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A REMARK ON SECOND ORDER FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. It is proved that under some conditions the set of solutions to initial value problem for second order functional differential system on an unbounded interval is a compact R_δ-set and hence nonvoid, compact and connected set in a Fréchet space. The proof is based on a Kubáček's theorem.

In the paper the initial value problem for second order functional differential system on an unbounded interval is studied. By means of the Kubáček's theorem in [2] which guarantees that the set of all fixed points of a compact map in a Fréchet space is a compact R_δ-set it is shown that under some restrictions on the growth of the right-hand side of that system the set of the solutions to that initial value problem is a compact R_δ-set in a properly chosen Fréchet space of C^1-functions. The result extend a similar theorem for first order functional differential systems which is proved in [2].

1. Introduction

Throughout the whole paper we shall use the following denotations and assumptions:

Let $h > 0$, $b \in R$, $\nu \in N$ and let $|\cdot|$ be a norm in R^ν. Further let $H = C^1([-h,0], R^\nu)$ be provided with the norm $\|x\| = \max\{|x(s)| + |x'(s)| : -h \leqq s \leqq 0\}$ for each $x \in H$ and let $H_0 = C([-h,0], R^\nu)$.

Let $X = C^1([b,\infty), R^\nu$ be equipped with the topology of locally uniform convergence of the functions and of their derivatives on $[b,\infty)$. The topology on the Fréchet space X is given by the metrics

$$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)},$$

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where $p_m(x) = \sup\{|x(t)| + |x'(t)| : b \leqq t \leqq b+m\}$, $x, y \in X$, $m \geqq 1$.

Let $X^* = C^1([b-h,\infty), R^\nu)$ be the Fréchet space provided with the seminorms $p_m^*(x) = \sup\{|x(t)| + x'(t)| : b-h \leqq t \leqq b+m\}$, $x \in X^*$, $m \geqq 1$.

For $x \in C([b-h,\infty), R^\nu)$ we denote by $x_t \in H_0$ the function $x_t(s) = x(t+s)$, $s \in [-h, 0]$, $t \geqq b$. Clearly $(x_t)'(s) = (x')_t(s)$, $s \in [-h,0]$ and $x \in X^*$, $t \geqq b$.

Let $f \in C([b,\infty) \times H \times H_0, R^\nu)$, $\psi \in H$. We shall consider the following initial value problem

(1) $x''(t) = f(t, x_t, x_t')$, $b \leqq t < \infty$

(2) $x_b = \psi$, $x_b' = \psi'$.

A solution x of (1), (2) is a function $x \in X^* \cap C^2([b,\infty), R^\nu)$ which satisfies (2) and the functional differential system (1) at each point $t \geqq b$.

Now we shall state the Kubáček theorem in [2] as Lemma 1. In that lemma a compact R_δ-set in a metric space (E, ϱ) means a nonempty subset F of E which is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. By [1], p. 92, a metric space G is called an absolute retract when for each metric space H and each closed $K \subset H$, each continuous map $f : K \to G$ has a continuous extension $g : H \to G$. E.g. a nonempty convex subset of a Fréchet space is an absolute retract.

Lemma 1. *Let M be a nonempty closed set in a Fréchet space (E, ϱ), $T : M \to E$ a compact map (i.e. T is continuous and $T(M)$ is a relatively compact set). Denote by S the map $I - T$, where I is the identity map on E. Let there exist a sequence $\{U_n\}$ of closed convex sets in E fulfilling*

(i) *$0 \in U_n$ for each $n \in N$;*

(ii) $\lim_{n\to\infty} \operatorname{diam} U_n = 0$

and a sequence $\{T_n\}$ of maps $T_n : M \to E$ fulfilling

(iii) *$T(x) - T_n(x) \in U_n$ for each $x \in M$ and each $n \in N$;*

(iv) *the map $S_n = I - T_n$ is a homeomorphism of the set $S_n^{-1}(U_n)$ onto U_n.*

Then the set F of all fixed points of the map T is a compact R_δ-set.

In a special case, when $E = X$, $\varrho = d$, Lemma 1 implies the following lemma.

Lemma 2. *Let (X, d) be the Fréchet space given above, let φ, $\varphi_n \in C([b,\infty), (0,\infty))$, $n \in N$ and let the following condition be satisfied:*

(v) *For each $t \in [b,\infty)$ the sequence $\{\varphi_n(t)\}$ is nonincreasing and $\lim_{n\to\infty} \varphi_n(t) = 0$.*

Let r, $s \in R^\nu$ and let

$$M = \{x \in X : |x(t) - r| + |x'(t) - s| \leqq \varphi(t), \quad t \geqq b, \quad x(b) = r, \quad x'(b) = s\}.$$

Suppose that $T : M \to X$ is a compact map with the property $T(x)(b) = r$, $(T(x))'(b) = s$ for each $x \in M$ and there exists a sequence $\{T_n\}$ of compact maps $T_n : M \to X$ such that $T_n(x)(b) = r$, $(T_n(x))'(b) = s$ for each $x \in M$ and

(vi) $|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - T(x)'(t)| \leqq \varphi_n(t), \quad x \in M,\ t \geqq b;$

(vii) *for every* $n \in N$ *there exists a function* $\varphi_{*n} \in C([b,\infty),(0,\infty))$ *such that*

$$\varphi_{*n} + \varphi_n \leqq \varphi \text{ on } [b,\infty)$$

and

$$|T_n(x)(t) - r| + |(T_n(x))'(t) - s| \leqq \varphi_{*n}(t), \quad x \in M,\ t \geqq b;$$

(viii) *the map* $S_n = I - T_n$ *is injective on* M *where* I *is the identity on* X.

Then the set F *of all fixed points of the map* T *is a compact* R_δ.

Proof. The set

$$U_n = \{x \in X : |x(t)| + |x'(t)| \leqq \varphi_n(t), \quad t \geqq b, x(b) = 0, x'(b) = 0\}$$

is convex and closed in X for each $n \in N$. We shall show that the sequence $\{U_n\}$ satisfies all conditions of Kubáček's theorem when $E = X$, $\varrho = d$. Then the statement of Lemma 2 follows from Lemma 1.

Clearly the condition (i) is fulfilled. Further for a given $\varepsilon > 0$ there exists an $m_0 \in N$ such that $\sum\limits_{m=m_0+1}^{\infty} (1/2^m) < \varepsilon/2$. (v) and the Dini theorem imply that the sequence $\{\varphi_n\}$ converges on $[b,\infty)$ locally uniformly to 0, therefore for $\varepsilon > 0$ and $m_0 \in N$ there exists an $n_0 \in N$ such that $q_m(\varphi_n) = \sup\{|\varphi_n(t)| : b \leqq t \leqq b+m\} \leqq \varepsilon/4m_0$ for $n \geqq n_0$ and $m = 1, 2, \ldots, m_0$. Thus for $n \geqq n_0$ and $x, y \in U_n$ we have

$$\begin{aligned} d(x,y) &= \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1 + p_m(x-y)} \leqq \sum_{m=1}^{m_0} p_m(x-y) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \\ &= \sum_{m=1}^{m_0} 2\, q_m(\varphi_n) + \sum_{m=m_0+1}^{\infty} \frac{1}{2^m} \leqq 2m_0\, \frac{\varepsilon}{4m_0} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

This implies that the condition (ii) is satisfied. The assumption (iii) follows from (vi) and from the definition of T, T_n, $n \in N$. To show that (iv) is fulfilled it suffices to prove the inclusion $U_n \subset S_n(M)$. (viii) then implies that S_n is a bijection of $S_n^{-1}(U_n)$ onto U_n and the continuity of $S_n^{-1}\big|_{U_n}$ is then a consequence of the compactness of T_n.

Thus we have to prove that for each $y \in U_n$ there is an $x_y \in M$ such that $x_y - T_n(x_y) = y$. This means that for every $y \in U_n$ the map $P_n(x) = y + T_n(x)$ has a fixed point. (vii) implies that

$$\begin{gathered} |y(t) + T_n(x)(t) - r| + |y'(t) + (T_n(x))'(t) - s| \leqq \\ |y(t)| + |y'(t)| + |T_n(x)(t) - r| + |(T_n(x))'(t) - s| \leqq \\ \varphi_n(t) + \varphi_{*n}(t) \leqq \varphi(t), \quad t \geqq b\,, \end{gathered}$$

and $P_n(x)(b) = r$, $(P_n(x))'(b) = s$ for each $x \in M$. Therefore $P_n(M) \subset M$. As M is a closed, convex bounded set and P_n is a compact map, by the Tichonov fixed point theorem, P_n has a fixed point. This completes the proof of the lemma. □

In what follows the function φ in Lemma 2 will be given as a solution of an integral equation. The existence of a solution to that equation will be discussed in the following two lemmas.

Lemma 3. *Let $\psi \in H$, $\omega \in C([b,\infty),[0,\infty))$, $g \in C([0,\infty),[0,\infty))$ be a nondecreasing function. Further denote*

$$U(x)(t) = |\psi'(0)|(t-b) + \int_b^t (t-s+1)\omega(s)g(\|\psi\| + x(s))\,ds,\ b \leqq t < \infty\,, \tag{3}$$

for each $x \in C([b,\infty),[0,\infty))$. Then the following statements are true:

1. A solution $\varphi \in C([b,\infty),[0,\infty))$ of the integral equation

$$\varphi(t) = U(\varphi)(t),\ b \leqq t < \infty\,, \tag{4}$$

exists iff there exists a function $\lambda \in C([b,\infty),[0,\infty))$ such that

$$\lambda(t) \geqq U(\lambda)(t),\ b \leqq t < \infty\,. \tag{5}$$

2. The solution φ of (4) (whenever it exists) is nondecreasing in $[b,\infty)$.

Proof. 1. Necessary condition is clear. Sufficient condition. We shall proceed by the method of steps. Hence we prove by mathematical induction that for each $m = 1, 2, \dots,$:

a) There exists a solution $y_m \in C([b,b+m],[0,\infty))$ of (4) on $[b,b+m]$ satisfying the inequalities $0 \leqq y_m(t) \leqq \lambda(t)$, $b \leqq t \leqq b+m$ and

b) $y_{m+1}(t) = y_m(t), \quad b \leqq t \leqq b+m.$

Consider the partially ordered Banach space $X_1 = C([b,b+1),R)$ with the sup-norm where $z_1 \leqq z_2$ if and only if $z_1(t) \leqq z_2(t)$ for each $t \in [b,b+1]$ and each pair z_1, z_2 from that space. Then, by definition, the interval $\langle z_1, z_2\rangle = \{y \in X_1 : z_1(t) \leqq y(t) \leqq z_2(t), \quad b \leqq t \leqq b+1\}$. The operator $U : X_1 \to X_1$ defined by (3) on $[b,b+1]$ is completely continuous, nondecreasing and in view of (5), it maps the interval $\langle 0, \lambda|_{[b,b+1]}\rangle$ into itself. Hence, by the Schauder fixed point theorem, there exists a fixed point y_1 of U which satisfies (4) in $[b,b+1]$.

Suppose, now, that there exists a solution y_m of (4) on $[b,b+m]$. Then we consider the space $X_{m+1} = C([b,b+m+1],R)$ with the sup-norm and the natural partial ordering. X_{m+1} is a partially ordered Banach space. The operator U given by (3) on [b, b+m+1] maps X_{m+1} into itself, it is completely continuous, nondecreasing and similarly as before, it maps the interval $\langle 0, \lambda|_{[b,b+m+1]}\rangle = \{y \in X_{m+1} : 0 \leqq y(t) \leqq \lambda(t), \quad b \leqq t \leqq b+m+1\}$ into itself. Moreover, U maps the closed and convex set $Y_{m+1} = \{x \in X_{m+1} : x(t) = y_m(t), \quad b \leqq t \leqq b+m\} \cap \langle 0, \lambda|_{[b,b+m+1]}\rangle$ into itself. Hence there exists a fixed point y_{m+1} of U in Y_{m+1} and this is the searched function y_{m+1} with the properties (a), and (b). Then the function $\varphi(t) = y_m(t)$ for $b \leqq t \leqq b+m$ and $m = 1, 2, \dots,$ is a solution of (4) in $[b,\infty)$ and it satisfies the inequalities

$$0 \leqq \varphi(t) \leqq \lambda(t), \quad b \leqq t < \infty\,.$$

2. The statement follows from (3), (4). □

Lemma 4. *Let $\psi \in H$ and let $g \in C([0,\infty),(0,\infty))$ be a nondecreasing function. Then to each function $\varrho \in C([b,\infty),[0,\infty))$ there exists a unique pair of functions $\varphi \in C([b,\infty),[0,\infty))$, $\omega \in C([b,\infty)),[0,\infty))$ such that*

$$\omega(t) = \frac{\varrho(t)}{g(\|\psi\| + \varphi(t))}, \quad t \geqq b \tag{6}$$

and φ is a solution of the equation (4).

Proof. Define

$$\varphi(t) = |\psi'(0)|(t-b) + \int_b^t (t-s+1)\,\varrho(s)\,ds, \quad b \leqq t < \infty\,. \tag{7}$$

If φ is a solution of (4) and ω satisfies (6), then necessarily φ has the form (7). Hence there exists at most one pair of functions φ, ω satisfying (4), and (6) simultaneously. On the other hand, if φ is determined by (7) and ω satisfies (6), then φ is a solution of (4). The statement of the lemma is proved. □

Example 1. If $\varrho(t) = e^t$, $\quad b \leqq t < \infty$, then by the last lemma the function

$$\varphi(t) = (|\psi'(0)| - e^b)(t-b) + 2(e^t - e^b), \quad b \leqq t < \infty$$

is increasing in $[b,\infty)$ and if $g(u) = ku + q$, $u \geqq 0$, where $k \geqq 0$, $q > 0$, then φ is a solution of (4) with

$$\omega(t) = \frac{e^t}{k(\|\psi\| + \varphi(t)) + q}, \quad t \geqq b\,.$$

2. Main Theorem

Now we shall state and prove the main theorem.

Theorem 1. *Let $\psi \in H$, $f \in C([b,\infty) \times H \times H_0, R^\nu)$. Let, further, $\omega \in C([b,\infty),[0,\infty))$, $g \in C([0,\infty),[0,\infty))$ be a nondecreasing function and let*

(ix) $\quad |f(t,\mathcal{X}_t,\mathcal{X}'_t)| \leqq \omega(t)\,g(\|\mathcal{X}_t\|)$ *for each* $\quad (t,\mathcal{X}) \in [b,\infty) \times M^*$

where

$$M^* = \{x \in X^* : |x(t) - \psi(0)| + |x'(t) - \psi'(0)| \leqq \varphi(t) \quad \text{for} \quad t \geqq b \quad \text{and } x_b = \psi, \quad x'_b = \psi'\}\,,$$

φ is a solution of the equation (4) in $[b,\infty)$.

Then the problem (1), (2) has a solution x satisfying the inequality

$$|x(t) - \psi(0)| + |x'(t) - \psi'(0) \leqq \varphi(t), \quad t \geqq b$$

and the set F^* of all such solutions is a compact R_δ-set in the space X^*.

Proof. Consider the set

$$M = \{x \in X : |x(t) - \psi(0)| + |x'(t) - \psi'(0)| \leqq \varphi(t) \text{ for } t \geqq b \text{ and } x(b) = \psi(0), x'(b) = \psi'(0)\}.$$

Evidently the map $P : X^* \to X$ determined by $P(x) = x\big|_{[b,\infty)}$ is a homeomorphism of M^* onto M. Let the map $T : M \to X$ be defined by

$$T(x)(t) = \psi(0) + \psi'(0)(t-b) + \int_b^t (t-s) f[s, (P^{-1}x)_s, (P^{-1}x)'_s]\, ds\,, \quad x \in M,\ t \geqq b\,. \tag{8}$$

Then $F^* = P^{-1}(F)$ where F is the set of all fixed points of the map T. As a homeomorphic image of a compact R_δ-set is again a compact R_δ-set, it suffices to prove that F is a compact R_δ-set in the space X. This will be done by using Lemma 2 where we put $r = \psi(0)$, $s = \psi'(0)$. The maps $T_n : M \to X$ defined by

$$T_n(x)(t) = \begin{cases} \psi(0) + \psi'(0)(t-b) & \text{if} \quad b \leqq t \leqq b + \frac{1}{n} \\ \psi(0) + \psi'(0)(t-b) + \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s) & \\ f[s, (P^{-1}x)_s, (P^{-1}x)'_s]\, ds & \text{if} \quad t \geqq b + \dfrac{1}{n}, \quad n \in N \end{cases} \tag{9}$$

as well as T are compact due to (ix) and, again by that assumption,

$$|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - (T(x))'(t)| \leqq$$
$$= \begin{cases} \int_b^t (t-s+1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds, & b \leqq t \leqq b + \frac{1}{n} \\ \int_b^t (t-s+1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds - \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\omega(s) g(\|(P^{-1}x)_s\|)\, ds, & b + \dfrac{1}{n} \leqq t < \infty\,. \end{cases}$$

As φ is nondecreasing in $[b, \infty)$, $\|(P^{-1}x)_s\| \leqq \|\psi\| + \varphi(s)$ and thus,

$$|T_n(x)(t) - T(x)(t)| + |(T_n(x))'(t) - (T(x))'(t)| \leqq \varphi_n(t), \quad x \in M, \quad t \geqq b$$

where

$$\varphi_n(t) = \begin{cases} \int_b^t (t-s+1)\omega(s) g(\|\psi\| + \varphi(s))\, ds, & b \leqq t \leqq b + \frac{1}{n} \\ \int_b^t (t-s+1)\omega(s) g(\|\psi\| + \varphi(s))\, ds - \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\omega(s)\,. & \\ g(\|\psi\| + \varphi(s))\, ds, & b + \dfrac{1}{n} \leqq t < \infty, \quad n \in N\,. \end{cases}$$

Clearly that $\varphi_n(t) \in C([b,\infty),[0,\infty))$, and $\lim\limits_{n\to\infty} \varphi_n(t) = 0$, $\varphi_{n+1}(t) \leqq \varphi_n(t)$ for each $t \geqq b$ can be proved. Hence these functions satisfy the assumptions (v), (vi) of Lemma 2.

Further

$$|T_n(x)(t) - \psi(0)| + |T_n(x)'(t) - \psi'(0)| \leqq \varphi_{*n}(t)$$
$$t \geqq b, \quad x \in M$$

where

$$\varphi_{*n}(t) = \begin{cases} |\psi'(0)|(t-b), & b \leqq t \leqq b + \frac{1}{n} \\ |\psi'(0)|(t-b) + \int_b^{t-\frac{1}{n}} (t - \frac{1}{n} - s + 1)\,\omega(s) g(\|\psi\| + \varphi(s))\,ds \end{cases}$$
$$b + \frac{1}{n} \leqq t < \infty, \quad n \in N\,,$$

are nonnegative continuous functions on $[b,\infty)$ and with respect to the meaning of φ we have that $\varphi_n(t) + \varphi_{*n}(t) = \varphi(t)$, $t \geqq b$, $n \in N$. Thus (vii) in Lemma 2 is satisfied, too. So it remains to show that the assumption (viii) in that lemma holds.

If $x, y \in M$, $x \neq y$, then there is a $t_0 \in [b,\infty)$ such that $x(t_0) \neq y(t_0)$. If $b \leqq t_0 \leqq b + \frac{1}{n}$, then $x(t_0) - T_n(x)(t_0) = x(t_0) - \psi(0) - \psi'(0)(t_0 - b) \neq y(t_0) - \psi(0) - \psi'(0)(t_0 - b) = y(t_0) - T_n(y)(t_0)$. In the other case there exists a $t_1 \geqq b + \frac{1}{n}$ such that $t_1 = \sup\{\tau > b : x(t) = y(t) \text{ for } b \leqq t < \tau\}$. Then there exists a $t_0 \in (t_1, t_1 + \frac{1}{n})$ such that $x(t_0) \neq y(t_0)$. This gives that

$$\begin{aligned} T_n(x)(t_0) &= \psi(0) + \psi'(0)(t_0 - b) + \int_b^{t_0 - \frac{1}{n}} (t_0 - \frac{1}{n} - s) f[s, (P^{-1}x)_s\,, (P^{-1}x)'_s\,ds \\ &= \psi(0) + \psi'(0)(t_0 - b) + \int_b^{t_0 - \frac{1}{n}} (t_0 - \frac{1}{n} - s) f[s, (P^{-1}y)_s\,, (P^{-1}y)'_s]\,ds \\ &= T_n(y)(t_0) \end{aligned}$$

and hence, $x(t_0) - T_n(x)(t_0) \neq y(t_0) - T_n(y)(t_0)$, $n \in N$. Thus all assumptions of Lemma 2 are fulfilled and the statement of Theorem 1 follows from that lemma.□

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