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OSCILLATORY PROPERTIES OF THE SOLUTIONS OF DIFFERENTIAL SYSTEM OF NEUTRAL TYPE

Eva Špániková

Abstract. The purpose of this paper is to obtain oscillation criterions for the differential system of neutral type.

In this paper we consider a differential system

$$
\begin{aligned}
&[y_i(t) + a_i(t)y_i(g_i(t))]' = p_i(t)f_i(y_{i+1}(h_{i+1}(t))), \quad i = 1,2 \\
&y_3'(t) = -p_3(t)f_3(y_1(h_1(t))), \quad t \in R_+ = [0,\infty)\,.
\end{aligned} \tag{S}
$$

The following conditions are always assumed to be fulfilled:

(a) $a_i : R_+ \longrightarrow [0,\lambda_i]$, $i = 1,2$, are continuous, λ_i is a constant, $0 < \lambda_i < 1$.
(b) $g_i : R_+ \longrightarrow R$, $i = 1,2$, are continuous, $g_i(t) \le t$ and $\lim_{t\to\infty} g_i(t) = \infty$.
(c) $h_i : R_+ \longrightarrow R$, $i = 1,2,3$, are continuous and $\lim_{t\to\infty} h_i(t) = \infty$.
(d) $f_i : R \longrightarrow R$, $i = 1,2,3$, are continuous and nondecreasing, $uf_i(u) > 0$ for $u \ne 0$.
(e) $p_i : R_+ \longrightarrow (0,\infty)$, $i = 1,2,3$, are continuous and $\int^\infty p_j(t)\,dt = \infty$ for $j = 1,2$.

The purpose of this paper is to obtain oscillation criterions for the differential system of neutral type. This paper is generalization of the results obtained in the paper [2].

Let $t_0 \ge 0$. Denote

$$\tilde t_0 = \min\{\inf_{t\ge t_0} g_i(t), \inf_{t\ge t_0} h_j(t),\ i = 1,2, \quad j = 1,2,3\}\,.$$

A function $y = (y_1,y_2,y_3)$ is a solution of the system (S), if there exists a $t_0 \ge 0$ such that y is continuous on $[\tilde t_0,\infty)$, $y_1(t) + a_i(t)y_i(g_i(t))$, $i = 1,2$ and $y_3(t)$ are continuously differentiable on $[t_0,\infty)$ and y satisfies (S) on $[t_0,\infty)$.

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Denote by W the set of all solutions $y=(y_1,y_2,y_3)$ of the system (S) which exist on some ray $[T_y,\infty)\subset R_+$ and satisfy

$$\sup\left\{\sum_{i=1}^{3}|y_i(t)| : t\ge T\right\}>0 \quad \text{for any} \qquad T\ge T_y\,.$$

A solution $y\in W$ is <u>nonoscillatory</u> if there exists a $T_y\ge 0$ such that its every component is different from zero for all $t\ge T_y$. Otherwise a solution $y\in W$ is said to be <u>oscillatory</u>.

Denote

$h_i^*(t)=\min\{t,h_i(t)\},\ \ i=1,2,3;$

$\gamma_i(t)=\sup\{s\ge 0, h_i^*(s)\le t\}),\ \ t\ge 0,\ i=1,2,3;$

$\beta_j(t)=\sup\{s\ge 0, g_j(s)\le t\},\ \ t\ge 0,\ \ j=1,2;$

$\gamma(t)=\max\{\gamma_1(t),\gamma_2(t),\gamma_3(t),\beta_1(t),\beta_2(t)\};$

$$u_i(t)=y_i(t)+a_i(t)y_i(g_i(t)),\ \ i=1,2\,. \tag{1}$$

Lemma 1. ([1, Lemma 5]). *Let $y_i(t)$ and $u_i(t)$ fulfil (1).*

A) If $y_i(t)u_i'(t)>0$ for $t\ge T_1$, then there exists $T_2\ge T_1$ such that

$$(1-\lambda_i)|u_i(t)|\le |y_i(t)| \quad \text{for} \quad t\ge T_2,\ i=1,2\,. \tag{2}$$

B) If $y_i(t)u_i'(t)<0$ for $t\ge T_1$ and $\lim\limits_{t\to\infty} u_i|(t)|=k_i>0$, then there exist $T_3\le T_1$ and a constant $r_i: 0<r_i<1$ such that

$$r_i|u_i(t)|\le |y_i(t)|\le |u_i(t)| \qquad \text{for} \quad t\ge T_3,\ \ i=1,2\,. \tag{3}$$

Lemma 2. *Let $y_i(t)$ and $u_i(t)$ fulfil (1) and $y_i(t)u_i'(t)<0$, $i=1,2$ for $t\ge T_1$. If $\lim\limits_{t\to\infty}u_i(t)=0$, then $\lim\limits_{t\to\infty}y_i(t)=0$, $i=1,2$.*

Proof of Lemma 2 is easy. □

Theorem 1. *Let the following conditions be satisfied:*

(4) $xyf_i(xy)\ge Kxyf_i(x)f_i(y) \qquad (0<K=\text{const.}) \qquad i=1,2,3.$

(5) *$h_j(t)$ are nondecreasing functions, $j=2,3$.*

(6) $h_3(h_2(h_1(t)))\le t$.

(7) $\int_{\gamma(0)}^{\infty} p_2(t)f_2\left(\int_{h_3(t)}^{\infty}p_3(s)\,ds\right)\,dt=\infty.$

(8) $\int_{\gamma(\gamma(0))}^{\infty} p_3(t)f_3\left(\int_{\gamma(0)}^{h_1(t)}p_1(s)f_1\left(\int_0^{h_2(s)}p_2(x)\,dx\right)\,ds\right)\,dt=\infty.$

(9) $\int_0^{\alpha}\frac{dt}{f_3(f_1(f_2(t)))}<\infty, \quad \int_0^{-\alpha}\frac{dt}{f_3(f_1(f_2(t)))}<\infty$ *for every constant $\alpha>0$.*

Then every solution $y\in W$ is either oscillatory or $\lim\limits_{t\to\infty}y_i(t)=0$, $i=1,2,3$.

Proof. Let $y\in W$ be a nonoscillatory solution of the system (S). Then there exists $t_1\ge 0$ such that each of its components is a constant sign on $[t_1,\infty)$. Without loss of generality we may suppose that $y_1(t) > 0$ for $t \geq t_1$. In the next we shall consider the following cases:

I) Let $y_1(t) > 0$, $y_3(t) < 0$, $t \geq t_1$.
In view of (S) and (1) we get

$$u_1(t) > 0, \quad u_2'(t) < 0, \quad y_3'(t) < 0, \quad t \geq t_2 = \gamma(t_1)\,. \tag{10}$$

Because $y_3(t)$ is negative and decreasing we have

$$y_3(h_3(t)) \leq -C_1 = y_3(t_1) < 0, \; t \geq t_3 = \gamma(t_2)\,.$$

The last inequality together (d) implies

$$f_2(y_3(h_3(t))) \leq -C_2, \quad t \geq t_3, \tag{11}$$

where $-C_2 = f_2(-C_1) < 0$.
Integrating the second equation of (S) and then using (11), we have

$$u_2(t) \leq u_2(t_3) - C_2 \int_{t_3}^{t} p_2(s)\,ds, \quad t \geq t_3\,. \tag{12}$$

From (12) and (e) for $t \to \infty$ we obtain $\lim\limits_{t\to\infty} u_2(t) = -\infty$. Then with regard to Lemma 1 we have $\lim\limits_{t\to\infty} y_2(t) = -\infty$ and $y_2(t) \leq -C_3 < 0, \quad t \geq t_4 \geq t_3$,

$$f_1(y_2(h_2(t))) \leq -C_4, \quad t \geq t_5 = \gamma(t_4)\,, \tag{13}$$

where $-C_4 = f_1(-C_3) < 0$.
Integrating the first equation of (S) and then using (13) and (e), we get $\lim\limits_{t\to\infty} u_1(t) = -\infty$, which contradicts (16). The case I) cannot occur.

IIa) Let $y_1(t) > 0$, $y_2(t) < 0$, $y_3(t) > 0$, $t \geq t_1$.
In view of (S) and (1) we get

$$\begin{aligned} &u_1(t) > 0, \quad u_2(t) < 0, \\ &u_1'(t) < 0, \quad u_2'(t) > 0, \quad y_3'(t) < 0, \quad t \geq t_2 = \gamma(t_1)\,. \end{aligned} \tag{14}$$

We shall prove that $\lim\limits_{t\to\infty} u_i(t) = 0$, $i = 1, 2$ and $\lim\limits_{t\to\infty} y_3(t) = 0$.

Let $\lim\limits_{t\to\infty} u_2(t) = -k_2 < 0$. In view of Lemma 1 there exists $t_3 \geq t_2$ such that $y_2(t) \leq -C_5$, $t \geq t_3$, where $C_5 = r_2 \cdot k_2 > 0$. We have

$$f_1(y_2(h_2(t))) \leq f_1(-C_5) < 0, \quad t \geq t_4 = \gamma(t_3)\,. \tag{15}$$

Integrating the first equation of (S) and then using (15) and (e), we get $\lim_{t\to\infty} u_1(t) = -\infty$, which contradicts (14) and hence $\lim_{t\to\infty} u_2(t) = 0$. Lemma 2 implies that $\lim_{t\to\infty} y_2(t) = 0$. Analogously we can show that $\lim_{t\to\infty} y_3(t) = 0$.

Let $\lim_{t\to\infty} u_1(t) = k_1 > 0$. Lemma 1 implies that there exist $t_5 \ge t_2$ and a constant $C_6 = r_1 \cdot k_1 > 0$ such that $y_1(t) \ge C_6$ for $t \ge t_5$. Then we get

$$f_3(y_1(h_1(t))) \ge C_7, \quad t \ge t_6 = \gamma(t_5), \quad \text{where } C_7 = f_3(C_6) > 0\,. \tag{16}$$

Integrating the third equation of (S) from t to ∞ and then using (16) we have

$$y_3(t) \ge C_7 \int_t^\infty p_3(s)\,ds, \quad t \ge t_6\,.$$

Then in view of (d), (4) and the last inequality we get

$$f_2(y_2(h_3(t))) \ge K f_2(C_7) f_2\left(\int_{h_3(t)}^\infty p_3(s)\,ds\right), \quad t \ge t_7 = \gamma(t_6)\,. \tag{17}$$

Integrating the second equation of (S) and then using (17) we get

$$u_2(t) \ge u_2(t_7) + K f_2(C_7) \int_{t_7}^t p_2(z) f_2\left(\int_{h_3(z)}^\infty p_3(s)\,ds\right)\,dz\,, \quad t \ge t_7\,.$$

By virtue of (7), the last inequality implies for $t \to \infty$ that $\lim_{t\to\infty} u_2(t) = \infty$, which contradicts (14). Therefore $\lim_{t\to\infty} u_1(t) = 0$ and $\lim_{t\to\infty} y_1(t) = 0$.

IIb) Let $y_1(t) > 0$, $y_2(t) > 0$, $y_3(t) > 0$, $t \ge t_1$.
In view of (S) and (1) we have

$$\begin{aligned} &u_1(t) > 0, \quad u_2(t) > 0\,, \\ &u_1'(t) > 0, \quad u_2'(t) > 0, \quad y_3'(t) < 0, \quad t \ge t_2 = \gamma(t_1)\,. \end{aligned}$$

Integrating the second equation of (S) we get

$$u_2(t) - u_2(t_2) = \int_{t_2}^t p_2(s) f_2(y_3(h_3(s)))\,ds, \quad t \ge t_2 \quad \text{and}$$

$$u_2(h_2(t)) \ge \int_{t_2}^{h_2(t)} p_2(s) f_2(y_3(h_3(s)))\,ds, \quad t \ge t_3 = \gamma(t_2)\,. \tag{18}$$

In view of Lemma 1 there exists $t_4 \ge t_3$ such that

$$(1-\lambda_2) u_2(h_2(t)) \le y_2(h_2(t))\,, \quad t \ge t_4\,. \tag{19}$$

Using (d), (4), (5), (18), (19) and the monotonicity of $f_2(y_3(h_3(s)))$, we get

$$y_2(h_2(t)) \geq (1-\lambda_2) f_2(y_3(h_3(h_2(t)))) \int_{t_2}^{h_2(t)} p_2(s)\,ds, \quad t \geq t_4 \quad \text{and}$$

$$f_1(y_2(h_2(t))) \geq C_8 f_1(f_2(h_3(h_2(t)))) f_1\left(\int_{t_2}^{h_2(t)} p_2(s)\,ds\right), \quad t \geq t_4\,,$$

where $C_8 = K^2 f_1(1-\lambda_2) > 0$.
Integrating the first equation of (S) and then using the last inequality, we have

$$u_1(t) \geq C_8 \int_{t_4}^{t} p_1(s) f_1(f_2(y_3(h_3(h_2(s))))) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\,, \qquad t \geq t_4\,. \tag{20}$$

Using (6), (20) and the monotonicity of $f_1(f_2(y_3(t)))$ we get

$$u_1(h_1(t)) \geq C_8 f_1(f_2(y_3(t))) \int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\,, \qquad t \geq t_5 = \gamma(t_4)\,. \tag{21}$$

In view of Lemma 1 there exists $t_6 \geq t_5$ such that

$$(1-\lambda_1) u_1(h_1(t)) \leq y_1(h_1(t))\,, \quad t \geq t_6\,. \tag{22}$$

In view of (d), (4), (21) and (22) we have

$$\begin{aligned} f_3(y_1(h_1(t))) &\geq C_9 f_3(f_1(f_2(y_3(t))))\times \\ &\times f_3\left(\int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\right), \quad t \geq t_6, \end{aligned} \tag{23}$$

where $C_9 = K^2 f_3((1-\lambda_1)C_8) > 0$. Multiplying (23) by $\frac{p_3(t)}{f_3(f_1(f_2(y_3(t))))}$, using the third equation of (S) and then integrating from t_6 to t, we get

$$\begin{aligned} &\int_{t}^{t_6} \frac{y_3'(z)\,dz}{f_3(f_1(f_2(y_3(z))))} \geq \\ &\geq C_9 \int_{t_6}^{t} p_3(z) f_3\left(\int_{t_4}^{h_1(z)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right) ds\right) dz\,, \quad t \geq t_6\,. \end{aligned} \tag{24}$$

The inequality (24) for $t \to \infty$ gives a contradiction to (8) with (9). This case cannot occur. The proof of Theorem 1 is complete. $\square$

Theorem 2. *Suppose that (4), (5), (6), (7) hold and in addition*

$$f_3(f_1(f_2(t))) = t \tag{25}$$

$$\int_{\gamma(\gamma(0))}^{\infty} p_3(t)\left[f_3\left(\int_{\gamma(0)}^{h_1(t)} p_1(s) f_1\left(\int_0^{h_2(s)} p_2(x)\,dx\right)ds\right)\right]^{1-\varepsilon} dt = \infty \tag{26}$$

where $0 < \varepsilon < 1$. Then the conclusion of Theorem 1 holds.

Proof. Let $y \in W$ be a nonoscillatory solution of the system (S). As in the proof of Theorem 1, we get three cases: I), IIa) and IIb). In the cases I) and IIa) we proceed in the same way as in the proof of Theorem 1. Consider now the case IIb). In this case the inequality (23) holds. Using (25), (23) implies

$$f_3(y_1(h_1(t))) \ge C_9 y_3(t) f_3\left(\int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right)ds\right) > 0\,, \quad t \ge t_6\,. \tag{27}$$

Raising (27) to the $(1-\varepsilon)$ power $(0 < \varepsilon < 1)$ we obtain

$$[C_9 y_3(t)]^{1-\varepsilon}\left[f_3\left(\int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right)ds\right)\right]^{1-\varepsilon} \le \le [f_3(y_1(h_1(t)))]^{1-\varepsilon}\,, \quad t \ge t_6\,. \tag{28}$$

Lemma 1 together (d) implies that there exist $t_7 \ge t_6$ and a constant $D_1 > 0$ such that

$$f_3(y_1(h_1(t))) \ge D_1\,, \quad t \ge t_7\,. \tag{29}$$

Now (29) implies

$$[f_3(y_1(h_1(t)))]^{1-\varepsilon} \le D_2 f_3(y_1(h_1(t)))\,, \quad t \ge t_7\,, \tag{30}$$

where $D_2 = D_1^{-\varepsilon} > 0$.
Combining (28) with (30), we get

$$[C_9 y_3(t)]^{1-\varepsilon}\left[f_3\left(\int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right)ds\right)\right]^{1-\varepsilon} \le \le D_2 f_3(y_1(h_1(t)))\,, \quad t \ge t_7\,. \tag{31}$$

Multiplying (31) by $p_3(t)[C_9 y_3(t)]^{\varepsilon-1}$, using the third equation of (S), integrating from t_7 to t and then using the fact that $y_3(t)$ is positive and decreasing, we have

$$\int_{t_7}^{t} p_3(z)\left[f_3\left(\int_{t_4}^{h_1(t)} p_1(s) f_1\left(\int_{t_2}^{h_2(s)} p_2(x)\,dx\right)ds\right)\right]^{1-\varepsilon} dz \le \le D_2 (C_9)^{\varepsilon-1} \cdot \varepsilon^{-1} \cdot [y_3(t_7)]^{\varepsilon} < \infty\,, \quad t \ge t_7\,,$$

which contradicts (26). Therefore the case IIb) cannot occur. The proof of Theorem 2 is complete. □

Theorem 3. *Suppose that (4), (7), (9) hold and in addition*

$$h_2(t) \geq t, \quad h_3(t) \leq t \tag{32}$$

$$\int_{\gamma(\gamma(0))}^{\infty} p_3(t) f_3 \left(\int_{\gamma(0)}^{h(t)} p_1(s) f_1 \left(\int_0^s p_2(x)\, dx \right) ds \right) dt = \infty \tag{33}$$

where $h(t) = h_1^*(t) = \min\{t, h_1(t)\}$. Then the conclusion of Theorem 1 holds.

Proof. Let $y \in W$ be a nonoscillatory solution of the system (S). Further proceeding in the same way as in the proof of Theorem 2, we consider only the case IIb). Lemma 1 together (d) and (4) implies that there exists $t_3 \geq t_2$ such that

$$f_1(y_2(h_2(t))) \geq D_3 f_1(u_2(h_2(t)))\,, \quad t \geq t_3\,, \tag{34}$$

where $D_3 = K f_1(1-\lambda_2) > 0$. Using (32), (34) and the monotonicity of $f_1(u_2(t))$ on $[t_3, \infty)$ the first equation of (S) implies

$$u_1'(t) \geq D_3 p_1(t) f_1(u_2(t))\,, \quad t \geq t_3\,. \tag{35}$$

In view of (32) and the monotonicity of $f_2(y_3(t))$ on $[t_3, \infty)$, the second equation of (S) implies

$$u_2'(t) \geq p_2(t) f_2(y_3(t))\,, \quad t \geq t_3\,. \tag{36}$$

Analogously as (35) we have

$$y_3'(t) \leq -D_4 p_3(t) f_3(u_1(h(t)))\,, \quad t \geq t_3\,, \tag{37}$$

where $D_4 = K f_3(1-\lambda_1) > 0$.
In view of (35), (36), (37), we modify the system (S) to the form

$$\begin{aligned} u_1'(t) &\geq D_3 p_1(t) f_1(u_2(t)) \\ u_2'(t) &\geq p_2(t) f_2(y_3(t)) \\ y_3'(t) &\leq -D_4 p_3(t) f_3(u_1(h(t)))\,, \quad t \geq t_3\,. \end{aligned} \tag{S*}$$

System (S*) implies

$$u_1(t) \geq D_3 \int_{t_3}^{t} p_1(s) f_1(u_2(s))\, ds\,, \quad t \geq t_3 \quad \text{and} \tag{38}$$

$$u_2(s) \geq \int_{t_3}^{s} p_2(x) f_2(y_3(x))\, dx\,, \quad s \geq t_3\,. \tag{39}$$

In view of (d), (4) and the monotonicity of $f_2(y_3(x))$ on $[t_3,\infty)$, from (39) we have

$$f_1(u_2(s)) \geq K f_1(f_2(y_3(s))) f_1\left(\int_{t_3}^{s} p_2(x)\,dx\right), \quad s \geq t_3 . \tag{40}$$

Combining (38) with (40), we get

$$u_1(t) \geq K D_3 \int_{t_3}^{t} p_1(s) f_1(f_2(y_3(s))) f_1\left(\int_{t_3}^{s} p_2(x)\,dx\right) ds, \quad t \geq t_3 . \tag{41}$$

Using (d), (4), the monotonicity of $f_1(f_2(y_3(s)))$ on $[t_3,\infty)$ and (41), we have

$$\begin{gathered} f_3(u_1(h(t))) \geq \\ \geq D_5 f_3(f_1(f_2(y_3(t)))) f_3\left(\int_{t_3}^{h(t)} p_1(s) f_1\left(\int_{t_3}^{s} p_2(x)\,dx\right) ds\right), \\ t \geq t_4 , \end{gathered} \tag{42}$$

where $D_5 = K^2 f_3(K D_3) > 0$.
Multiplying (42) by $D_4 p_3(t)[f_3(f_1(f_2(y_3(t))))]^{-1}$, integrating from t_4 to t, using the third inequality of (S*) and (9), for $t \to \infty$ we get

$$\begin{aligned} D_4 D_5 \int_{t_4}^{t} p_3(z) f_3\left(\int_{t_3}^{h(z)} p_1(s) f_1\left(\int_{t_3}^{s} p_2(x)\,dx\right) ds\right) dz \leq \\ \leq \int_{y_3(t)}^{y_3(t_4)} \frac{dz}{f_3(f_1(f_2(z)))} < \infty , \end{aligned}$$

which contradicts (33). Therefore the case IIb) cannot occur. The proof of Theorem 3 is complete. □

Theorem 4. *Suppose that (4), (7), (25), (32) hold and in addition*

$$\int_{\gamma(\gamma(0))}^{\infty} p_3(t)\left[f_3\left(\int_{\gamma(0)}^{h(t)} p_1(s) f_1\left(\int_{0}^{s} p_2(x)\,dx\right) ds\right)\right]^{1-\varepsilon} dt = \infty , \quad 0 < \varepsilon < 1 , \tag{43}$$

where $h(t) = h_1^*(t)$. Then the conclusion of Theorem 1 holds.

We can prove Theorem 4 analogously as Theorem 2 and Theorem 3.

Remark. Theorem 1 – Theorem 4 we can easily extend for the following system:

$$\begin{gathered} [y_i(t) + a_i(t) y_i(g_i(t))]' = (-1)^{\nu_i} p_i(t) f_i(y_{i+1}(h_{i+1}(t))), \quad i = 1,2 \\ y_3'(t) = (-1)^{\nu_3} p_3(t) f_3(y_1(h_1(t))), \quad t \in R_+ \\ \nu_j \in \{0,1\} \quad j = 1,2,3 \quad \text{and} \quad \nu_1 + \nu_2 + \nu_3 \equiv 1 \pmod 2 . \end{gathered}$$

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Eva Špániková
Department of Mathematics VŠDS
J. M. Hurbana 15
010 26 Žilina, SLOVAK REPUBLIC