines an isomorphism of A_i with D_{ij}, and the map

$$g_j(x) \to x/\delta_{ij}(x \in A)$$

defines an isomorphism from B_j onto D_{ij}. It is easy to see that the mapping h_i defined on A_i by $h_i(f_i(x)) = g_j(x)$ is an isomorphism from A_i onto B_j.

To prove (D2), let $x, y \in A$. We have

$$i \in I(f(x), f(y)) \leftrightarrow f_i(x) \neq f_i(y) \leftrightarrow h_i \circ f_i(x) \neq h_i \circ f_i(y) \leftrightarrow \\ \leftrightarrow g_{\sigma(i)}(x) \neq g_{\sigma(i)}(y) \leftrightarrow \sigma(i) \in J(g(x), g(y)).$$

Therefore, (D2) is satisfied. □

Theorem 3. *Under the assumptions of Lemma 2, if $\langle (A_i : i \in I), f \rangle$ is an $(L_1, 1_A)$-representation of A and $\langle (B_j : j \in J), g \rangle$ is an $(L_2, 1_A)$-representation of A, with each A_i and each B_j directly indecomposable, then there is a bijection $\sigma : I \to J$ and for each $i \in I$ there is an isomorphism h_i from A_i onto $B_{\sigma(i)}$ such that $g_{\sigma(i)} = h_i \circ f_i$ for all $i \in I$.*

Proof. The proof is similar to that of Theorem 1. Here we apply Lemmas 2, 3 and 4. □

By Theorem 2 and Lemma 4 we obtain

Corollary 2. *Let A be an algebra whose congruence lattice is completely distributive. If $\langle (A_i : i \in I), f \rangle$ and $\langle (B_j : j \in J), g \rangle$ are two irredundant finitely restricted subdirect representations of A with subdirectly irreducible factors, then there is a bijection σ from I onto J and for each $i \in I$ there is an isomorphism h_i of A_i with $B_{\sigma(i)}$ such that $g_{\sigma(i)} = h_i \circ f_i$ for all $i \in I$.*

From Theorem 3 we have

Corollary 3. *Let A be an algebra with $\mathrm{Con}(A)$ distributive. Let two full subdirect representations $\langle (A_i : i \in I), f \rangle$ and $\langle (B_j : j \in J), g \rangle$ of A be given. If each $A_i (i \in I)$ and each $B_j (j \in J)$ is directly indecomposable, then there is a bijection $\sigma : I \to J$ and for each $i \in I$ there exists an isomorphism h_i from A_i onto $B_{\sigma(i)}$ such that $g_{\sigma(i)} = h_i \circ f_i$ for all $i \in I$.*

Moreover, as an immediate consequence of Theorem 3 we get

Corollary 4. *Let A be an algebra whose congruence lattice is distributive. If $\langle (A_i : i \in I), f \rangle$ and $\langle (B_j : j \in J), g \rangle$ are two weak direct representations of A with all factors directly indecomposable, then there is a bijection $\sigma : I \to J$ and for each $i \in I$ there exists an isomorphism $h_i : A_i \to B_{\sigma(i)}$ such that $g_{\sigma(i)} = h_i \circ f_i$ for all $i \in I$.*

Let $\varphi \in \mathrm{Con}(A)$. We say that the congruences of an algebra A φ-permute iff for every congruences α and β on A, $\alpha \wedge \varphi$ and $\beta \wedge \varphi$ permute.

It is obvious that for every algebra A the congruences of A 0_A-permute and that 1_A-permuting is the same thing as permuting.

Theorem 4. *Let φ be a dually distributive element of* $\mathrm{Con}(A)$. *Suppose that the congruences of A φ-permute and $\mathrm{Con}(A)$ is modular and complemented. Then there exists a system $(A_i : i \in I)$ of simple algebras and an embedding f from A into $\prod(A_i : i \in I)$ such that $\langle (A_i : i \in I), f\rangle$ is an irredundant (L,φ)-representation of A, where L is an ideal of $\mathcal{P}(I)$ containing all finite subsets of I.*

Proof. By Theorem 4.3 of [1], $\mathrm{Con}(A)$ is atomic. Let Γ be the set of all atoms of $\mathrm{Con}(A)$, and let $\{\alpha_i : i \in I\}$ be a maximal subset of Γ such that $\alpha_i \wedge \bigvee(\alpha_j : j \in I - \{i\}) = 0_A$ for all $i \in I$. (The existence of such maximal subset of Γ follows easily by Zorn's Lemma.) For $i \in I$, we set

$$\theta_i = \bigvee(\alpha_j : j \neq i) \quad \text{and } \bar{\theta}_i = \bigwedge(\theta_j : j \neq i)\,.$$

From Theorem 6.6 of [1] it follows that

$$0_A = \bigwedge(\theta_i : i \in I)\,. \tag{7}$$

As a consequence of Theorem 4.3 and 6.5. of [1] we have

$$1_A = \bigvee(\alpha_i : i \in I)\,.$$

Since $\alpha_i \leq \bar{\theta}_i$ for all $i \in I$, we obtain

$$1_A \leq \bigvee(\bar{\theta}_i : i \in I) = \bigvee(\theta(\{i\}) : i \in I) \leq \bigvee(\theta(M) : M \in L)\,.$$

Hence $1_A = \bigvee(\theta(M) : M \in L)$, and therefore (C2) is satisfied. Let i be an element of I. Obviously we have

$$1_A = \alpha_i \vee \theta_i \leq \bar{\theta}_i \vee \theta_i\,.$$

Since φ is dually distributive and the congruence of A φ-permute, we get

$$\varphi = \varphi \wedge (\theta_i \vee \bar{\theta}_i) = (\varphi \wedge \theta_i) \vee (\varphi \wedge \bar{\theta}_i) = (\varphi \wedge \theta_i) \circ (\varphi \wedge \bar{\theta}_i)\,.$$

From this we conclude that $\varphi \subseteq \theta_i \circ \bar{\theta}_i$, i.e. (C3) holds. Thus the system $(\theta_i : i \in I)$ of congruences on A satisfies conditions (7), (C2) and (C3). Therefore, $0_A = \prod_{(L,\varphi)}(\theta_i : i \in I)$. We put $A_i = A/\theta_i$ for $i \in I$ and define the mapping $f : A \to \prod(A_i : i \in I)$ by setting $f(x) = (x/\theta_i : i \in I)$. By Theorem 1, $\langle (A_i : i \in I), f\rangle$ is an (L,φ)-representation of A. This representation of A is irredundant, because the set $\{\theta_i : i \in I\}$ is meet irredundant. Since θ_i is a coatom of $\mathrm{Con}(A)$, we obtain that A_i is simple. The proof is complete. □

As an immediate consequence of Theorem 4 we obtain

Corollary 5. (see [3], Theorem 5.1) *If congruence lattice of an algebra A is complemented and modular, then there is an irredundant finitely restricted subdirect representation of A with simple factors.*

It is well known that every algebra whose congruences permute has modular congruence lattice. Therefore, we get

Corollary 6. (cf. [3], Theorem 5.2) *Let A be any algebra whose congruences permute and whose congruence lattice is complemented. Then there exists a weak direct (and also a full subdirect) representation of A with simple factors.*

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Andrzej Walendziak
Department of Mathematics
Agricultural and Pedagogical University
PL - 08110 Siedlce, POLAND