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SHORT PROOFS OF SOME INEQUALITIES OF HORST ALZER

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Abstract. We provide elementary proofs of some inequalities of Horst Alzer.

Let the real numbers a_i be such that $0 < a_i \le \frac{1}{2}$ for all $i = 1, \dots, n$, and let

$$\begin{aligned} A_n &= \frac{1}{n}\sum_{i=1}^{n} a_i, & G_n &= \prod_{i=1}^{n} a_i^{1/n}, & H_n &= n/\sum_{i=1}^{n} 1/a_i \\ A'_n &= \frac{1}{n}\sum_{i=1}^{n}(1-a_i), & G'_n &= \prod_{i=1}^{n}(1-a_i)^{1/n}, & H'_n &= n/\sum_{i=1}^{n} 1/(1-a_i)\,. \end{aligned}$$

In [1] Horst Alzer establishes the inequalities

$$1/H'_n - 1/H_n \le 1/G'_n - 1/G_n \le 1/A'_n - 1/A_n\,, \tag{1}$$

where the sign of equality holds if and only if $a_1 = \dots = a_n$. Furthermore, in [1] he refers to the inequalities

$$H_n/H'_n \le G_n/G'_n \le A_n/A'_n \tag{2}$$

discovered by W.-L. Wang and P.-F. Wang, and Ky Fan. (See [3] and [2, p. 5].)

In this note we use the classical arithmetic, geometric, harmonic mean inequalities

$$A_n \ge G_n \ge H_n \tag{3}$$

together with (2) to establish the inequalities (1). We begin by showing that

$$\frac{1}{H_n}\left(1 - \frac{H_n}{H'_n}\right) \ge \frac{1}{G_n}\left(1 - \frac{G_n}{G'_n}\right) \tag{4}$$

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which is equivalent to the left hand inequality in (1). Clearly, since $0 < a_i \leq \frac{1}{2}$ for all $i = 1, \ldots, n$ we have $H'_n \geq H_n > 0$ and $G'_n \geq G_n > 0$, and using (2) we deduce that

$$1 - \frac{H_n}{H'_n} \geq 1 - \frac{G_n}{G'_n} \geq 0 \,. \tag{5}$$

In addition, the right hand inequality in (3) yields

$$\frac{1}{H_n} \geq \frac{1}{G_n} > 0 \,, \tag{6}$$

and forming the products of corresponding sides in the inequalities (5) and (6) readily produces (4). The right hand inequality in (1) is proved in an analogous way.

Remark. It is worth noting that the above argument may be used to show that if $p > 0$ then

$$(1/H'_n)^p - (1/H_n)^p \leq (1/G'_n)^p - (1/G_n)^p \leq (1/A'_n)^p - (1/A_n)^p \,.$$

References

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