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ON DIVICCARO, FISHER AND SESSA OPEN QUESTIONS

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Abstract. Let K be a closed convex subset of a complete convex metric space X and $T, I : K \to K$ two compatible mappings satisfying following contraction definition: $d^p(Tx, Ty) \leq \mathrm{a}d^p(Ix, Iy) + (1 - a)\max\{d^p(Ix.Tx),\ d^p(Iy, Ty)\}$ for all x, y in K, where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If I is continuous and $I(K)$ contains $\mathrm{Co}[T(K)]$, then T and I have a unique common fixed point in K and at this point T is continuous. This result gives affirmative answers to open questions set forth by Diviccaro, Fisher and Sessa in connection with necessarity of hypotheses of linearity and non-expansivity of I in their Theorem [3] and is a generalisation of that Theorem. Also this result generalizes theorems of Delbosco, Ferrero and Rossati [2], Fisher and Sessa [4], Gregus [5], G. Jungck [7] and Mukherjee and Verma [8]. Two examples are presented, one of which shows the generality of this result.

Introduction

Let X be a Banach space and C a closed convex subset of X. Generalizing Theorem of Gregus [5], Diviccaro, Fisher and Sessa [3] proved the following theorem:

Theorem A. *Let T and I be two weakly commuting mappings of C which satisfy the inequality*

$$(1) \quad \|Tx - Ty\|^p \leq a\|Ix - Iy\|^p + (1 - a)\max\{\|Tx - Ix\|^p,\ \|Ty - Iy\|^p\}$$

for all $x, y \in C$, where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If I is linear, non-expansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point and at this point T is continuous.

Diviccaro Fisher and Sessa [3, pp 88] pointed out that they do not know if their Theorem holds assuming I is continuous instead of non-expansive. Moreover, they also pointed out that it is not yet known if the hypothesis of the linearity of I is necessary in their Theorem.

Many theorems which are closely related to Gregus's Theorem have appeared in recend years ([1]-[4], [6]-[8]).

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In this paper we shall show that in Theorem A the hypothesis of non-expansivity of I can be replaced by continuity of I, and that some form of the hypothesis of linearity of I is necessary. Namely, we shall show that the hypothesis of linearity of I can be replaced by the much more general hypothesis that $I(C)$ contains $\mathrm{Co}[T(C)]$ (Co=convex hull), but as Example 2.1. below shows, this new hypothesis can not be omitted. So we give the affirmative answers to the above open questions.

We point out that the new hypothesis, which does not include the linearity of mapping enables to generalize the results of the type in Theorem A, from Banach space to more general setting of non-linear convex metric spaces. Also we shall relax the hypothesis of weak commutativity by compatibility of the two mappings.

1. Main result

Before stating the main result, we shall recall the following definitions.

Definition 1.1. (G. Jungck [6]). Self-maps T and I of a metric space (X, d) are *compatible* iff $\lim_n d(TIx_n, ITx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some t in X.

Note that T and I are weakly commuting, where T and I are self-maps of X, if $d(TIx, ITx) \leq d(Ix, Tx)$ for each $x \in X$. Clearly, commuting maps are weakly commuting and weakly commuting maps are compatible, but neither implication is reversible, as examples in [6] and [9] show.

Definition 1.2. (Takahashi [10]). Let X be a metric space and $I = [0, 1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a *convex structure* on X if for all x, y in X, λ in I, $d[u, W(x, y, \lambda)] \leq \lambda d(u, x) + (1-\lambda)d(u, y)$ for all u in X. X together with a convex structure is called a *convex metric space*.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x, y, \lambda) = \lambda x + (1-\lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x.0) + (1-\lambda)d(y, 0)$, then X is a convex metric space. There are many other examples but we consider these as paradigmatic.

Now we are in a position to state our main result.

Theorem 1.1. *Let K be a closed convex subset of a complete convex metric space X and $T, I : K \to K$ two compatible mappings satisfying the following condition:*

$$d^p(Tx, Ty) \leq \mathrm{ad}^{\mathrm{p}}(Ix, Iy) + (1-a)\max\{d^p(Ix, Tx), d^p(Iy, Ty)|\} \tag{2}$$

for all x, y in K, where $0 < a < 1/2^{p-1}$ and $p \geq 1$. If I is continuous and $T(K) \cup W[T(K) \times T(K) \times \{1/2\}] \subseteq I(K)$, where W is a convex structure on K, then T and I have a unique common fixed point in K at which T is continuous.

Proof. Let $x \in K$ be an arbitrary point. Then Ix and Tx are defined.
Choose points x_1, x_2, x_3 in K such that

$$Ix_1 = Tx,\ Ix_2 = Tx_1,\ Ix_3 = W(Tx_1, Tx_2, 1/2).$$

This choice can be done since $Tx, Tx_1, Tx_2, W(Tx_1, Tx_2, 1/2)$ are in $I(K)$.
From (2)

$$\begin{aligned} d^p(Ix_1, Ix_2) &= d^p(Tx, Tx_1) \\ &\leq \mathrm{ad}^p(Ix, Ix_1) + (1-a)\max\{d^p(Ix, Tx), d^p(Ix_1, Tx_1)\} \\ &= \mathrm{ad}^p(Ix, Ix_1) + (1-a)\max\{d^p(Ix, Ix_1), d^p(Ix_1, Ix_2)\}. \end{aligned}$$

Hence we have

$$d(Ix_1, Ix_2) \leq d(Ix, Ix_1). \tag{3}$$

From (2) and (3),

$$\begin{aligned} d^p(Ix_2, Tx_2) = d^p(Tx_1, Tx_2) &\leq \mathrm{ad}^p(Ix_1, Ix_2) \\ &+ (1-a)\max\{d^p(Ix_1, Tx_1), d^p(Ix_2, Tx_2)\} \leq \mathrm{ad}^p(Ix, Ix_1) \\ &+ (1-a)\max\{d^p(Ix, Ix_1), d^p(Ix_2, Tx_2)\} \end{aligned}$$

which implies

$$d(Ix_2, Tx_2) \leq d(Ix, Ix_1). \tag{4}$$

Using that $f(x) = x^p$ is increasing for $x \geq 0$, from (2) we have

$$\begin{aligned} d^p(Ix_1, Tx_2) &= d^p(Tx, Tx_2) \\ &\leq \mathrm{ad}^p(Ix, Ix_2) + (1-a)\max\{d^p(Ix, Tx), d^p(Ix_2, Tx_2)\} \\ &\leq a[d(Ix, Ix_1) + d(Ix_1, Ix_2)]^p \\ &\quad + (1-a)\max\{d^p(Ix, Ix_1), d^p(Ix_2, Tx_2)\} \end{aligned}$$

Hence, using (3) and (4), we have

$$d^p(Ix_1, Tx_2) \leq (2^p a + 1 - a) d^p(Ix, Ix_1). \tag{5}$$

Using Definition 2 and convexity of $f(x) = x^p$ $(p \geq 1)$ we have

$$\begin{aligned} d^p(Ix_1, Ix_3) &= d^p[Ix_1, W(Tx_1, Tx_2, 1/2)] \\ &\leq [1/2 \cdot d(Ix_1, Tx_1) + 1/2 \cdot d(Ix_1, Tx_2)]^p \\ &\leq 1/2 \cdot d^p(Ix_1, Ix_2) + 1/2 \cdot d^p(Ix_1, Tx_2) \end{aligned}$$

and hence, from (3) and (5),

$$d^p(Ix_1, Ix_3) \leq [1 + 2^{p-1}a(1 - 2^{-p})] d^p(Ix, Ix_1). \tag{6}$$

Since

$$d^p(Ix_2, Ix_3) = d^p[Ix_2, W(Tx_1, Tx_2, 1/2)] \leq [1/2 \cdot d(Ix_2, Ix_2) + 1/2 \cdot d(Ix_2, Tx_2)]^p,$$

by (4) we get

$$d(Ix_2, Ix_3) \leq 1/2 \cdot d(Ix, Ix_1). \tag{7}$$

Choose now $x_4 \in K$ such that $Ix_4 = Tx_3$. Then from (2), (3) and (4) we have

$$\begin{aligned} d^p(Ix_3, Ix_4) = d^p(Tx_3, Ix_3) &= d^p[Tx_3, W(Tx_1, Tx_2, 1/2)] \\ &\leq [1/2 \cdot d(Tx_1, Tx_3) + 1/2 \cdot d(Tx_2, Tx_3)]^p \\ &\leq 1/2 d^p(Tx_1, Tx_3) + 1/2 \cdot d^p(Tx_2, Tx_3) \\ &\leq 1/2[\mathrm{ad^p}(Ix_1, Ix_3) + (1-a)\max\{d^p(Ix_1, Ix_2), d^p(Ix_3, Ix_4)]\} \\ &\quad + 1/2[\mathrm{ad^p}(Ix_2, Ix_3) + (1-a)\max\{d^p(Ix_2, Tx_2), d^p(Ix_3, Ix_4)]\} \\ &\leq a/2 \cdot [d^p(Ix_1, Ix_3) + d^p(Ix_2, Ix_3)] \\ &\quad + (1-a)\max\{d^p(Ix, Ix_1), d^p(Ix_3, Ix_4)\}. \end{aligned}$$

Hence, using (6) and (7), we have

$$d^p(Ix_3, Ix_4) \leq \lambda^p \max\{d^p(Ix, Ix_1), d^p(Ix_3, Ix_4)\},$$

where $\lambda^p = a/2 \cdot [1 + 2^{p-1}a(1-2^{-p}) + 2^{-p}] + 1 - a$. Since $p \geq 1$ and $0 < a < 1/2^{p-1}$, we obtain

$$\lambda^p < a/2 \cdot [1 + (1 - 2^{-p}) + 2^{-p}] + 1 - a = 1\,.$$

Therefore,

$$d(Ix_3, Ix_4) \leq \lambda d(Ix, Ix_1) \qquad (0 < \lambda < 1). \tag{8}$$

Now we shall consider the sequence $\{Ix_n\}_{n=0}^{\infty}$ which possess the properties (3), (4), (7) and (8). i.e. the sequence defined as follows:

$$Ix_{3k+1} = Tx_{3k};\ Ix_{3k+2} = Tx_{3k+1};\ Ix_{3(k+1)} = W(Tx_{3k+1}, Tx_{3k+2}, 1/2), \\ (k = 0, 1, 2 \ldots).$$

It is easily shown by induction that form (8), (3) and (7) we have

$$\begin{aligned} d(Ix_{3k}, Ix_{3k+1}) &\leq \lambda d(Ix_{3(k-1)}, Ix_{3(k-1)+1}) \leq \cdots \leq \lambda^k d(Ix, Ix_1), \\ d(Ix_{3k+1}, Ix_{3k+2}) &\leq d(Ix_{3k}, Ix_{3k+1}) \leq \lambda^k d(Ix, Ix_1), \\ d(Ix_{3k+2}, Ix_{3(k+1)}) &\leq 1/2 \cdot d(Ix_{3k}, Ix_{3k+1}) \leq 1/2 \cdot \lambda^k d(Ix, Ix_1). \end{aligned} \tag{9}$$

Hence for $m > n > N$,

$$d(Ix_m, Ix_n) \leq \sum_{i=N}^{\infty} d(Ix_i, Ix_{i+1}) \leq 5/2 \cdot d(Ix, Ix_1)\lambda^{(N/3)}/(1-\lambda),$$

where $(N/3)$ means the greatest integer not exceeding $N/3$. Thus $\{Ix_n\}_{n=0}^{\infty}$, with $x_0 = x$, is a Cauchy sequence in K, hence convergent. Call the limit u.

Since $Tx_{3k} = Ix_{3k+1}, Tx_{3k+1} = Ix_{3k+2}$, from (4) and (9) we have

$$d(Tx_{3k+2}, Ix_{3k+2}) \leq d(Ix_{3k}, Ix_{3k+1}) \leq \lambda^k d(Ix, Ix_1).$$

Therefore,

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ix_n = u\,. \tag{10}$$

Then by continuity of I

$$\lim_{n\to\infty} ITx_n = \lim_{n\to\infty} IIx_n = Iu. \tag{11}$$

Since T and I are compatible, (10) implies

$$\lim d(ITx_n, TIx_n) = 0. \tag{12}$$

Using (11) and (12) we have $\lim\limits_{n\to\infty} TIx_n = Iu$. From (2),

$$d^p(TIx_n, Tu) \leq \mathrm{a}d^{\mathrm{p}}(IIx_n, Iu) + (1-a)\max\{d^p(IIx_n, TIx_n), d^p(Iu, Tu)\}.$$

Taking the limit as $n \to \infty$ we obtain $d^p(Iu, Tu)\} \leq a \cdot o + (1-a)$ max $\{0, d^p(Iu, Tu)\}$, which implies (as $a > o$) $d(Iu, Tu) = 0$. Hence $Tu = Iu$. Then by (2) we have

$$d^p(Tx_n, Tu) \leq \mathrm{a}d^{\mathrm{p}}(Ix_n, Iu) + (1-a)\max\{d^p(Ix_n, Tx_n), d^p(Iu, Tu)\}.$$

Taking the limit as $n \to \infty$ yields $d^p(u, Tu) \leq \mathrm{a}d^{\mathrm{p}}(u, Iu) = \mathrm{a}d^{\mathrm{p}}(u, Tu)$, which implies $Tu = u$. Therefore, we have $Tu = Iu = u$. Condition (2) ensures that u is the unique common fixed point of T and I.

Now assume that $\{u_n\}$ is a sequence in K with limit u. Using (2), we have

$$d^p(Tu_n, Tu) \leq \mathrm{a}d^{\mathrm{p}}(Iu_n, Iu) + (1-a)\max\{d^p(Iu_n, Tu_n), 0\},$$

and hence, as I is continuous, we note that

$$\lim_{n\to\infty} \sup d^p(Tu_n, Tu) \leq (1-a) \lim_{n\to\infty} \sup d^p(Tu, Tu_n).$$

Hence $\lim\limits_{n\to\infty} d(Tu_n, Tu) = 0$, as $a > 0$. Therefore, T is continuous at u. This completes the proof. $\square$

2. Corollaries and examples

Remark 2.1. The condition that $W[T(K) \times T(K) \times \{1/2\}]$ is contained in $I(K)$ is necessary in our Theorem 1.1. This shows the following example.

Example 2.1. Let X be the set of reals with the usual distance and $K = [0, 1]$. Define $T, I : K \to K$ as follows:

$$Tx = 1 \quad \text{for } 0 \le x \le 1/2 \quad \text{and } Tx = 0 \quad \text{for } 1/2 < x \le 1;$$
$$Ix = 0 \quad \text{for } 0 \le x \le 1/2 \quad \text{and } Ix = 1 \quad \text{for } 1/2 < x \le 1.$$

Then all the assumptions of our Theorem are trivially satisfied except that $W[T(K) \times T(K) \times 1/2] \subseteq E(K)$, but T and I do not have common fixed points.

The following consequence of Theorem 1.1 is an extension of Theorem A.

Corollary 2.1. *Let T and I be two compatible mappings of a closed convex subset C of Banach space satisfying (1) with $p \ge 1$ and $0 < a < 1/2^{p-1}$. If I is linear and continuous in C and $I(C)$ contains $T(C)$, then T and I have a unique common fixed point and at this point T is continuous.*

Proof. The linearity of I and the condition $T(C) \subseteq I(C)$ imply $W[T(C) \times T(C) \times [0, 1]] \subseteq I(C)$. □

Remark 2.2. Corollary 2.1 reduces to the main theorem of Jungck [7] in the case $p = 1$.

Remark 2.3. The following example shows that our Theorem 1.1 is a genuine generalization of theorems [3] [4], [7] and [8].

Example 2.2. Let $K = [0, 1]$ be the closed unit interval and $T, I : K \to K$ be defined by $Tx = x/4$ and $Ix = x^{1/2}$. Clearly $\mathrm{Co}[T(K)] \subseteq I(K,$ I is continuous and T and E are weakly commutative, hence compatible. As

$$d(Tx, Ty) = 1/4 \cdot |x - y| \le 1/4 \cdot |x - y| 2/(x^{1/2} + y^{1/2}) = 1/2 \cdot d(Ix, Iy)$$

for all $x, y \in K$, we conclude that all the hypotheses of Theorem 1.1 are satisfied and 0 is a unique common fixed point. But I is neither linear nor nonexpansive.

Corollary 2.2. *Let T and I be two compatible self-mappings of K satisfying*

$$d^p(Tx, Ty) \le \mathrm{ad}^p(Ix, Iy) + 2^{-p}(1 - a) \max\{d^p(Ix, Ty), d^p(Iy, Tx)\} \tag{13}$$

for all x, y in K, where $0 < a < 1/2^{p-1}$ and $p \ge 1$. If $Tx, Ty, W(Tx, Ty, 1/2) \in I(K)$ for all x, y in K, and I is continuous in K, then T and I have a unique common fixed point and at this point T is continuous.

Proof. Convexity of $x^p (p \ge 1)$ and inequality (13) imply

$$\begin{aligned} d^p(Tx, Ty) \le \mathrm{ad}^p(Ix, Iy) + 2^{-p}(1 - a) \max\{&2^p[1/2 \cdot d(Ix, Iy) + 1/2 \cdot d(Iy, Ty)]^p, \\ &2^p[1/2 \cdot d(Iy, Ix) + 1/2 \cdot d(Ix, Tx)]^p\} \\ \le (1 + a)/2 \cdot d^p(Ix, Iy) &+ (1 - a)/2 \cdot \max\{d^p(Ix, Tx), d^p(Iy, Ty)\} \end{aligned}$$

for all x, y in K. Since $(1 - a)/2 = 1 - (1 + a)/2$, the statement follows by Theorem 1.1. □

Corollary 2.3. *Let T be a mapping of K into itself satisfying*

$$d^p(Tx,Ty) \leq \mathrm{ad}^{\mathrm{p}}(x,y) + (1-a)\max\{d^p(x,Tx), d^p(y,Ty)\} \tag{14}$$

for all x, y in K, where $0 < a < 1/2^{p-1}$ and $p \geq 1$. Then T has a unique fixed point.

Corollary 2.4. *Let T be a mapping of K into itself satisfying*

$$d^p(Tx,Ty) \leq \mathrm{ad}^{\mathrm{p}}(x,y) + bd^p(x,Tx) + cd^p(y,Ty) \tag{15}$$

for all x, y in K, where $0 < a < 1/2^{p-1}$, $p \geq 1$, $b \geq 0$, $c \geq 0$ and $a+b+c=1$. Then T has a unique fixed point.

Proof. Due to the symmetry, it follows that if T satisfies (15), then it also satisfies

$$d^p(Tx,Ty) \leq \mathrm{ad}^{\mathrm{p}}(x,y) + \mathrm{hd}^p(x,Tx) + \mathrm{hd}^p(y,Ty) \tag{15'}$$

with the same a and $h = (b+c)/2$. Clearly, (15') and $a + 2h = 1$ imply (14). $\square$

Remark 2.4. We note that Corollary 2.4 reduces to Theorem 1.1 of Delbosco, Ferrero and Rossati [2] in the case that K is a closed convex subset of a Banach space X.

References

[1] Ćirić, Lj.B., *On a common fixed point theorem of a Greguš type*, Publ. Inst. Math. **49**(63) (1991), 174-178, Beograd.

[2] Delbosco, D., Ferrero, O., Rossati, F., *Teoreme di punto fisso per applicazioni negli spazi di Banach*, Boll. Un. Mat. Ital. (6) 2-A (1983), 297-303.

[3] Diviccaro, M. L., Fisher, B., Sessa, S., *A common fixed point theorem of Greguš type*, Publ. Math. Debrecen **34** (1987), No. 1-2.

[4] Fisher, B., Sessa, S., *On a fixed point theorem of Greguš*, Internat. J. Math. Math. **9** (1986), No. 1, 23-28.

[5] Greguš, M., *A fixed point theorem in Banach space*, Boll. Un. Mat. Ital. (5) 7-A (1980), 193-198.

[6] Jungck, G., *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), 771-779.

[7] Jungck, G., *On a fixed point theorem of Fisher and Sessa*, Internat. J. Math. Math. Sci **13** (1988), 497-500.

[8] Mukherjee, R. N., Verma, V., *A note on a fixed point theorem of Greguš*, Math. Japon. **33** (1988), 745-749.

[9] Sessa, S., *On a week commutativity condition in fixed point considerations*, Publ. Inst. Math. (Beograd) (N.S.) **32**(46) (1982), 149-153.
[10] Takahashi, W., *A convexity in metric space and nonexpansive mappings I*, Kodai Math. Sem. Rep. **22** (1970), 142-149.

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