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Z–EQUILIBRIA IN MANY–PLAYER STOCHASTIC DIFFERENTIAL GAMES

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Abstract. In this paper N-person nonzero-sum games are considered. The dynamics is described by Ito stochastic differential equations. The cost-functions are conditional expectations of functionals of Bolza type with respect to the initial situation. The notion of Z-equilibrium is introduced in many-player stochastic differential games. Some properties of Z-equilibria are analyzed. Sufficient conditions are established guaranteeing the Z-equilibrium for the strategies of the players. In a particular case of a linear-quadratic game the Z-equilibrium strategies are found in an explicit form.

1. Introduction

In this paper we follow the approach of Fleming and Rishel [1] to the optimal control of stochastic dynamic system, but applied in situations of conflicts, i.e. to stochastic differential games. Let $\{1, \dots, N\}$ be the set of players. The dynamics is described by the following Ito stochastic differential equation:

$$dx\,(t) = f\,(t, x(t), u_1, \dots, u_N)\,dt + g(t, x(t), u_1, \dots, u_N)\,dw(t), \quad t \in [t_0, T]\,.$$

The control u_i is chosen by the i-th player in the feedback form $u_i = u_i(t, x(t))$ with the objective of minimizing the personal cost-function

$$J_i(u_i, \dots, u_N) = \mathbb{E}_{t_0, x_0}\{\Psi_i(T, x(T)) + \textstyle\int_{t_0}^{T} L_i(t, x(t), u_1, \dots, u_N)\,dt\}, \quad i \in I\,.$$

As a solution of the game the concept of Z-equilibrium is proposed. In deterministic differential games this notion is introduced by Zhukovskii in [8] and in two-player stochastic differential games by the author in [2]. The Z-equilibrium is based on the concept of Pareto-optimality, see Gaidov [3], [4] and represents a further development in the theory in comparison with the Nash-equilibrium , see Gaidov [3], [5].

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The present paper is organized as follows. In Section 2 we consider accurately the formalization of the game and a model of a linear-quadratic game. In Section 3 we recall some definitions and quote some results from our papers [3 – 6]. In Section 4 we introduce the notion of Z-equilibrium in many-player stochastic differential games and analyze some of its properties. Sufficient conditions for the Z-equilibrium strategies of the players are established in Section 5. Finally in Section 6 in the linear-quadratic game the Z-equilibrium strategies are found in an explicit form.

2. Formalization of the games

Let us consider the game

$$\Gamma = \langle I, \sum, \{\mathcal{U}_i\}_{i\in I}, \{J_i\}_{i\in I} \rangle \,.$$

Here $I = \{1, \ldots, N\}$ is the set of players participating in the game Γ. The evolution of the dynamic system $\sum$ is described by Ito stochastic differential equation of the type

$$(*) \qquad dx(t) = f(t, x(t), u_i, \ldots, u_N)\, dt + g(t, x(t), u_1., \ldots, u_N)\, dw(t), \quad t \in [t_0, T]$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}^n$ where $T > t_0 \geq 0$. The process $W = \{w(t),\, t \in [t_0, T]\}$ is a standard m-dimensional Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is adapted to a family $F = \{\mathcal{F}_t, t \in [t_0, T]\}$ of nondecreasing sub-σ-algebras of $\mathcal{F}$. The vector $x(t) \in \mathbb{R}^n$ is the state process and $u_i \in U_i \subset \mathbb{R}^{n_i}$ is the control of the i-th player, $i \in I$. Now let us make the following assumptions about the functions $f(t, x, u_i, \ldots, u_N)$ and $g(t, x, u_1, \ldots, u_N)$. Suppose

$$f : [t_0, T] \times \mathbb{R}^n \times U_1 \times \cdots \times U_N \to \mathbb{R}^n$$

and

$$g : [t_0, T] \times \mathbb{R}^n \times U_1 \times \cdots \times U_N \to \mathbb{R}^n \times \mathbb{R}^m$$

have continuous partial derivatives in $x, u_1, \ldots, u_N$ and let $C > 0$ be a constant such that

$$|f(t, 0, \ldots, 0)| + |g(t, 0, \ldots, 0)| \leqq C\,,$$
$$|f_x| + |g_x| + \textstyle\sum_{i\in I}(|f_{u_i}| + |g_{u_i}|) \leqq C\,.$$

Here $|\cdot|$ is a general symbol for the norms in the respective spaces.

Each player has complete information about the state vector $x(t)$ at every moment $t \in [t_0, T]$ and constructs his strategy in the game Γ as an admissible feedback control, i.e.

$$u_i = u_i(t, x(t))$$

where

$$u_i(\cdot,\cdot) : [t_0, T] \times \mathbb{R}^n \to U_i$$

is a Borel function satisfying the conditions:

(i) There exists a constant $M_i > 0$ such that

$$|u_i(t,x)| \leqq M_i(1+|x|) \text{ for all } (t,x) \in [t_0,T] \times \mathbb{R}^n ;$$

(ii) For each bounded set $B \subset \mathbb{R}^n$ and $T^* \in (t_0, T)$ there exists a constant $K_i > 0$ such that for arbitrary $x, y \in B$ and $t \in [t_0, T^*]$

$$|u_i(t,x) - u_i(t,y)| \leqq K_i |x-y| .$$

Denote by $\mathcal{U}_i$ the set of strategies of the i-th player, $i \in I$ and $\mathcal{U} = \prod_{i \in I} \mathcal{U}_i$, $U = \prod_{i \in I} U_i$. Let a vector of strategies $u = (u_i, \ldots, u_N) \in \mathcal{U}$ be called for brevity simply a strategy.

The assumptions made above imply the existence and sample path uniqueness of the solution $X = \{x(t), t \in [t_0, T]\}$ of Ito equation $(*)$ corresponding to the control $u \in \mathcal{U}$, see Fleming and Rishel [1]. Moreover, X is an a.s. continuous Markov process and its infinitesimal operator $\mathcal{A}(u)$ has the form

$$\mathcal{A}(u)\,V(t,x) = f'(t,x,u)\,V_x(t,x) + \frac{1}{2}\operatorname{tr}\left[a(t,x,u)\,V_{xx}(t,x)\right],$$

where $a = gg'$ and prime denotes vector or matrix transpose. Here $V(t,x)$ is a real-valued function with continuous partial derivatives up to second order for all $t \in [t_0, T]$, $x \in \mathbb{R}^n$.

Let L_i, Ψ_i be continuous functions satisfying the polynomial growth conditions:

$$|L_i(t,x,u_i,\ldots,u_N)| \leqq C_i(1+|x|+\textstyle\sum_{i\in I}|u_i|)^k$$
$$|\Psi_i(t,x)| \leqq C_i(1+|x|)^k$$

where C_i, k are positive constants. Introduce now the cost-function $J_i(u)$ of the i-th player:

$$J_i(u) = \mathbb{E}_{t_0,x_0}\{\Psi_i(T,x(T)) + \textstyle\int_{t_0}^T L_i(t,x(t),u_1,\ldots,u_N)\,dt\}, \quad i \in I .$$

The object of each player in the game Γ is to minimize his own cost-function.

Now let us consider one particular but important case of the game described above. Let

$$\Gamma_{lq} = \langle I, \textstyle\sum^l, \{\mathcal{U}_i^l\}_{i\in I}\,\{J_i^q\}_{i\in I}\rangle .$$

Here again $I = \{1, \ldots, N\}$. The evolution of the dynamic system $\sum^l$ is described by the linear stochastic differential equation of the type

$$dx(t) = [A(t)\,x(t) + \textstyle\sum_{i\in I} B_i(t) u_i]\,dt + g(t,x(t),u_1,\ldots,u_N)\,dw(t), \quad t \in [t_0,T]$$

with initial condition $x(t_0) = x_0 \in \mathbb{R}$. Here $x(t) \in \mathbb{R}$ is the state process, $W = \{w(t),\, t \in [t_0, T]\}$ is an $(N+2)$-dimensional standard Wiener process and $u_i \in U_i \subset \mathbb{R}$ is the control of the i-th player, $i \in I$. $g(t, x(t), u_i, \dots, u_N)$ is an $1 \times (N+2)$-matrix of the form

$$g = (g_0(t)\, x(t) \quad g_1(t)\, u_1 \quad \dots \quad g_N(t)\, u_N \quad g_{N+1}(t)) \ .$$

Henceforth $A(t),\ B_i(t),\ i \in I,\ g_0(t),\ g_{N+1}(t),\ g_i(t),\ i \in I$ are continuous real-valued functions. The strategies of the i-th player are identified to functions of the type $u_i(t,x) = F_i(t)x$ where $F_i(t)$ is a continuous real-valued function, $i \in I$. The cost-function $J_i^q(u)$ of the i-th player is the functional

$$J_i^q(u) = \mathbb{E}_{t_0, x_0}\{D_i x^2(T) + \textstyle\int_{t_0}^{T} [M_i(t)\, x^2(t) + \sum_{j \in I} N_j^{(i)}(t)\, u_j^2]\, dt\}, \quad i \in I \ .$$

Here D_i are constants and $M_i(t),\ i \in I,\ N_j^{(i)}(t),\ i, j \in I$ are real-valued continuous functions.

3. Auxiliary notions and results

For the completeness of presentation we need some facts from previous papers.

Definition 3.1. ([3], [4]). The strategy $u^P \in \mathcal{U}$ is said to be Pareto-optimal in the game Γ if the relations

$$J_i(u) \leqq J_i(u^P), \quad i \in I$$

for some strategy $u \in \mathcal{U}$ imply the equalities

$$J_i(u) = J_i(u^P), \quad i \in I \ .$$

Theorem 3.2. ([3], [4]). *The strategy $u^P \in \mathcal{U}$ is Pareto-optimal in the game Γ if there exist a vector $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$, $\lambda_i > 0$, $i \in I$, $\lambda_1 + \dots + \lambda_N = 1$ and real-valued function $V(t,x)$ such that for all $t \in [t_0, T]$, $x \in \mathbb{R}^n$ the following conditions jointly hold:*

(a) $V,\ \ V_t,\ V_x,\ V_{xx}$ *are continuous;*
(b) $H_\lambda(t, x, u^P) = 0$;
(c) $H_\lambda(t, x, u) \geqq 0$ *for each strategy* $u \in \mathcal{U}$;
(d) $V(T, x) = \sum_{i \in I} \lambda_i \Psi_i(T, x)$.

Here for all $t \in [t_0, T]$, $x \in \mathbb{R}^n$, $u \in \mathcal{U}$:

$$H_\lambda(t, x, u) = V_t(t, x) + \mathcal{A}(u)\, V(t, x) + \textstyle\sum_{i \in I} \lambda_i L_i(t, x, u).$$

Denote

$$D_\lambda = \textstyle\sum_{i \in I} \lambda_i D_i, \ \ M_\lambda(t) = \sum_{i \in I} \lambda_i M_i(t) \ \text{and} \ N_\lambda^{(i)}(t) = \sum_{j \in I} \lambda_j N_i^{(j)}(t)\,, \quad i \in I\ .$$

Proposition 3.3. ([3], [4]). *Let there exist a vector* $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ *such that* $\lambda_i > 0$, $i \in I$, $\lambda_1 + \cdots + \lambda_N = 1$, D_λ *is a non-negative constant,* $M_\lambda(t)$ *is a non-negative function and* $N_\lambda^{(i)}(t)$ *is a positive function for each* $t \in [t_0, T]$. *Then*

$$u_i^P = -[g_i^2(t)K_\lambda(t) + N_\lambda^{(i)}(t)]^{-1} B_i(t)K_\lambda(t)x, \quad i \in I$$

are Pareto-optimal strategies in the game Γ_{lq} *where* $K_\lambda(t)$ *is the solution of the nonlinear differential equation*

$$\begin{gathered}\dot{K}_\lambda(t) + 2A(t)K_\lambda(t) + M_\lambda(t) + g_0^2(t)K_\lambda(t) \\ -K_\lambda^2(t)\textstyle\sum_{i\in I}[g_i^2(t)K_\lambda(t) + N_\lambda^{(i)}(t)]^{-1}B_i^2(t) = 0\end{gathered}$$

with the boundary condition $K_\lambda(T) = D_\lambda$.

Remark 3.4. It is important to mention that the existence of the function $K_\lambda(t) \geqq 0$, $t \in [t_0, T]$ follows e.g. from the well-known Bellman quasilinearization method, see Roitenberg [7].

Let us recall two other definitions.

Definition 3.5. ([3], [5]). The strategy $u^n \in \mathcal{U}$ is a Nash-equilibrium strategy in the game Γ if for each $u_i \in \mathcal{U}_i$

$$J_i(u_1^n, \ldots, u_{i-1}^n, u_i, u_{i+1}^n, \ldots, u_N^n) = J_i(u^n \| u_i) \geqq J_i(u^n), \quad i \in I\,.$$

Definition 3.6. ([6]). The strategy $u_i^g \in \mathcal{U}_i$ is a guaranteeing strategy of the i-th player in the game Γ if

$$\min_{u_i} \max_{u_{I\setminus i}} J_i(u_i, u_{I\setminus i}) = \max_{u_{I\setminus i}} J_i(u_i^g, u_{I\setminus i}).$$

Here $I \setminus i = \{1, \ldots, i-1, i+1, \ldots, N\}$ and $u_{I\setminus i} = (u_i, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N) \in \prod_{j\in I\setminus i}\mathcal{U}_j = \mathcal{U}_{I\setminus i}$. Let also $(u_i, u_{I\setminus i}) = u$.

4. Z-equilibrium. Basic properties

Now we generalize for many-player games the concept of Z-equilibrium , considered for two-player games in Gaidov [2].

Definition 4.1. The strategy $u^* \in \mathcal{U}$ is an active equilibrium strategy in the game Γ if for each player $i \in I$ we have: for any strategy $u_i \in \mathcal{U}_i$ there exists a collection of strategies $\hat{u}_{I\setminus i} \in \mathcal{U}_{I\setminus i}$ such that

$$J_i(u_i, \hat{u}_{I\setminus i}) \geqq J_i(u^*)\,.$$

Definition 4.2. The strategy $u^Z \in \mathcal{U}$ is a Z-equilibrium strategy in the game Γ if u^Z is both Pareto-optimal and an active equilibrium strategy.

Now we analyze some properties of the Z-equilibrium strategies and compare them with other optimal strategies.

Property 4.3. (Pareto-optimality). By Definition 4.2 we have that Z-equilibria are Pareto-optimal, i.e. they look after (guarantee) the collective interests of the players.

Property 4.4. (Active stability of Z-equilibria against unilateral deviation of a player.) Let $u^Z \in \mathcal{U}$ be a Z-equilibrium point in the game Γ. Then Definition 4.1 implies that for every strategy $u_i \in \mathcal{U}_i$ of the i-th player ($i \in I$) there is a collection of strategies $\hat{u}_{I \setminus i} \in \mathcal{U}_{I \setminus i}$ such that

$$J_i(u_i, \hat{u}_{I \setminus i}) \geqq J_i(u^Z)\,.$$

Thus, if the i-th player uses a strategy u_i different from u_i^Z, then the other players $I \setminus i$ can punish the deflecting one. Moreover, $I \setminus i$ generates an active response $\hat{u}_{I \setminus i}$ to each u_i. Let us note that Nash-equilibria (see Definition 3.5) are also stable versus the deflection of one player: for each $u_i \in \mathcal{U}_i$

$$J_i(u_i, u^n_{I \setminus i}) \geqq J_i(u^n), \quad i \in I$$

where $u^n \in \mathcal{U}$ is a Nash-equilibrium point. However, here the penalty $u^n_{I \setminus i}$ is passive. In fact the players $I \setminus i$ simply stick to their strategies from u^n.

Property 4.5. (Individual rationality). Let $u_i^g \in \mathcal{U}_i$ be a guaranteeing (minimax) strategy of the i-th player (see Definition 3.6) and let $u^Z \in \mathcal{U}$ be a Z-equilibrium. Then for u_i^g there exists $\hat{u}_{I \setminus i}$ by Definition 4.2 such that

$$J_i(u^Z) \leqq J_i(u_i^g, \hat{u}_{I \setminus i}) \leqq \max_{u_{I \setminus i}} J_i(u_i^g, u_{I \setminus i}) = \min_{u_i} \max_{u_{I \setminus i}} J_i(u), \quad i \in I\ .$$

Thus, the values of the cost-functions in a Z-equilibrium point are at most equal to the minimax values.

Property 4.6. (Pareto-optimal Nash-equilibria are Z-equilibria). The Pareto-optimality is required for the Z-equilibrium. Thus we have to prove that the Nash-equilibrium implies the active equilibrium. Let $u^n \in \mathcal{U}$ be a Nash-equilibrium point in the game Γ. Then for each $u_i \in \mathcal{U}_i$

$$J_i(u_i, u^n_{I \setminus i}) \geqq J_i(u^n), \quad i \in I\ .$$

Thus, for each $u_i \in \mathcal{U}_i$ we can choose $\hat{u}_{I \setminus i} = u^n_{I \setminus i}$ and by Definition 4.1 we conclude that u^n is an active equilibrium.

Property 4.7. (Saddle-points in two-person zero-sum games are Z-equilibrium points). Let us consider the two-person zero-sum game

$$\Gamma_0 = \langle \{1,2\}, \sum, \{\mathcal{U}_1, \mathcal{U}_2\}, J(u_1, u_2)\rangle$$

with the objection of minimizing $J(u_1, u_2)$ for the first player and maximizing $J(u_1, u_2)$ for the second one. Let (u_1^0, u_2^0) be a saddle-point of Γ_0:

$$J(u_1^0, u_2) \leqq J(u_1^0, u_2^0) \leqq J(u_1, u_2^0)$$

for each $u_1 \in \mathcal{U}_1, u_2 \in \mathcal{U}_2$.

We consider also the game

$$\Gamma_2 = \langle \{1,2\}, \sum, \{\mathcal{U}_1, \mathcal{U}_2\}, \{J_1, J_2\}\rangle$$

where $J_1(u_1, u_2) = J(u_1, u_2)$ and $J_2(u_1, u_2) = -J(u_1, u_2)$. Here both players choose their strategiess with the aim of minimizing their own cost-functions.

First we prove that the saddle-point (u_1^0, u_2^0) of Γ_0 is Pareto-optimal in Γ_2. Suppose (u_1^0, u_2^0) is not Pareto-optimal in Γ_2. Then there exists a pair of strategies $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J_i(u_1, u_2) \leqq J_i(u_1^0, u_2^0), \quad i = 1, 2$$

where at least one of these two inequalities is strict. Hence

$$J_1(u_1, u_2) + J_2(u_1, u_2) < J_1(u_1^0, u_2^0) + J_2(u_1^0, u_2^0)$$

i.e.

$$0 = J(u_1, u_2) - J(u_1, u_2) < J(u_1^0, u_2^0) - J(u_1^0, u_2^0) = 0$$

which is wrong. Therefore the Pareto-optimality of (u_1^0, u_2^0) is established.

Second we show the active equilibrium property of (u_1^0, u_2^0) in Γ_2. Indeed, for each $u_1 \in \mathcal{U}_1$ we put $\hat{u}_2 = u_2^0$ and for each $u_2 \in \mathcal{U}_2$ we put $\hat{u}_1 = u_1^0$. Thus we get

$$J_1(u_1, \hat{u}_2) = J_1(u_1, u_2^0) = J(u_1, u_2^0) \geqq J(u_1^0, u_2^0) = J_1(u_1^0, u_2^0)$$

and

$$J_2(\hat{u}_1, u_2) = J_2(u_1^0, u_2) = -J(u_1^0, u_2) \geqq -J(u_1^0, u_2^0) = J_2(u_1^0, u_2^0)\ .$$

Therefore we arrive at the conclusion that the notion of a Z-equilibrium includes the notion of a saddle-point for zero-sum two-players games.

5. Sufficient conditions

In this section we shall find conditions which are sufficient for the Z-equilibrium strategies. Denote

$$G_i(t,x,u) = V_t^{(i)}(t,x) + \mathcal{A}(u)V^{(i)}(t,x) + L_i(t,x,u), \quad i \in I$$

where $t \in [t_0, T]$, $x \in \mathbb{R}^n$, $u \in U$.

Theorem. *Suppose for the strategy $u^Z \in U$ the next three groups of conditions jointly hold:*

1) There exist a vector $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$, $\lambda_i > 0$, $\quad i \in I$, $\lambda_1 + \dots + \lambda_N = 1$ and a real-valued function $V(t,x)$ such that for all $t \in [t_0, T]$, $x \in \mathbb{R}^n$ the following conditions jointly hold:

- (a_1) *V, V_t, V_x, V_{xx} are continuous;*
- (b_1) *$H_\lambda(t,x,u^Z) = 0$;*
- (c_1) *$H_\lambda(t,x,u) \geqq 0$ for each strategy $u \in \mathcal{U}$;*
- (d_1) *$V(T,x) = \sum_{i\in I} \lambda_i \Psi_i(T,x)$.*

2) There exist real-valued functions $V^{(i)}(t,x)$, $i \in I$ such that for all $t \in [t_0, T]$, $x \in \mathbb{R}^n$ and $i \in I$ the following conditions jointly hold:

- (a_2) *$V^{(i)}$, $V_t^{(i)}$, $V_x^{(i)}$, $V_{xx}^{(i)}$ are continuous;*
- (b_2) *$G_i(t,x,u^Z) = 0$;*
- (c_2) *$V^{(i)}(T,x) = \Psi_i(T,x)$.*

3) For each $i \in I$ and arbitrary strategy $u_i \in \mathcal{U}_i$ there exists a collection of strategies $\hat{u}_{I\setminus i} \in \mathcal{U}_{I\setminus i}$ such that

$$G_i(t,x,u_i,\hat{u}_{I\setminus i}) \geqq 0 \ .$$

Then the strategy $u^Z \in \mathcal{U}$ is a Z-equilibrium strategy in the game Γ.

Proof. Conditions 1) are equivalent to the conditions of Theorem 3.2, i.e. the strategy $u^Z \in \mathcal{U}$ is Pareto-optimal. Let the set of functions $V^{(i)}(t,x)$, $i \in I$ with continuous derivatives be the solution of the system of equations (b_2) with boundary conditions (c_2). Suppose $X^Z = \{x^Z(t),\, t \in [t_0,T]\}$ and $X^{(i)} = \{x^{(i)}(t),$ $t \in [t_0,T]\}$ are the solutions of Ito equation $(*)$ corresponding to the strategies u^Z and $(u_i, \hat{u}_{I\setminus i})$, respectively.

Next write the formula of Ito-Dynkin for $V^{(i)}(t,x)$, u^Z and X^Z:

$$V_{(t,x)}^{(i)} = \mathbb{E}_{t,x}\{V^{(i)}(T,x(T)) - \textstyle\int_t^T [V_t^{(i)}(\tau,x^Z(\tau)) + \mathcal{A}(u^Z)V^{(i)}(\tau,x^Z(\tau))]\,d\tau\}, \quad i \in I\,.$$

This representation in conjunction with (b_2) and (c_2) implies that

$$V^{(i)}(t,x) = \mathbb{E}_{t,x}\{\Psi_i(T,x^Z(T)) + \textstyle\int_t^T L_i(\tau,x^Z(\tau),u^Z)\,d\tau\}, \quad i \in I\,,$$

and hence

$$V^{(i)}(t_0, x_0) = \mathbb{E}_{t_0,x_0}\{\Psi_i(T, x^Z(T)) + \int_{t_0}^T L_i(t, x^Z(t), u^Z)\, dt\}, \quad i \in I\ .$$

Now write again the formula of Ito-Dynkin for $V^{(i)}(t,x)$ but with $(u_i, \hat{u}_{I\setminus i})$ and $X^{(i)}$:

$$\begin{aligned} V^{(i)}(t,x) = \mathbb{E}_{t,x}\{V^{(i)}(T, x^i(T)) - \textstyle\int_t^T [V_t^{(i)}(\tau, x^{(i)}(\tau)) \\ + \mathcal{A}(u_i, \hat{u}_{I\setminus i})\, V^{(i)}(\tau, x^{(i)}(\tau))]\, d\tau\}, \quad i \in I\,. \end{aligned}$$

Taking into account conditions 3) and (c_2), we get

$$V^{(i)}(t,x) = \mathbb{E}_{t,x}\{\Psi_i(T, x^{(i)}(T)) + \int_t^T L_i(\tau, x^{(i)}(\tau), u_i, \hat{u}_{I\setminus i})\, d\tau\}, \quad i \in I$$

which leads to

$$V^{(i)}(t_0, x_0) = \mathbb{E}_{t_0,x_0}\{\Psi_i(T, x^{(i)}(T)) + \int_{t_0}^T L_i(t, x^{(i)}(t), u_i, \hat{u}_{I\setminus i})\, dt\}, \quad i \in I\ .$$

Finaly we have

$$V^{(i)}(t_0, x_0) = J_i(u^Z) \leqq J_i(u_i, \hat{u}_{I\setminus i}), \quad i \in I\ .$$

This means that u^Z is an active equilibrium strategy in Γ and hence u^Z is a Z-equilibrium strategy. So the proof of the Theorem is completed. □

6. Linear-quadratic game

Now consider the linear-quadratic stochastic differential game Γ_{lq}, described in Section 2. Let the conditions of Proposition 3.3 hold. Then the strategies

$$u_i^Z = u_i^P = -[g_i^2(t) K_\lambda(t) + N_\lambda^{(i)}(t)]^{-1}\, B_i(t)\, K_\lambda(t)\, x, \quad i \in I$$

are Pareto-optimal strategies in the game Γ_{lq}.

Further we follow the procedure of searching the Z-equilibrium from the Theorem of Section 5 and construct the functions

$$\begin{aligned} G_i(t,x,u) = V_t^{(i)}(t,x) + [A(t)\, x + \textstyle\sum_{j\in I} B_j(t) u_j]\, V_x^{(i)}(t,x) \\ + \frac{1}{2}[g_0^2(t)\, x^2 + \textstyle\sum_{j\in I} g_j^2(t)\, u_j^2 + g_{N+1}(t)]\, V_{xx}^{(i)}(t,x) \\ + M_i(t)\, x^2 + \textstyle\sum_{j\in I} N_j^{(i)}(t)\, u_j^2,\ i \in I\,. \end{aligned}$$

We search $V^{(i)}(t,x)$ as a solution of the equation

$$G_i(t,x,u^Z) = 0$$

with the boundary condition $V^{(i)}(T,x) = D_i x^2$ in the following special form

$$V^{(i)}(t,x) = \theta_i(t)\,x^2 + r_i(t)$$

where $\theta_i(t)$ and $r_i(t)$ are real-valued functions, $i \in I$. thus we get for each $i \in I$ the nonhomogeneous linear differential equation

$$\begin{aligned}\dot\theta_i(t) + 2A(t)\theta_i(t) - 2\theta_i(t)\,K_\lambda(t)\textstyle\sum_{j\in I} B_j^2(t)[g_j^2(t)\,K_\lambda(t) + N_\lambda^{(j)}(t)]^{-1}\\ +g_0^2(t)\theta_i(t) + \textstyle\sum_{j\in I} g_j^2(t)[g_j^2(t)\,K_\lambda(t) + N_\lambda^{(j)}(t)]^{-2}\,B_j^2(t)\,K_\lambda^2(t)\,\theta_i(t)\\ +M_i(t) + \textstyle\sum_{j\in I} N_j^{(i)}(t)[g_j^2(t)\,K_\lambda(t) + N_\lambda^{(j)}(t)]^{-2}\,B_j^2(t)\,K_\lambda^2(t) = 0\end{aligned}$$

with boundary condition $\theta_i(T) = D_i$ and

$$\dot r_i(t) + g_{N+1}^2(t)\theta_i(t) = 0\ .$$

Suppose D_i is a non-negative constant and $M_i(t)$, $N_j^{(i)}(t)$, $j \in I$ are non-negative functions, $i \in I$. Taking into account this assumption and the continuity of the coefficients of the last equation we get the existence and uniqueness of its continuous solution $\theta_i(t)$ which is non-negative for each $t \in [t_0, T]$.

Now we can have the following representation

$$\begin{aligned}G_i(t,x,u) = x^2\{\dot\theta_i(t) + 2A(t)\,\theta_i(t) + g_0^2(t)\,\theta_i(t) + M_i(t)\}\\ +2\textstyle\sum_{j\in I} B_j(t)\,u_j\theta_i(t)\,x + \textstyle\sum_{j\in I} g_j^2(t)\,u_j^2\theta_i(t) + \textstyle\sum_{j\in I} N_j^{(i)}(t)\,u_j^2\ .\end{aligned}$$

Next, take arbitrary $i \in I$ and let $u_i = F_i(t)\,x$. Fix $j \in I \setminus i$ and let $u_j = ax$ where a is a positive constant. Also let $u_k = u_k^Z$, $k \in I \setminus \{i,j\}$. Then for a suitable $S(t)$ we can write $G_i(t,x,u)$ in the form

$$G_i(t,x,u) = x^2\{S(t) + 2aB_j(t)\theta_i(t) + a^2[g_j^2(t)\,\theta_i(t) + N_j^{(i)}(t)]\}\,.$$

Suppose $N_j^{(i)}(t)$ is a positive function for each $t \in [t_0, T]$. Then $g_j^2(t)\,\theta_i(t) + N_j^{(i)}(t)$ is a positive as well. This implies the existence of a positive number a^* such that the quantity

$$S(t) + 2a^* B_j(t)\,\theta_i(t) + (a^*)^2[g_j^2(t)\,\theta_i(t) + N_j^{(i)}(t)]$$

is positive for all $t \in [t_0, T]$. Hence, if we put $u_j^* = a^* x$ we get

$$G_i(t,x,u_i,\hat u_{I\setminus i}) \geqq 0$$

where $\hat u_{I\setminus i} = \{u_j^*, u_k^Z,\ k \in I \setminus \{i,j\}\}$. Thus $u^Z = u^P$ is an active equilibrium in the game Γ_{lq} and we come to the following result.

Proposition. *Let D_i, $i \in I$ be non-negative constants, $M_i(t)$, $i \in I$ and $N_j^{(i)}(t)$, $i, j \in I$ be non-negative functions for each $t \in [t_0, T]$. Let there exist a vector $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ such that $\lambda_i > 0$, $i \in I$, $\lambda_1 + \cdots + \lambda_N = 1$ and $N_\lambda^{(i)}(t)$, $i \in I$ are positive functions for each $t \in [t_0, T]$. Let for every $i \in I$ there exist $j \in I \setminus i$ such that the function $N_j^{(i)}(t)$ is positive for all $t \in [t_0, T]$. Then u^Z is a Z-equilibrium strategy in the game Γ_{lq}.*

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