ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 161 – 166

OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF ADVANCED FUNCTIONAL EQUATIONS

Jozef Džurina

Abstract. In this paper we compare the asymptotic behaviour of the advanced functional equation

$$(*) \qquad L_n u(t) - F\big(t, u[g(t)]\big) = 0$$

with the asymptotic behaviour of the set of ordinary functional equations

$$\alpha_i u(t) - F\big(t, u(t)\big) = 0.$$

On the basis of this comparison principle the sufficient conditions for property (B) of equation (*) are deduced.

This paper is concerned with the oscillatory and asymptotic behaviour of the solutions of the functional differential equation with advanced argument

$$L_n u(t) - F\big(t, u[g(t)]\big) = 0, \tag{1}$$

where $n \ge 3$ and L_n denotes the disconjugate differential operator

$$L_n = \frac{1}{r_n(t)} \frac{d}{dt} \frac{1}{r_{n-1}(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{r_1(t)} \frac{d}{dt} \frac{\cdot}{r_0(t)}. \tag{2}$$

It is assumed that

(i) $r_i,\, g : [t_0, \infty) \to \mathbb{R}$ are continuous $r_i(t) > 0$, $0 \le i \le n$ and $g(t) \ge t$;
(ii) $F : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and $\operatorname{sgn} F(t,x) = \operatorname{sgn} x$ for each $t \in [t_0, \infty)$.

1991 *Mathematics Subject Classification*: Primary 34C10.
Key words and phrases: comparison theorem, advanced argument, property (B).
Received June 8, 1992.

In the sequel we will suppose that

$$\int^{\infty} r_i(s)\,ds = \infty \quad \text{for} \quad 1 \le i \le n-1. \tag{3}$$

The operator L_n satisfying (3) is said to be in canonical form. It is well-known that any differential operator of the form (2) can always be represented in canonical form in an essentially unique way (see Trench [6]).

The following notation is employed:

$$\begin{aligned} L_0 u(t) &= \frac{u(t)}{r_0(t)}, \\ L_i u(t) &= \frac{1}{r_i(t)} \frac{d}{dt} L_{i-1} u(t), \quad 1 \le i \le n. \end{aligned}$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $u : [T_u, \infty) \to \mathbb{R}$ such that $L_i u(t)$, $0 \le i \le n$ exist and are continuous on $[T_u, \infty)$. A nontrivial solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

If $u(t)$ is a nonoscillatory solution of (1) then according to a generalization of a lemma of Kiguradze [3, Lemma 3]. there exists a $t_1 \in [t_0, \infty)$ and an integer $\ell \in \{0, 1, \cdots, n\}$ such that $\ell \equiv n \pmod 2$ and

$$\begin{aligned} u(t) L_i u(t) &> 0, \quad 1 \le i \le \ell, \\ (-1)^{i-\ell} u(t) L_i u(t) &> 0, \quad \ell + 1 \le i \le n, \end{aligned} \tag{4}$$

for all $t \ge t_1$.

A function $u(t)$ satisfying (4) is said to be a function of degree ℓ (see Foster and Grimmer [1]). The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solution of (1), then

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_n \qquad \text{if } n \text{ is odd,}$$

and

$$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_n \qquad \text{if } n \text{ is even.}$$

Definition 1. Equation (1) is said to have property (B) if for n odd every nonoscillatory solution of (1) is of degree n, i. e. $\mathcal{N} = \mathcal{N}_n$ and for n even every nonoscillatory solution of (1) is either of degree n or of degree 0, i. e. $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_n$.

The objectives of this paper is to establish a comparison principle between advanced equation (1) and corresponding ordinary equation and to obtain sufficient conditions for equation (1) to have property (B).

We remark that for delay equations ($g(t) \le t$) of the form (1), efforts in this direction have been undertaken by Kusano and Naito [4] in which delay equation of the form (1) is compared with ordinary equation without delay and on the basis of such comparison theorem we can deduce criteria for property (B) of delay equation (1).

Let us consider the set of the disconjugate differential operators

$$\alpha_i = \frac{1}{r_n(t)}\frac{d}{dt}\frac{1}{r_{n-1}(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{r_i(t)} \\ \times\frac{d}{dt}\frac{1}{r_{i-1}[g(t)]g'(t)}\frac{d}{dt}\cdots\frac{d}{dt}\frac{1}{r_1[g(t)]g'(t)}\frac{d}{dt}\frac{\cdot}{r_0[g(t)]}.$$

for $i = 1, 2, \cdots, n-2$.

Theorem 1. *Suppose that*

$$F(t,x) \text{ is nondecreasing in } x. \tag{5}$$

$$g(t) \in C^1([t_0,\infty)), \quad g'(t) > 0, \quad g(t) \geq t. \tag{6}$$

Further assume that for $i = 2, 4, \cdots, n-2$ if n is even and for $i = 1, 3, \cdots, n-2$ if n is odd, the functional equation

$$\alpha_i u(t) - F\big(t, u(t)\big) = 0 \tag{E_i}$$

has not any solution of degree i. Then equation (1) has property (B).

Proof. Let $u(t)$ be a solution of (1), which is eventually positive and let $\ell \in \{0, 1, \cdots, n\}$ be the integer such that $\ell \equiv n \pmod 2$ and (4) holds for all large t, say $t \geq t_1$. We claim that $\ell = 0$ (if n is even) or $\ell = n$. Assume that $\ell \in \{1, 2, \cdots, n-2\}$. An integration of (1) yields

$$-L_{n-1}u(t) \geq \int_t^\infty r_n(s)F\big(s, u[g(s)]\big)\,ds, \quad t \geq t_1.$$

Continuing in this manner we obtain

$$L_\ell u(t) \geq \int_t^\infty r_{\ell+1}(s_{\ell+1}) \\ \times\int_{s_{\ell+1}}^\infty\cdots\int_{s_{n-1}}^\infty r_n(s_n)F\left(s_n, u[g(s_n)]\right)\,ds_n\cdots ds_{\ell+1}, \tag{7}$$

for $t \geq t_1$. We multiply (7) by $r_\ell(t)$ and integrate over $[t_1, t]$ to obtain

$$L_{\ell-1}u[g(t)] \geq L_{\ell-1}u(t) \geq \int_{t_1}^t r_\ell(s_\ell)\int_{s_\ell}^\infty r_{\ell+1}(s_{\ell+1}) \\ \times\int_{s_{\ell+1}}^\infty\cdots\int_{s_{n-1}}^\infty r_n(s_n)F\left(s_n, u[g(s_n)]\right)\,ds_n\cdots ds_\ell. \tag{8}$$

The first inequality in (8) follows from the facts that $g(t) \geq t$ and $L_{\ell-1}u(t)$ is an increasing function as $\ell \geq 1$. If $\ell \geq 2$ then we multiply (8) by $r_{\ell-1}[g(t)]g'(t)$ and integrate the resulting inequality over $[t_1,t]$. Repeating this procedure, we arrive at

$$L_0u[g(t)] \geq \int_{t_1}^{t} r_1[g(s_1)]g'(s_1)\int_{t_1}^{s_1}\cdots\int_{t_1}^{s_{\ell-2}} r_{\ell-1}[g(s_{\ell-1})]g'(s_{\ell-1})$$

$$\times\int_{t_1}^{s_{\ell-1}} r_\ell(s_\ell)\int_{s_\ell}^{\infty}\cdots\int_{s_{n-1}}^{\infty} r_n(s_n)F\big(s_n,u[g(s_n)]\big)\,ds_n\cdots ds_1,\quad t\geq t_1. \tag{9}$$

Denote the right hand side of (9) by $v(t)$ and define $z(t)=r_0[g(t)]v(t)$. Repeated differentiation of $z(t)$, shows $z(t)$ is a function of degree ℓ and, on the other hand,

$$\alpha_\ell z(t) - F\big(t,u[g(t)]\big) = 0. \tag{10}$$

Since $u[g(t)]\geq z(t)$, we obtain in view of (10) that $z(t)$ is a solution of the differential inequality

$$\big\{\alpha_\ell z(t) - F\big(t,z(t)\big)\big\}\operatorname{sgn} z(t)\geq 0.$$

But then Corollary 1 of Kusano and Naito in [4], ensures that equation (E_ℓ) has also a solution of degree ℓ, which contradicts the hypotheses.

If $\ell=1$ then (8) implies

$$L_0u[g(t)] \geq \int_{t_1}^{t} r_1(s_1)\int_{s_1}^{\infty} r_2(s_2)\int_{s_2}^{\infty}\cdots\int_{s_{n-1}}^{\infty} r_n(s_n)F\left(s_n,u[g(s_n)]\right)ds_n\cdots ds_1.$$

Again, denote the right hand side of above inequality by $v(t)$ and define $z(t)=r_0[g(t)]v(t)$. Proceeding similarly as above we can verify that equation (E_1) has a solution of degree 1, which contradicts the hypotheses. The proof is complete now. □

Now, we apply our comparison principle to the linear form of equation (1), namely, to the advanced equation

$$L_nu(t) - p(t)u[g(t)] = 0, \tag{11}$$

where function $p(t)$ is continuous and positive on $[t_0,\infty)$.

Let $1\leq i\leq n-1$ and $t,s\in[t_0,\infty)$, for convenience we make use the following notations:

$$I_0 = 1,$$

$$I_i(t,s;r_i,\cdots,r_1) = \int_s^t r_i(x)I_{i-1}(x,s;r_{i-1},\cdots,r_1)\,dx.$$

For simplicity of notation we put

$$J_i(t,s) = r_0[g(t)]\,I_i\big(t,s;r_1(g)g',\cdots,r_i(g)g'\big),$$
$$K_i(t,s) = r_n(t)\,I_i\left(t,s;r_{n-1},\cdots,r_{n-i}\right).$$

First note that the following formula holds for the function $J_i(t,s)$ defined above

$$J_i(t,s) = r_0[g(t)]I_i\big(g(t),g(s);r_1,\cdots,r_i\big).$$

Theorem 2. *Suppose that (6) holds. Let*

$$q_i(t) = r_{i+1}(t) \int_t^\infty K_{n-i-2}(s,t) J_{i-1}(s,t) p(s)\, ds, \quad i = 1, 2, \cdots, n-2.$$

Assume that the second order equations

$$(\tilde{E}_i) \qquad \left(\frac{1}{r_i(t)} z'(t) \right)' + q_i(t) z(t) = 0$$

are oscillatory for $i = 2, 4, \cdots, n-2$ if n is even and for $i = 1, 3, \cdots, n-2$ if n is odd. Then equation (11) has property (B).

Proof. Let $\ell \in \{1, 2, \cdots, n-2\}$ be fixed. By Theorem 1 equation (11) has not any solution of degree ℓ if the equation

$$\alpha_\ell u(t) - p(t) u(t) = 0$$

has not any solution of degree ℓ, which according to Theorem 2 in [5] comes if equation $(\tilde{E}_\ell)$ is oscillatory. The proof is complete. □

As application of Theorem 2 we give the following result:

Corollary 1. *Let all conditions of Theorem 2 hold. If*

$$(12) \qquad \liminf_{t\to\infty} \left(\int_{t_0}^t r_i(s)\, ds \right) \left(\int_t^\infty q_i(s)\, ds \right) > \frac{1}{4}$$

for $i = 2, 4, \cdots, n-2$ if n is even and for $i = 1, 3, \cdots, n-2$ if n is odd. Then equation (11) has property (B).

Proof. Let $i \in \{1, 2, \cdots, n-2\}$, such that $n + i$ is even, be fixed. By the well-known criterion of Hille [2] condition (12) is sufficient so that all solutions of equation $(\tilde{E}_i)$ are oscillatory. Hence Corollary 1 follows from Theorem 2. □

Let us consider the fourth order advanced equation

$$(13) \qquad \left(\frac{1}{r_3(t)} \left(\frac{1}{r_2(t)} \left(\frac{1}{r_1(t)} u'(t) \right)' \right)' \right)' - p(t) u[g(t)] = 0.$$

Let us denote

$$R_i(t) = \int_{t_0}^t r_i(s)\, ds, \quad \text{for } i = 1, 2.$$

Then from Corollary 1 we obtain:

Corollary 2. *Suppose that (6) holds. Equation (13) has property (B) if*

$$\liminf_{t\to\infty} R_2(t)\int_t^\infty r_3(s)\int_s^\infty \Big(R_1[g(x)]-R_1[g(s)]\Big)p(x)\,dx ds > \frac{1}{4}. \tag{14}$$

As a matter of fact we are able to relax condition of monotonicity imposed on the advanced argument in Theorem 2. Let us consider functional equation of the form (1) with larger advanced argument $Q(t)$, where $Q(t) : [t_0,\infty) \to \mathbb{R}$ is continuous.

Theorem 4. *Suppose that (6) holds and $Q(t) \ge g(t)$. Further assume that equations $(\tilde{E}_i)$ are oscillatory for $i = 1, 3, \cdots, n-1$ if n is odd and for $i = 2, 4, \cdots, n-1$ if n is even. Then the equation*

$$L_n u(t) - p(t)u[Q(t)] = 0$$

has property (B).

Proof. By Theorem 2 equation (11) has property (B). Our assertion now follows from Theorem 1 in [4]. □

Example 3. Let us consider the advanced equation

$$\left(\frac{1}{r_3(t)}\left(\frac{1}{r_2(t)}\left(\frac{1}{r_1(t)}u'(t)\right)'\right)'\right)' - p(t)u[2t+\cos t] = 0. \tag{15}$$

Letting $g(t) = 2t - 1$ and applying Corollary 1 and Theorem 4, one gets that equation (15) has property (B) if (6) holds and (14) is satisfied with $g(t) = 2t - 1$.

References

[1] Foster, K. E., Grimmer, R. C., *Nonoscillatory solutions of higher order differential equations*, J. Math. Anal. Appl. **71** (1979), 1-17.

[2] Hille, E., *Non-oscillation theorems*, Trans. Amer. Math. Soc. **64** (1948), 234-258.

[3] Kiguradze, I. T., *On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u|^n sign\, u = 0$*, Mat. Sb. **65** (1964), 172-187. (Russian)

[4] Kusano, T., Naito, M., *Comparison theorems for functional differential equations with deviating arguments*, J. Math. Soc. Japan **3** (1981), 509-532.

[5] Kusano, T., Naito, M., Tanaka, K., *Oscillatory and asymptotic behaviour of solutions of a class of linear ordinary differential equations*, Proc. Roy. Soc. Edinburg **90** (1981), 25-40.

[6] Trench, W. F., *Canonicals form and principal systems for general disconjugate equations*, Trans. Amer. Math. Soc. **189** (1974), 319-327.

Jozef Džurina
Department of Mathematical Analysis
Šafárik University
Jesenná 5
041 54 Košice, SLOVAKIA