ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 221 – 226

ATOMS IN LATTICE OF RADICAL CLASSES OF LATTICE–ORDERED GROUPS

Dao-Rong Ton

Abstract. There are several special kinds of radical classes. For example, a product radical class is closed under forming product, a closed-kernel radical class is closed under taking order closures, a K-radical class is closed under taking K-isomorphic images, a polar kernel radical class is closed under taking double polars, etc. The set of all radical classes of the same kind is a complete lattice. In this paper we discuss atoms in these lattices. We prove that every nontrivial element in these lattices has a cover.

For the definitions and the standard results concerning ℓ-groups, the reader is referred to [1, 3, 6]. Let G be an ℓ-group. $\mathscr{C}(G)$, $\mathscr{L}(G)$ and $K(G)$ will be denoted the complete lattices of all convex ℓ-subgroups, all ℓ-ideals and closed convex ℓ-subgroups of G, respectively. Let $C \subseteq G$. By $\overline{C}_G$ $C_G^{\perp\perp}$ we denote the order closure of C in G and the double polar of C in G, respectively. Two ℓ-groups G and G' are said to be K-isomorphic, if $K(G)$ and $K(G')$ are isomorphic as lattices. Join in a lattice L is denoted by $\vee^{(L)}$.

Let $\mathscr{G}$ be the set of all ℓ-groups. For $X \subseteq \mathscr{G}$ we denote by
$J_K(X)$ — the class of all ℓ-groups G having a system $\{G_\lambda | \lambda \in \wedge\} \subseteq X \cap K(G)$ such that $G = \bigvee\limits_{\lambda\in\wedge}{}^{(K(G))} G_\lambda$;
$L(X)$ — the class of all ℓ-groups G such that $K(G)$ is isomorphic to $K(G_1)$ for some $G_1 \in X$.

We can make new ℓ-groups from some original ℓ-groups. These constructions include:

1. taking convex ℓ-subgroups,
2. forming joins of convex ℓ-subgroups,
3. forming completely subdirect products,
3′. forming direct products,
4. taking ℓ-homomorphic images,
4′. taking complete ℓ-homomorphic images,

1991 *Mathematics Subject Classification*: Primary 06F15.
Key words and phrases: lattice-ordered group, radical class, closure operator, atom.
Received September 2, 1992.

5. forming extensions, that is, G is an extension of A by using B if A is an ℓ-ideal of G and $B = G/A$
6. taking order closures, that is, G is an order closure of A if A is a convex ℓ-subgroup of an ℓ-group H and $G = \overline{A}_H$.
7. taking double polars, that is, G is a double polar of A if A is a convex ℓ-subgroup of an ℓ-group H and $G = A_H^{\perp\perp}$.
8. taking K-isomorphic images.

A family $\mathscr{U}$ of ℓ-groups is called a class, if it is closed under some constructions. If a class $\mathscr{U}$ is closed under the constructions $i_1, \dots, i_k$, we call $\mathscr{U}$ a $i_1 \dots i_k$-class, where $i_1, \dots, i_k \in \{1, 2, 3, 3', 4, 4', 5, 6, 7, 8\}$ and $1 \le k \le 8$. All our classes are always assumed to contain along with a given ℓ-group all its ℓ-isomorphic copies.

Thus, a radical class [7] is a 12-class, a quasi-torsion class [9] is a $124'$-class, a torsion class [10] is a 124-class, a closed-kernel radical class [5] is a 126-class, a polar kernel radical class [5] is a 127-class, a K-radical class [8] is a 128-class. We call a $123'$ (123-class) a product radical class (a subproduct radical class). We call a 125-class a complete (idempotent) radical class.

In this paper we call $12i_3 \dots i_k$-classes radical classes. Let $T_{12i_3\dots i_k}$ be the set of all $12i_3 \dots i_k$-classes. For any family $\{\mathscr{R}_\lambda | \lambda \in \wedge\}$ of $12i_3 \dots i_k$-classes, $\bigcap_{\lambda\in\wedge} \mathscr{R}_\lambda \in T_{12i_3\dots i_k}$. So we can define

$$\bigwedge_{\lambda\in\wedge} \mathscr{R}_\lambda = \bigcap_{\lambda\in\wedge} \mathscr{R}_\lambda ,$$

$$\bigvee_{\lambda\in\wedge} \mathscr{R}_\lambda = \cap\{\mathscr{U} \in T_{12i_3\dots i_k} | \mathscr{U} \supseteq \mathscr{R}_\lambda \quad \text{for each} \quad \lambda \in \wedge\} ,$$

and $T_{12i_3\dots i_k}$ becomes a complete lattice.

Let $\mathscr{R}_{12i_3\dots i_k}$ be a $12i_3 \dots i_k$-class and G be an ℓ-group. Then there exists a largest convex ℓ-subgroup of G belonging to $\mathscr{R}_{12i_3\dots i_k}$. We denote it by $\mathscr{R}_{12i_3\dots i_k}(G)$ and call it a $\mathscr{R}_{12i_3\dots i_k}$-radical. It is invariant under all the ℓ-automorphisms of G. It is clear that an ℓ-group G belongs to $\mathscr{R}_{12i_3\dots i_k}$ if and only if $G = \mathscr{R}_{12i_3\dots i_k}(G)$. If $\mathscr{R}_1, \mathscr{R}_2 \in T_{12i_3\dots i_k}$, then $\mathscr{R}_1 \le \mathscr{R}_2$ if and only if $\mathscr{R}_1(G) \subseteq \mathscr{R}_2(G)$ for each ℓ-group (G).

Lemma 1. *Every closed-kernel radical class is a subproduct radical class.*

Proof. Suppose that $\mathscr{R}$ is a closed-kernel radical class and G is a completely subdirect product of $\{G_\lambda | \lambda \in \wedge\}$ where $\{G_\lambda | \lambda \in \wedge\} \subseteq \mathscr{R}$. That is,

$$\sum_{\lambda\in\wedge} G_\lambda \subseteq G \subseteq \prod_{\lambda\in\wedge} G_\lambda .$$

For each $\lambda \in \wedge$ put $\overline{G}_\lambda = \{g \in \prod_{\lambda\in\wedge} G_\lambda | \lambda' \ne \lambda \Longrightarrow g_{\lambda'} = 0\}$. Then $\mathscr{R}(G) \cap \overline{G}_\lambda = \mathscr{R}(\overline{G}_\lambda) = \overline{G}_\lambda$ and so $G \supseteq \mathscr{R}(G) \supseteq \overline{G}_\lambda$ for each $\lambda \in \wedge$. Let $0 < a = (\dots, a_\lambda, \dots) \in G$. Then

$$a = \bigvee_{\lambda\in\wedge}{}^{(G)} \overline{a}_\lambda$$

where $\overline{a}_\lambda = (0,\dots,0,a_\lambda,0,\dots,0) \in \overline{G}_\lambda(\lambda \in \wedge)$. Since $\mathscr{R}$ is closed-kernel, $a \in \mathscr{R}(G)$. Hence $G = \mathscr{R}(G)$ and $G \in \mathscr{R}$. $\square$

Suppose that $\mathscr{R}, \mathscr{T} \in T_{12}$. We define the product $\mathscr{R}\cdot\mathscr{T} = \{G \in \mathscr{G} | G/\mathscr{R}(G) \in \mathscr{T}\}$. Let $\mathscr{T} \in T_{12}$ and σ be an ordinal number. We define an assending sequence $\mathscr{T}, \mathscr{T}^2, \dots, \mathscr{T}^\sigma, \dots$ as follows:

$$\mathscr{T}^\sigma \begin{cases} \mathscr{T}\cdot\mathscr{T}^{\sigma-1} & \text{if } \sigma \text{ is not a limit ordinal} \\ \{G | G = \bigcup_{\alpha<\sigma} \mathscr{T}^\alpha(G)\} & \text{if } \sigma \text{ is a limit ordinal}. \end{cases}$$

It is easy to show that $\mathscr{T}^\sigma$ is a 12-class for each ordinal σ. Define $\mathscr{T}^* = \bigcup_\sigma \mathscr{T}^\sigma$. Similarly to the proof of Theorem 1.6 and Theorem 1.7 of [10] we can prove

Lemma 2. *Let $\mathscr{R}$ be a 12-class. Then $\mathscr{R}^*$ is the smallest complete 12-class containing $\mathscr{R}$. $\mathscr{R}$ is complete if and only if $\mathscr{R} = \mathscr{R}^*$. $\mathscr{R}^* \subseteq \mathscr{R}^{\perp\perp}$.*

Proposition 3. *For $12i_3\dots i_k$-classes of ℓ-groups we have the following relations:*

$$\begin{array}{ccccccccccccc} T_{128} & \subseteq & T_{126} & \subseteq & T_{123} & \subseteq & T_{123'} & \subseteq & T_{12} & \supseteq & T_{124'} & \supseteq & T_{124} \\ &&&&&&&& \cup\!| &&&& \\ &&&&&&&& T_{125} &&&& \\ &&&&&&&& \cup\!| &&&& \\ &&&&&&&& T_{127}. &&&& \end{array}$$

Proof. $T_{123} \subseteq T_{123'} \subseteq T_{12} \supseteq T_{124'}$ are clear. By Lemma 1 and Lemma 2 we get $T_{126} \subseteq T_{123}$ and $T_{127} \subseteq T_{125}$. It follows from Lemma 2.2 of [8] or Lemma 1.5 of [2] that $T_{128} \subseteq T_{126}$. $\square$

Now suppose that $\mathscr{R} \in T_{12}$. Put

$$\mathscr{R}^{i_3\dots i_k} = \cap\{\mathscr{U} \in T_{12i_3\dots i_k} | \mathscr{U} \supseteq \mathscr{R}\}.$$

Then $\mathscr{R}^{i_3\dots i_k} \in T_{12i_3\dots i_k}$. It is called the $12i_3\dots i_k$-closure of $\mathscr{R}$ or $12i_3\dots i_k$-class generated by $\mathscr{R}$ and we have the closure operator $\mathscr{R} \to \mathscr{R}^{i_3\dots i_k}$ on T_{12}. By Proposition 3 we have

Proposition 4. *Let $\mathscr{R}$ be a radical class. Then*

$$\begin{array}{ccccccccccccc} \mathscr{R}^8 & \supseteq & \mathscr{R}^6 & \supseteq & \mathscr{R}^3 & \supseteq & \mathscr{R}^{3'} & \supseteq & \mathscr{R} & \subseteq & \mathscr{R}^{4'} & \subseteq & \mathscr{R}^4 \\ &&&&&&&& \cap\!| &&&& \\ &&&&&&&& \mathscr{R}^5 &&&& \\ &&&&&&&& \cap\!| &&&& \\ &&&&&&&& \mathscr{R}^7. &&&& \end{array}$$

In [5] M. Darnel determined some closure operators. Let G be an ℓ-group. Then

$$\mathscr{R}^4(G) = \vee^{(\mathscr{C}(G))}\{C \in \mathscr{C}(G) | \text{ there exists } H \in \mathscr{R} \text{ and } L \in \mathscr{L}(H) \text{ such that } C \cong H/L\}, \tag{1}$$

$$\mathscr{R}^6(G) = \overline{\mathscr{R}(G)}_G, \tag{2}$$

$$\mathscr{R}^7(G) = \mathscr{R}(G)_G^{\perp\perp}. \tag{3}$$

By Lemma 2 we have $\mathscr{R}^5 = \mathscr{R}^*$. In the following we will determine the closure operator $\mathscr{R} \to \mathscr{R}^8$ on T_{12}.

Theorem 5. *Suppose that $\mathscr{R}$ is a K-radical class. Then*
(I) if $A \in \mathscr{C}(G)$, then $\mathscr{R}(A) = A \cap \mathscr{R}(G)$;
(II) if φ is a K-isomorphism between G and G', then $\varphi(\mathscr{R}(G)) = \mathscr{R}(G')$.

Conversely, if we associate to each ℓ-group G an ℓ-ideal $\mathscr{T}(G) \in K(G)$ subject to (I) and (II) above, and let $\mathscr{R} = \{G | \mathscr{T}(G) = G\}$, then $\mathscr{R}$ is a K-radical class, and for each ℓ-group G, $\mathscr{R}(G) = \mathscr{T}(G)$.

Proof. The assertion (I) is known (cf. e.g. [5]). If $K(G)$ is isomorphic to $K(G')$ with K-isomorphism φ, $\varphi(\mathscr{R}(G)) = \mathscr{R}(G')$ by the property b) of [4, p. 187].

Conversely, suppose that we associate to each ℓ-group G an ℓ-ideal $\mathscr{T}(G) \in K(G)$ subject to (I) and (II) above, and let $\mathscr{R} = \{G \in \mathscr{G} | \mathscr{T}(G) = G\}$. It is easy to see that $\mathscr{R}$ is a radical class. Let T be the class of all lattice L such that there exists $G \in \mathscr{R}$ and L is isomorphic to $K(G)$. Thus, (II) implies that $\mathscr{R}$ is a K-radical class. Let G be an ℓ-group. $\mathscr{T}(G) \in \mathscr{R}$ implies $\mathscr{R}(G) \supseteq \mathscr{T}(G)$. On the other hand, $\mathscr{R}(G) = \mathscr{T}(\mathscr{R}(G)) = \mathscr{R}(G) \cap \mathscr{T}(G)$, so $\mathscr{R}(G) \subseteq \mathscr{T}(G)$. Therefore $\mathscr{R}(G) = \mathscr{T}(G)$. $\square$

Any mapping $f : G \to \mathscr{R}(G)$ on $\mathscr{G}$ satisfying the above properties (I) and (II) is called a K-radical mapping. Theorem 5 indicates that a K-radical class is uniquely determined by its K-radical mapping.

Theorem 6. *Let $\mathscr{R}$ be a radical class and G be an ℓ-group. Then $G \to \mathscr{R}^8(G) = \vee^{(K(G))}\{A \in K(G) | A$ is K-isomorphic to some $A' \in \mathscr{R}\}$ is a K-radical mapping and $\mathscr{R}^8 = \{G | \mathscr{R}^8(G) = G\}$ is the K-radical class generated by $\mathscr{R}$.*

This theorem is a corollary of Theorem 2.9 in [8], hence the proof is omitted.

Corollary 7. *Let $\mathscr{R}$ be a radical class. Then the K-radical class generated by $\mathscr{R}$ is $\mathscr{R}^8 = J_K L(\mathscr{R})$.*

This corollary is also a result of Theorem 2.9 of [8].

Suppose that $\mathscr{R}_1 \neq \mathscr{R}_2 \in T_{12i_3\dots i_k}$. If the interval $[\mathscr{R}_1, \mathscr{R}_2] = \{\mathscr{R}_1, \mathscr{R}_2\}$, we say that $\mathscr{R}_2$ covers $\mathscr{R}_1$ or that $\mathscr{R}_2$ is an atom over $\mathscr{R}_1$. The set of all atoms over $\mathscr{R}_1$ will be denoted by $A_{12i_3\dots i_k}(\mathscr{R}_1)$. Let $\mathscr{R}_0 = \{\{0\}\}$ be the least element of $T_{12i_3\dots i_k}$. We put $A_{12i_3\dots i_k}(\mathscr{R}_0) = A_{12i_3\dots i_k}$. In [7] J. Jakubik proved that, if $\mathscr{G} \neq \mathscr{R} \in T_{12}$, then $A_{12}(\mathscr{R})$ is a proper class. In particular A_{12} is a proper class. In this paper we will prove that, if $\mathscr{R} \in T_{125}$ (T_{126}, T_{127} and T_{128}) and $\mathscr{R} \neq \mathscr{G}$, then $A_{125}(\mathscr{R})$ ($A_{126}(\mathscr{R})$, $A_{127}(\mathscr{R})$ and $A_{128}(\mathscr{R})$) is nonempty.

Lemma 8. *Suppose that $\mathscr{R} \in T_{12i_3\ldots i_k}$ and $\mathscr{R}_1 \in A_{12}(\mathscr{R})$. If for any $\mathscr{R}' \in T_{12i_3\ldots i_k}$ with $\mathscr{R} < \mathscr{R}' \leq \mathscr{R}_1^{i_3\ldots i_k}$, $\mathscr{R}' \cap \mathscr{R}_1 \neq \mathscr{R}$. Then $\mathscr{R}_1^{i_3\ldots i_k} \in A_{12i_3\ldots i_k}(\mathscr{R})$.*

Proof. Let $\mathscr{R}' \in T_{12i_3\ldots i_k}$ such that $\mathscr{R} < \mathscr{R}' \leq \mathscr{R}_1^{i_3\ldots i_k}$. Then $\mathscr{R} < \mathscr{R}' \cap \mathscr{R}_1 \leq \mathscr{R}_1$. Since $\mathscr{R}_1 \in A_{12}(\mathscr{R})$, $\mathscr{R}' \cap \mathscr{R}_1 = \mathscr{R}_1$. That is, $\mathscr{R}' \geq \mathscr{R}_1$. But $\mathscr{R}' \in T_{12i_3\ldots i_k}$, so $\mathscr{R}' \geq \mathscr{R}_1^{i_3\ldots i_k}$. Therefore $\mathscr{R}' = \mathscr{R}_1^{i_3\ldots i_k}$ and $\mathscr{R}_1^{i_3\ldots i_k} \in A_{12i_3\ldots i_k}(\mathscr{R})$. □

Lemma 9. (Proposition 3.3 of [7]) *Let $\mathscr{G} \neq \mathscr{R} \in T_{12}$. Then $A_{12}(\mathscr{R})$ is a proper class.*

Theorem 10. *Let $\mathscr{G} \neq \mathscr{R} \in T_{126}$. Then $A_{126}(\mathscr{R})$ is nonempty.*

Proof. . Since $\mathscr{R} \neq \mathscr{G}$, $A_{12}(\mathscr{R})$ is a proper class by Lemma 9. For any $\mathscr{R}_{12} \in A_{12}(\mathscr{R})$, let $\mathscr{R}' \in T_{126}$ such that $\mathscr{R} < \mathscr{R}' \leq \mathscr{R}_{12}^6$. By the formula (2) we have $\mathscr{R}_{12}^6 = \{G \in \mathscr{G} | G = \overline{\mathscr{R}_{12}(G)}\}$. So the element G of $\mathscr{R}'$ has the form $G = \overline{\mathscr{R}_{12}(G)}$. If $\mathscr{R}_{12}(G) \in \mathscr{R}$ for all elements G of $\mathscr{R}'$, then since $\mathscr{R} \in T_{126}$, $\mathscr{R}' = \mathscr{R}$. This contradicts to $\mathscr{R} < \mathscr{R}'$. Hence there exists $G_1 = \overline{\mathscr{R}(G_1)} \in \mathscr{R}'$ such that $\mathscr{R}_{12}(G_1) \in \mathscr{R}_{12} \backslash \mathscr{R}$. But $\mathscr{R}_{12}(G_1) \in \mathscr{C}(G_1)$, so $\mathscr{R}_{12}(G_1) \in \mathscr{R}' \cap \mathscr{R}_{12}$. This means $\mathscr{R}' \cap \mathscr{R}_{12} \neq \mathscr{R}$. The Lemma 8 implies $\mathscr{R}_{12}^6 \in A_{126}(\mathscr{R})$. □

Theorem 11. *Let $\mathscr{G} \neq \mathscr{R} \in T_{128}$. Then $A_{128}(\mathscr{R})$ is nonempty.*

Proof. $A_{12}(\mathscr{R})$ is a proper class. Let $\mathscr{R}_{12} \in A_{12}(\mathscr{R})$ and $\mathscr{R}' \in T_{128}$ such that $\mathscr{R} < \mathscr{R}' \leq \mathscr{R}_{12}^8$. By Proposition 3 $\mathscr{R}' \in T_{126}$ and $\mathscr{R}_{12}^8 \in T_{126}$. From the proof of Theorem 10 we see that $\mathscr{R}' \cap \mathscr{R}_{12} \neq \mathscr{R}$. So Lemma 8 implies $\mathscr{R}_{12}^8 \in A_{128}(\mathscr{R})$. □

Theorem 12. *Let $\mathscr{G} \neq \mathscr{R} \in T_{125}$. Then $A_{125}(\mathscr{R})$ is nonempty.*

Proof. Let $\mathscr{R}_{12} \in A_{12}(\mathscr{R})$ and $\mathscr{R}' \in T_{125}$ such that $\mathscr{R} < \mathscr{R}' \leq \mathscr{R}_{12}^5 = \mathscr{R}_{12}^*$. It follows from the definition of $\mathscr{R}_{12}^*$ that $\mathscr{R}' \cap \mathscr{R}_{12} \neq \mathscr{R}$. So by Lemma 8 we have $\mathscr{R}_{12}^5 \in A_{125}(\mathscr{R})$. □

Theorem 13. *Let $\mathscr{G} \neq \mathscr{R} \in T_{127}$. Then $A_{127}(\mathscr{R})$ is nonempty.*

The proof of this theorem is similar to that for Theorem 11.

References

[1] Anderson, M., Feil, T., *Lattice-Ordered Groups (An Introduction)*, D. Reidel Publishing Company, 1988.
[2] Bleier, R. D., Conrad, P., *a^*-closures of lattice-ordered groups*, Trans. Math. Soc. **209** (1975), 367-387.
[3] Conrad, P., *Lattice-Ordered Groups*, Tulane Lecture Notes (1970), Tulane University.
[4] Conrad, P., *K-radical classes of lattice ordered groups*, Algebra, Proc. Conf. Carbondale (1980), Lecture Notes Math., 848, 186-207.
[5] Darnel, M., *Closure operators on radical classes of lattice-ordered groups*, Czech. Math. J. **37**(112) (1987), 51-64.
[6] Glass, A. M. W., Holland, W. C., *Lattice-Ordered Groups (Advances and Techniques)*, Kluwer Academic Publisher, 1989.
[7] Jakubik, J., *Radical mappings and radical classes of lattice ordered groups*, Symposia Math. **21** (1977), 451-477, Academic Press.
[8] Jakubik, J., *On K-radical classes of lattice ordered groups*, Czech. Math. J. **33**(108) (1983), 149-163.
[9] Kenny, G. O., *Lattice-Ordered Groups*, Ph.D. dissertation, University of Kansas (1975).
[10] Martinez, J., *Torsion theory for lattice ordered groups*, Czech. Math. J. **25**(100) (1975), 284-299.
[11] Ton, Dao-Rong, *Product radical classes of ℓ-groups*, Czech. Math. J. **42**(117) (1992), 129-142.

Dao-Rong Ton
Department of Mathematics and Physics
Hohai University
Nanjing, 210024
THE PEOPLE'S REPUPLIC OF CHINA