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3–CONFIGURATIONS WITH SIMPLE EDGE BASIS AND THEIR CORRESPONDING QUASIGROUP IDENTITIES

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Abstract. There is described a procedure which determines the quasigroup identity corresponding to a given 3-coloured 3-configuration with a simple edge basis.

Part 1 3-configurations

1.1. Main notations

Under a *(3)- configuration* we shall understand a finite incidence structure $(\mathcal{V}, \mathcal{E}, \mathrm{I})$ such that $\#\{y \in \mathcal{E} \,|\, x \,\mathrm{I}\, y\} \leqq 3$ for all $x \in \mathcal{V}$. If, moreover, $\#\{y \in \mathcal{E} \,|\, x \,\mathrm{I}\, y\} = 3$ for all $x \in \mathcal{V}$ and $\#\{x \in \mathcal{V} | x \,\mathrm{I}\, y\} \geqq 2$ for all $y \in \mathcal{E}$, then we get a *3-configuration.* If one (3)-configuration is an incidence substructure of another (3)-configuration we speak of a *subconfiguration.* Elements of $\mathcal{V}$ will be called *vertices* and elements of $\mathcal{E}$ *edges* of a given (3)-configuration $(\mathcal{V}, \mathcal{E}, \mathrm{I})$. After all, such (3)-configuration are hypergraphs with vertices of degree ≤ 3 and with the property that two distinct vertices cannot lie simultaneously on two distinct edges.

If $x \,\mathrm{I}\, y$ then x, y are said to be *adjacent.* Two distinct vertices adjacent to the same edge; respectively two or three mutually distinct edges adjacent to the same vertex, are said to be *neighbouring.* A (3)-configuration is called *connected* if for any distinct vertices a, b there is a finite sequence of edges $e_1, e_2, \dots, e_k$ such that $a \,\mathrm{I}\, e_1$, $b \,\mathrm{I}\, e_k$ and that e_i, e_{i+1} are neighbouring for all $i \in \{1, \dots, k-1\}$. Troughout the paper every 3-configuration will be supposed to be connected.

Let there be given a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$. We shall define a halfoperation on $\mathcal{E}$: for every couple (x, y) of neighbouring edges x, y their product $x \cdot y$ is equal to the remaining edge which forms together with x and y a neighbouring triple. We obtain the *edge halfgroupoid* $(\mathcal{E}, \cdot)$ *associated* to $\mathfrak{C}$.

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1.2. Bases of 3-configurations

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-configuration. If $\mathcal{X}$ is a non-void set of edges then denote by $[\mathcal{X}]$ the set of such vertices which are adjacent to least two edges from $\mathcal{X}$.

Now start with a non-void set $\mathcal{A}$ of edges, construct a sequence $\mathcal{A}_0 := \mathcal{A}$, $\mathcal{A}_1 := [\mathcal{A}_0]$, $\mathcal{A}_2 := [\mathcal{A}_1]$, $\mathcal{A}_3 := [\mathcal{A}_2]$, $\ldots$ and form the unions $\mathcal{E}_{\mathcal{A}} := \bigcup_{i=0}^{\infty} \mathcal{A}_{2i}$, $\mathcal{V}_{\mathcal{A}} := \bigcup_{i=0}^{\infty} \mathcal{A}_{2i+1}$. Then $(\mathcal{V}_{\mathcal{A}}, \mathcal{E}_{\mathcal{A}}, \mathrm{I})$ is a subconfiguration in $\mathfrak{C}$ *generated* by $\mathcal{A}$; we denote it by $\langle\mathcal{A}\rangle$. We shall call the sequence $(\mathcal{A}_i)_{i=0}^{\infty}$ the *pillar* over $\mathcal{A}$. Further let the *stage* of elements of $\mathcal{A}_0$, respectively $\mathcal{A}_1$ be defined as 0, respectively 1. If the stages of elements of $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_i$ are already known then define the stage of elements of$\mathcal{A}_{i+2} \backslash \mathcal{A}_i$ to be equal to $i+2$. The edge, respectively vertex a of $\langle\mathcal{A}\rangle$ is said to be *terminal* if there is no edge, respectively no vertex of greater stage neighbouring to a. Under a *constituent* we shall understand a term expression (by means of $\cdot$) of edges over $\mathcal{A}$ within the framework of the pillar. The set $\mathcal{A}$ will be called *independent* if $\langle\mathcal{A}\backslash\{a\}\rangle \neq \mathcal{A}$ for all $a \in \mathcal{A}$. The set $\mathcal{A}$ is called an *edge basis* of $\mathfrak{C}$ if it is independent and if $\langle\mathcal{A}\rangle = \mathfrak{C}$. The set of all element bases of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ cannot be void: we can namely delete, step by step, all dependent elements from the full edge set $\mathcal{E}$. If $\mathcal{B}$ is an edge basis and $\mathcal{T}$ the set of all terminal elements (vertices, as well as edges, if they exist) then $\#\mathcal{V} + \#\mathcal{B} = \#\mathcal{E} + \#\mathcal{T}$ (a well-known condition, cf. [1], pp. 50-51). Different edge bases of $\mathfrak{C}$ can we have different number of edges. Bases which lead to just one terminal vertex or edge will be denoted as *simple*. Thus for simple bases we have

$$\#\mathcal{T} = \#\mathcal{V} + \#\mathcal{B} - \#\mathcal{E} = 1, \quad \text{i.e.} \quad \#\mathcal{B} = \#\mathcal{E} - \#\mathcal{V} + 1\,.$$

If $\mathcal{B}$ is a simple basis of $\mathfrak{C}$ then in the corresponding sequence $(\mathcal{B}_i)_{i=0}^{\infty}$ the terminal vertex t lies on two edges e, e' with maximal stage (equal to stage of t minus one) and still on the third edge e''. This gives the equation $e \cdot e' = e''$ and consequently the equation between corresponding terms over $\mathcal{B}$ in the halfgroupoid $(\mathcal{B}, \cdot)$. Analogously, if there is the terminal edge t then it has maximal stage and determines an equality between two corresponding terms over $\mathcal{B}$ in $(\mathcal{E}, \cdot)$.

We give two examples: First let us investigate the 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ on Fig. 1 with $\#\mathcal{V} = 10$, $\#\mathcal{E} = 12$, $\mathcal{B} = \{x, y, z\}$.

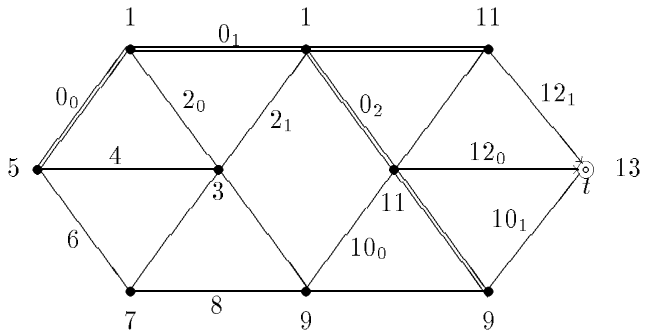
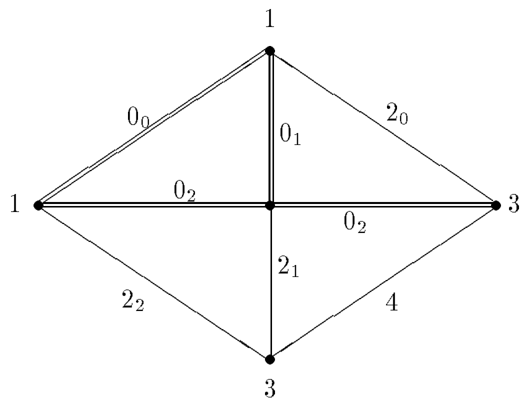


Fig.1

Here: $0_0 \dots x$; $0_1 \dots y$; $0_2 \dots z$; $2_0 \dots xy$; $2_1 \dots yz$; $4 \dots (xy)(yz)$; $6 \dots x((xy)(yz))$; $8 \dots (x((xy)(yz)))(yz)$; $10_0 \dots ((x((xy)(yz)))(yz))(xy)$; $10_1 \dots ((x((xy)(yz))(yz))z$; $12_0 \dots (((x((xy)(yz)))(yz))(xy))$; $12_1 \dots (((x((xy)(yz)))(yz))(xy))y$.

This basis is simple and determines just one terminal vertex t of stage 13, which results as the vertex adjacent to two neighbouring edges of stages 12. The remaining edge through t is one of edges of stage 10. The corresponding term equality is thus $((((x((xy)(yz))(yz))(xy))y) \cdot (((x((xy)(yz))(yz))xy))z) = ((x((xy)(yz))(yz))z$.

The second example deals with a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ on Fig. 2 with $\#\mathcal{V} = 5$, $\#\mathcal{E} = 7$, $\mathcal{B} = \{x, y, z\}$. There is just one terminal element t. It has stage 4 and determines the equality among terms $(xy)(yz)$, $(xy)z$.



$0_0 \dots x$, $0_1 \dots y$, $0_2 \dots z$, $2_0 \dots xy$, $2_1 \dots yz$, $2_2 \dots xz$, $4 \dots (xy)z \dots (xy)(yz)$

Fig. 2

Now let $\mathcal{B}$ be a simple edge basis of a given 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ and $t_1 = t_2$ the term equality over $\mathcal{B}$ in $(\mathcal{E}, \cdot)$ corresponding to the terminal element (vertex or edge).

We shall find some properties of the equality $t_1 = t_2$:

(i) The set $\mathcal{B}$ must contain at least three edges:
In fact, by the existence of just one edge in $\mathcal{B}$ or just two edges in $\mathcal{B}$, respectively, the pillar over $\mathcal{B}$ should have no vertices of stage 1 respectively 3 contrary to further size of the pillar.

(ii) Every $x \in \mathcal{B}$ occurs at least twice in $t_1 = t_2$:
By exactly one occurence (in the subterm $u \cdot x$ of the pillar or in $t_1 = x$, respectively $t_2 = x$) we should have on x only one vertex or we should express $x \in \mathcal{B}$ by the edges from $\mathcal{B}$ (which contradicts the independence of edges of the basis).

(iii) No constituent in $(\mathcal{B}_i)_{i=0}^{\infty}$ can be of the form $u(uv)$ with u and v as constituents in $(\mathcal{B}_i)_{i=0}^{\infty}$:
This follows at once from the definition of the pillar over $\mathcal{B}$.

(iv) Equations $t_1 = uv_1$, $t_2 = uv_2$, respectively $t_1 = u$, $t_2 = uv$, respectively $t_1 = uv$, $t_2 = u$ with constituents in $(\mathcal{B}_i)_{i=0}^{\infty}$ are not possible:
The equality $uv_1 = uv_2$ or $u = uv$, respectively, should violate the fundamental properties of the 3-configuration ($uv_1 = uv_2 \Rightarrow v_1 = v_2$, but on the other side, v_1 and v_2 should be distinct edges as constituents in the pillar; moreover, $u = uv$ contradicts the fact that uv must be the third edge neighbour to u and v).

(v) Let t_1 and t_2 be not formed by exactly one edge of $\mathcal{B}$ and let $x \in \mathcal{B}$ enters in $v = ux$, where u, v are constituents of the pillar. Then x enters still in some term $v' = u'x \neq v$ where also u', v' are constituents.

We can define a closure condition in a 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ derived from a 3-configuration $\mathfrak{C}_* = (\mathcal{V}_*, \mathcal{E}_*, \mathrm{I}_*)$ having a simple basis $\mathcal{B}$ with terminal element t_2 and with the *main subconfiguration* $\hat{\mathfrak{C}}_* = (\mathcal{V}_*, \mathcal{E}_* \setminus \{t\}, \mathrm{I}_*)$. The corresponding *closure condition* claims:
For every incidence structure homomorphism $\sigma : \hat{\mathfrak{C}}_* \to \mathfrak{C}$ there exists its prolongation onto a homomorphism $\bar{\sigma} : \mathfrak{C}_* \to \mathfrak{C}$.

In the second part of our considerations we shall adapt this definition for 3-web configurations and 3-webs. As we are familiar with the subject, only less is known about closure conditions in general 3-configurations (or, more largely, in webs of arbitrary number of parallel line pencils).

Part 2 Closure conditions in 3-webs

2.1. Coloured configurations

We say that a (3)-configuration is 3-coloured if for every vertex x an injective map of the set of all edges through x into the colour set $\{1, 2, 3\}$ is given such that for all neigbouring vertices x, y the colour assigned to the common adjacent edge is the same. The edges with colour 1, 2, or 3, respectively will be called *vertical*, *horizontal* or *skew*, respectively.

A 3-coloured (3)-configuration $\mathfrak{C}_1$ is a *subconfiguration* of a coloured (3)-configuration $\mathfrak{C}_2$ if it is an incidence substructure preserving the colour of every edge from $\mathfrak{C}_1$.

Now we shall define a 3-web independently on 3-configurations though it is possible to take a (finite) 3-web as a special 3-configuration.
A *3-web* is defined as an incidence structure $(\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\#\mathcal{P} \leqq 2$, together with a decomposition of the set $\mathcal{L}$ onto mutually disjoint sets $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ such that for every $x \in \mathcal{P}$ and every $i \in \{1, 2, 3\}$ there is just one $y \in \mathcal{L}_i$ such that $x \mathrel{\mathrm{I}} y$ and for every $x \in \mathcal{L}_i$ and every $y \in \mathcal{L}_j$ for distinct $i, j \in \{1, 2, 3\}$ there is just one $z \in \mathcal{P}$ such that $z \mathrel{\mathrm{I}} x$ and $z \mathrel{\mathrm{I}} y$. Elements of $\mathcal{P}$ are *points*, elements of $\mathcal{L}$ *lines*, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are *pencils* (of *parallel lines*). Lines from $\mathcal{L}_1$, $\mathcal{L}_2$ or $\mathcal{L}_3$, respectively, are said to be *vertical*, *horizontal* or *skew*, respectively. If $z \mathrel{\mathrm{I}} x$ and $z \mathrel{\mathrm{I}} y$ is true for distinct lines x, y then z is called the *intersection point* of both lines.

A *homomorphism* of a 3-coloured (3)-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ into a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is an incidence structure homomorphism which maps every vertical, horizontal or skew edge, respectively, onto a vertical, horizontal or skew line, respectively.

On Fig. 3 we present a non-3-colourable 3-configuration $(\mathcal{V}, \mathcal{E}, \mathrm{I})$,

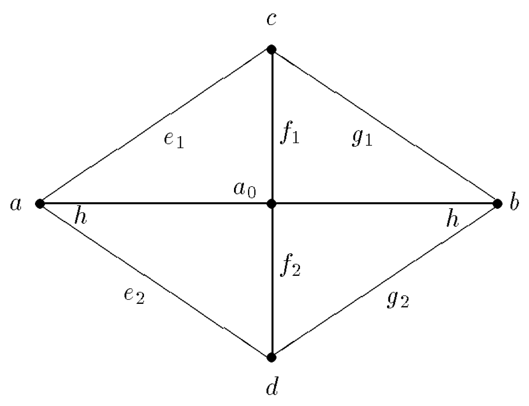


Fig. 3

with $\mathcal{V} = \{a_0, a, b, c, d\}$, $\mathcal{E} = \{e_1, f_1, g_1, e_2, f_2, g_2, h\}$ having one edge adjacent to just three vertices. If the colours of e_1, f_1, g_1, h are $i, j, k, l \in \{1, 2, 3\}$ then $i \neq j \neq k \neq i$ so that every possibility for l leads to a contradiction.

Let $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ be a 3-coloured 3-configuration and $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ sets of all its horizontal, vertical or complementar edges, respectively. Then define a half-operation $\binom{k}{i\,j} : \mathcal{E}_i \times \mathcal{E}_j \to \mathcal{E}_k$ such that $\binom{k}{i\,j}(x_i, x_j) = x_k$ is defined if and only if $x_i \in \mathcal{E}_i$, $x_j \in \mathcal{E}_j$, $x_k \in \mathcal{E}_k$ are neighbouring edges ($\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right)$ is an arbitrary permutation of the set $\{1, 2, 3\}$). So we obtain six *parastrophic* halfoperations associated to $\mathfrak{C}$ which from a *three-sorted edge halfquasigroup* $\left(\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \left\{\binom{k}{i\,j} \,\middle|\, \left(\begin{smallmatrix} 1 & 2 & 3 \\ i & j & k \end{smallmatrix}\right) \in \mathcal{S}_3\right\}\right)$ *associated* to $\mathfrak{C}$.

Let $\mathcal{B}$ be a basis of a simple 3-coloured 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ with the terminal element t. We express the edges of $\mathfrak{C}$ as constituents in the pillar $(\mathcal{B}_i)_{i=0}^{\infty}$ over $\mathcal{B}$ with use of halfoperations $\binom{k}{i\,j}$. The existence of t leads to an equality between constituents of greatest stage.

From the successive construction of constituents in $(\mathcal{B}_i)_{i=0}^{\infty}$ we obtain the validity of the *Position Property*: In every occurence of the edge e as a constituent $\binom{k_1}{i_1 j_1}(\cdot, \cdot)$ or as a member of other constituents $\binom{k_2}{i_2 j_2}(e, \cdot)$, $\binom{k_3}{i_3 j_3}(\cdot, e)$ the equalities $k_1 = i_2 = j_3$ must hold.

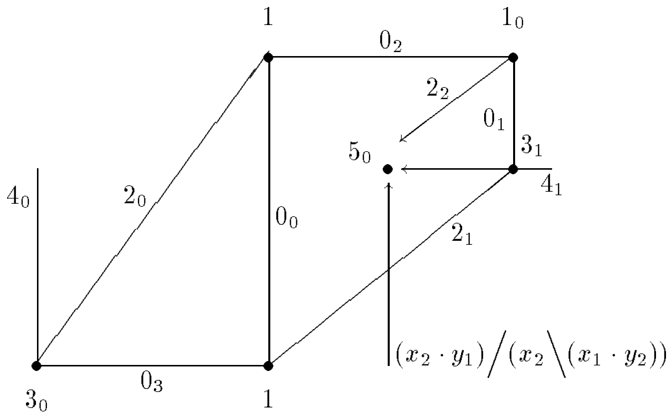
Now we see that Position Property implies that constituents of the form $\binom{k_1}{i_1 j_1}$ $(u, \binom{k_2}{i_2 j_2}(u, v))$ with $i_1 \neq i_2$ do not occur. Property (iii) affirms that constituents of the form $\binom{k_1}{i j_1}(u, \binom{k_2}{i j_2}(u, v))$ do not occur. Property (iv) excludes the equality $t_1 = t_2$ with $t_1 = \binom{k}{i\,j}(u, v_1)$, $t_2 = \binom{k}{i\,j}(u, v_2)$ whereas the case $t_2 = \binom{k}{i\,j}(t_1, v)$ with $i \neq 3$ is excluded again by Position Property.

If Position Property holds (v) can be reduced onto the occurence of a constituent $\binom{k}{i\,j}(x,u')$, $u'\neq u$ when a constituent $\binom{k}{i\,j}(x,u)$ occurs (with t_1 and t_2 different from a single $x\in\mathcal{B}$).

When a 3-sorted quasigroup $\mathbb{Q}=(Q_1,Q_2,Q_3,\ \{(\binom{k}{i\,j})|\left(\begin{smallmatrix}1&2&3\\ i&j&k\end{smallmatrix}\right)\in\mathcal{S}_3\})$ is given with $\#Q_1=\#Q_2=\#Q_3>1$ then let $t_1=t_2$ be a quasigroup identity for $\mathbb{Q}$ with a finite set $\mathcal{X}$, $\#\mathcal{X}\leq 3$ of variables (the formal construction of terms t_1, t_2 is supposed as known, see e.g.[1], pp. 30-31). The subterms of $t_1=t_2$ fulfill necessarily this *Position Property*: In every occurence of a subterm $\tau=\binom{k_1}{i_1 j_1}(\cdot,\cdot)$ and subterms $\binom{k_2}{i_2 j_2}(\tau,\cdot)$, $\binom{k_3}{i_3 j_3}(\cdot,\tau)$ the equalities $k_1=i_2=j_3$ must hold.

We assume that following conditions are fulfilled: every $x\in\mathcal{X}$ occurs in $t_1=t_2$ at least twice; the case $t_1=\binom{k}{i\,j}(u,v_1)$, $t_2=\binom{k}{i\,j}(u,v_2)$ is excluded; if t_1, t_2 are not single variables and a subterm $\binom{k}{i\,j}(x,u)$ occurs for $x\in\mathcal{X}$ then also a further subterm $\binom{k}{i\,j}(x,u')$ with $u'\neq u$ occurs; subterms are reduced by fundamental quasigroups identities $\binom{j}{ik}(\xi,\binom{k}{i\,j}(\xi,\eta)=\eta)$ where $\binom{3}{12}(\xi_1,\xi_2)=\xi_3\Leftrightarrow\binom{k}{i\,j}(\xi_i,\xi_j)=\xi_k$ for all permutations $\left(\begin{smallmatrix}1&2&3\\ i&j&k\end{smallmatrix}\right)$. Under these assumptions the subterm buildup of the identity coincides with the edge generation of a convenient 3-coloured 3-configuration over a simple basis corresponding to $\mathcal{X}$. We leave the details aside.

As an example we shall investigate a 3-coloured 3-configuration $\mathfrak{C}=(\mathcal{V},\mathcal{E},\ \mathrm{I})$ on Fig. 4 with a simple basis $\mathcal{B}=\{x_1,x_2,y_1,y_2\}$, where x_1,x_2 are vertical edges, y_1,y_2 horizontal edges and the terminal element is the vertex b. The edges of stages 4 determine the remaining edge through b. This leads to the equality $\big((x_1\cdot y_1)/y_2\big)\cdot\big(x_2\backslash(x_1\cdot y_2)\big)=x_2\cdot y_1$ where $\binom{3}{12}=\cdot$, $\binom{1}{32}=/,\binom{2}{13}=\backslash$. We can write equivalently $(x_1\cdot y_1)/y_2=(x_2\cdot y_1)/(x_2\backslash(x_1\cdot y_2))$.



$0_0\ldots x_1$, $0_1\ldots x_2$, $0_2\ldots y_1$, $0_3\ldots y_2$, $1_0\ldots a'$, $2_0\ldots x_1\cdot y_1$, $2_1\ldots x_1\cdot y_2$, $2_2\ldots x_2\cdot y_1$, $3_0\ldots a$, $3_1\ldots a''$, $4_0\ldots(x_1\cdot y_1)/y_2$, $4_1\ldots x_2\backslash(x_1\cdot y_2)$, $5_0\ldots b$

Fig. 4

The stages of single edges and vertices are written as labels on Fig. 4 as well as the corresponding constituents. We can readily confirm that properties (i)-(v) are fulfilled.

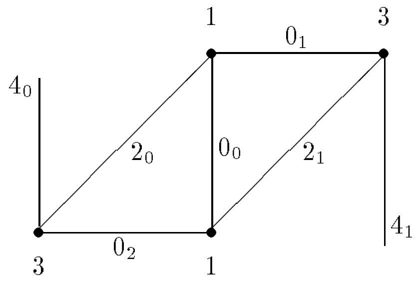
We say that a 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ satisfies the closure condition $\Gamma(\mathfrak{C}, \mathcal{B})$ derived from a 3-coloured 3-configuration $\mathfrak{C} = (\mathcal{V}, \mathcal{E}, \mathrm{I})$ with a simple basis $\mathcal{B}$ if every homomorphism $\sigma : \hat{\mathfrak{C}} \to \mathcal{W}$ can be prolonged onto a homomorphism $\bar{\sigma} : \mathfrak{C} \to \mathcal{W}$. Here $\hat{\mathfrak{C}} = (\mathcal{V}\backslash\{t\}, \mathcal{E}, \mathrm{I})$ or $\hat{\mathfrak{C}} = (\mathcal{V}, \mathcal{E}\backslash\{t\}, \mathrm{I})$ where t is the terminal vertex or edge.

To a given 3-web $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with pencils $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ we associate the *coordinatizing 3-sorted quasigroup* $(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \{\binom{k}{i\,j} | \binom{1\ 2\ 3}{i\ j\ k} \in \mathcal{S}_3\})$ which is defined by $\binom{k}{i\,j} : \mathcal{L}_1 \times \mathcal{L}_j \to \mathcal{L}_k, (x_i, x_j) \mapsto x_k$ where x_i, x_j, x_k are concurrent lines.

Let a given 3-web $\mathcal{W}$ with pencils $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ satisfy the closure condition derived from a 3-coloured 3-configuration $\mathfrak{C}$ with a simple basis $\mathcal{B}$. Then the corresponding term equality in $\mathfrak{C}$ over $\mathcal{B}$ can be interpreted as an identity with respect to the coordinatizing 3-sorted quasigroup $\mathbb{Q}$ and $\mathcal{B}$ as the set of variables. Thus $\mathcal{W}$ determines the corresponding identity over $\mathbb{Q}$.

We shall return to the 3-configuration $\mathfrak{C}$ on Fig. 4 and suppose that a 3-web $\mathcal{W}$ satisfies a closure condition $\Gamma(\mathcal{C}, \mathcal{B})$. Then the coordinatizing 3-sorted quasigroup satisfies the identity $(x_1 \cdot y_1)/y_2 = (x_2 \cdot y_1)\big/(x_2\backslash(x_1 \cdot y_2))$ with variables x_1, x_2 of "position" 1 and variables y_1, y_2 of "position" 2.

If we take a homomorphism $\varrho : \hat{\mathfrak{C}} \to \mathcal{W}$ with $\varrho(b) = \varrho(a') = \varrho(a'')$ we get a subconfiguration $\overline{\mathfrak{C}}$ of $\mathfrak{C}$ on Fig. 5 with a simple basis $\overline{\mathcal{B}} = \{x_1, y_1, y_2\}$. The closure condition $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ determines the identity $(x_1 \cdot y_1)/y_2 = (x_1 \cdot y_2)/y_1$. Thus in $\mathcal{W}$ from $\Gamma(\mathfrak{C}, \mathcal{B})$ it follows that $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$.



$0_0 \ldots x_1$, $0_1 \ldots y_1$, $0_2 \ldots y_2$, $2_0 \ldots x_1 \cdot y_1$, $2_1 \ldots x_1 \cdot y_2$, $4_0 \ldots (x_1 \cdot y_1)/y_2$, $4_1 \ldots (x_1 \cdot y_2)/y_1$

Fig. 5

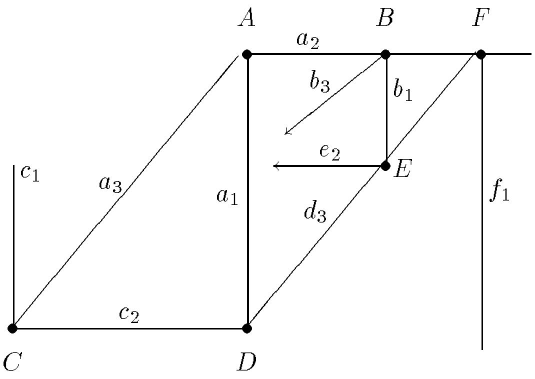


Fig. 6

Now we map $\hat{\mathcal{C}}$ under a homomorphism into $\mathcal{W}$ as on Fig. 6 and suppose that $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ holds in $\boldsymbol{\mathcal{W}}$. We construct the intersection point F of the lines a_2 and d_3 and apply $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ onto the images a_1, a_2, a_3 of basis edges so that $c_1 = f_1$. We apply $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ once more namely onto the images b_1, a_2, e_2 of basis elements so that f_1 must go through the intersection point G of b_3 and e_2. Thus G lies on c_1 and $\Gamma(\mathfrak{C}, \mathcal{B})$ holds when applied onto the images a_1, b_1, a_2, c_2 of basis edges. Thus $\Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ implies $\Gamma(\mathfrak{C}, \mathcal{B})$ so that $\Gamma(\mathfrak{C}, \mathcal{B}) \Leftrightarrow \Gamma(\overline{\mathfrak{C}}, \overline{\mathcal{B}})$ in $\boldsymbol{\mathcal{W}}$ and consequently $(x_1 \cdot y_1)\big/y_2 = (x_2 \cdot y_1)\big/(x_2\backslash(x_1 \cdot y_2))$, $(x_1 \cdot y_1)/y_2 = (x_1 \cdot y_2)\big/y_1$ are equivalent identities in $\mathbb{Q}$. Note that the second identity has the same left side as the first one whereas on the right side there occur variables x_1, x_2, y_1, y_2, respectively only variables x_1, y_1, y_2.

These "geometric" considerations will be now complemented by direct algebraic ones.

Let us consider quasigroup identities

(i) $\quad (x_1y_1)\big/y_2 = (x_2y_1)\big/(x_2\backslash(x_1y_2))$,

(ii) $\ (x \cdot y_1)\big/y_2 = (x \cdot y_2)/y_1$

once again.

Let (i) be valid in a 3-sorted quasigroup $\mathbb{Q}$. Putting $x_1y_2 = x_2y_1$, $x_1 = x$ we rewrite the right side of (i): $\quad (x_2y_1\big/(x_2\big\backslash(x_1y_2)) = (x \cdot y_2)\big/(x_2\backslash(x_2y_1)) = (x \cdot y_2)\big/y_1$. The left side of (i) is $(x \cdot y_1)\big/y_2$ so that (ii) is valid in $\mathbb{Q}$.

Conversely, let (ii) be valid in $\mathbb{Q}$. Write a duplicate of (ii)

(ii′) $$(x' \cdot y_1')\big/y_2' = (x' \cdot y_2')\big/y_1'$$

with $x' \cdot y_2' = x \cdot y_2$, $y_1' = y_1$. Thus $y_2' = x' \Big\backslash (x \cdot y_2)$ and, after substitution into (ii$'$), we get $(x' \cdot y_1' \Big/ (x' \Big\backslash (x \cdot y_2)) = (x' \cdot y_2) \Big/ y_1'$, i.e. $(x' \cdot y_1) \Big/ (x' \Big\backslash (x \cdot y_2')) = (x \cdot y_2) \Big/ y_1$.

Writing here $x = x_1$, $x' = x_1$ we get (i).

References

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