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ON THE EQUIVALENCE OF VARIATIONAL PROBLEMS II

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Abstract. Elements of general theory of infinitely prolonged underdetermined systems of ordinary differential equations are outlined and applied to the equivalence of one-dimensional constrained variational integrals. The relevant infinite-dimensional variant of Cartan's moving frame method expressed in quite elementary terms proves to be surprisingly efficient in solution of particular equivalence problems, however, most of the principal questions of the general theory remains unanswered. New concepts of Poincaré-Cartan form and Euler-Lagrange system without Lagrange multiplies appearing as a mere by-product seem to be of independent interest in connection with the 23rd Hilbert problem.

After the previous part [3] exhibiting some advantages of a certain unorthodox approach to the equivalence of variational problems on examples, we pluck up the currage to outline our task in full generality. In the space of variables x, y^i_s $(i = 1, \dots, m;\ s = 0, 1, \dots)$, we have a *variational integral*

$$\int f(x, y^1_0, \dots, y^m_0, \dots, y^1_n, \dots, y^m_n)dx \rightarrow \text{ extremum } (y^i_s \equiv d^s y^i / dx^s) \tag{1}$$

considered on *admissible curves* $y^i_s \equiv y^i_s(x) = d^s y^i(x)/dx^s$ which satisfy an underdetermined system of differential equations (the *constraints*)

$$g^j(x, y^1_0, \dots, y^m_0, \dots, y^1_n, \dots, y^m_n) \equiv 0 \quad (j = 1, \dots, c;\ c < m)$$

and consequently its *infinite prolongation*

$$\partial^k g^j \equiv 0 \quad (j = 1, \dots, c;\ k = 0, 1, \dots; \partial = \partial/\partial x + \sum y^i_{s+1} \partial/\partial y^i_s). \tag{2}$$

On the other hand, we have analogous objects

$$\int F\, dx \rightarrow \text{ extremum }, \quad (\partial/\partial X + \sum Y^i_{s+1} \partial/\partial Y^i_s)^k G^j \equiv 0 \tag{3}$$

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in the copy-space of capital variables X, Y^i_s $(i = 1, \dots, m;\ s = 0, 1 \dots)$. Then the question can be raised whether an invertible transformation exists that changes the original data (1), (2) into the capital ones (3). However, this is *not* the true setting for two reasons. First, only the subspace of *admissible points* satisfying (2) and the capital counterpart are in reality important (and not their behaviour in the ambient space of all variables). Second, even on this subspace, a mere "conditional equivalence" of variational integrals is interesting since the arguments in f, F are of a special kind (see below).

In order to delete the first trouble, only the points satisfying (2) should be taken to constitute the correct underlying space. It is to be noted that the latter (infinite-dimensional) space endoved with the restrictions of the *contact forms* $\vartheta^i_s \equiv dy^i_s - y^i_{s+1}dx$ (the system $\vartheta^i_s \equiv 0$ serves for a coordinate-free transcription of the relations $y^i_s \equiv d^s y^i/dx^s$) can be characterized in abstract terms and we shall speak of a *diffiety*. In the correct setting of the problem, this diffiety is to be identified with the relevant "capital diffiety" by an invertible mapping between the new underlying spaces. In more detail, every (restriction of the) form ϑ^i_s is to be changed into (the restriction of) a linear combination of the capital contact forms $\Theta^j_s = dY^j_s - Y^j_{s+1}dX$.

As the second trouble is concerned, one can observe that if ξ is such a 1-form that the value of the integral $\int \xi$ is equal to $\int f\,dx$ for all admissible curves then $\xi = f\,dx + \Sigma a^i_s \vartheta^i_s$ (finite sum) for appropriate functions a^i_s. It follows that in the correct setting of equivalence problems, every such ξ is to be transformed into a certain capital counterpart $\Xi = F\,dX + \Sigma A^i_s \Theta^i_s$.

Continuing [3], our approach is rather elementary and avoids the common machinery of G-structures [6,9]. We try to find certain quite definite forms $\bar\xi = f\,dx + \Sigma \bar a^i_s \vartheta^i_s$, $\bar\omega^i_s \equiv \Sigma \bar a^{ij}_{sr} \vartheta^j_r$ (so called *specifications*) that constitute a coframe (the *Frenet coframe*) and can be intrinsically related to the given data. If is quite clear that they are changed into the relevant capital counterparts $\bar\Xi$, $\bar\Omega^i_s$ by the equivalence transformations (if the latter exist). In this sense, the equivalence problem in "in principle" resolved if the Frenet coframe is known. (In particular, a lot of other functions and differential forms which are corresponding to the relevant capital counterparts can be derived by the well-known methods, first of all from the developments of $d\bar\xi$, $d\bar\omega^i_s$ in terms of the Frenet coframe. We omit these investigations since they are of a purely technical nature.)

We shall also mention the *divergence equivalence problem* by assuming that a mere differential $d\xi$ is changed into $d\Xi$, that is, the above form ξ is transformed into $\Xi + dH$ (H is unknown in advance). On the contrary, there are *subordinated equivalences* if certain additional objects are selected for invariants in advance. From our point of view, the classical setting of equivalence is of the latter kind since it is developed in an apriori prescribed (finite-dimensional) space of variables.

Our reasonings will be carried out in real C^∞-smooth category near *generic points* where ranks of certain matrices are locally constant, submanifolds are embedded, certain functions do not change sign, various modules over the ring $\mathcal{F}$ of all C^∞-smooth functions have free bases which turn into bases of $\mathbf{R}$-linear spaces after taking the values at a point (the generalized *regularity* concept, cf. Section

1), etc. We shall not specify the definition domains. The common tools of classical analysis will be used in a somewhat unusual infinite-dimensional case but it does not cause any difficulties here (cf. however [4]). But the terminology and notation differ from the common usage and for this reason, the introductory part (Sections 1-9) should be followed with a certain care. The body of the paper is devoted to particular and self-contained examples of the equivalence problems, the concluding Sections briefly mention 23rd Hilbert problem and some related topics.

Ordinary differential equations

1. Some fundamental concepts, [4]. We shall deal with the space $\mathbf{R}^\infty$ of all infinite sequences $t^* = (t^1, t^2, \dots)$ of real numbers equipped with the ring $\mathcal{F}$ of all (real-valued and C^∞-smooth) functions $f = f(t^1, \dots, t^m)$, $m = m(f)$. Another coordinates $\bar{t}^* = (\bar{t}^1, \bar{t}^2, \dots)$ can be introduced by invertible formulae

$$\bar{t}^i \equiv f^i(t^1, \dots, t^{p(i)}), \quad t^i \equiv g^i(\bar{t}^1, \dots, \bar{t}^{q(i)}) \qquad (f^i, g^i \in \mathcal{F}) \tag{4}$$

which may be also regarded as an invertible transformation (often denoted by $t^i \to \bar{t}^i \equiv f^i$). Let Φ be the $\mathcal{F}$-module of all differential forms $\varphi = \Sigma f^i dg^i$ ($f^i, g^i \in \mathcal{F}$; finite sum). We shall deal with various submodules $\Psi \subset \Phi$. Then a (finite or infinite) family $\psi_1, \psi_2, \dots \in \Psi$ is called a *basis* of Ψ if every $\psi \in \Psi$ admits a unique representation $\psi = \Sigma g^i \psi_i$ ($g^i \in \mathcal{F}$, finite sum) and Ψ is a *regular* module if values of $\psi_1, \psi_2, \dots$ at a fixed point are linearly independent (over $\mathbf{R}$). The existence of a basis and the regularity will be tacitly supposed for all Ψ under consideration. By $\boldsymbol{\ell}(\Psi)$ we denote the *dimension* of Ψ, i.e., the number of elements of a basis of Ψ. The notation $\Psi = \{\varphi_1, \varphi_2, \dots\}$ signifies the *generators* of Ψ, i.e., Ψ consists of all forms $\varphi = \Sigma f^i \varphi_i$ ($f^i \in \mathcal{F}$, finite sum). Vector fields are expressed by infinite series $Z = \Sigma z^i \partial/\partial t^i$ and $Zf = \Sigma z^i \partial f/\partial t^i$ makes a good sense for every $f \in \mathcal{F}$. Here $z^i \equiv Zt^i \in \mathcal{F}$ can be (in principle) arbitrarily chosen. The common rules for Lie brackets and Lie derivatives $\mathcal{L}_Z = Z\rfloor d + dZ\rfloor$ are accepted. Denoting by $\Psi^\perp$ the module of all vector fields satisfying $\psi(Z) \equiv 0 (\psi \in \Psi)$, clearly $\Psi^{\perp\perp} = \Psi$ in the obvious sense.

2. Diffieties, [4]. Let $\Omega \subset \Phi$ be a submodule of *codimension one*, i.e., $\Omega^\perp$ is consisting of multiplies of a non-vanishing vector field ∂. Then $\mathcal{L}_\partial \Omega = \partial \rfloor d\Omega$, hence $\partial \rfloor \mathcal{L}_\partial \Omega = 0$, $\mathcal{L}_\partial \Omega \subset \Omega$. Assume that there is a filtration

$$\Omega^* : \dots \Omega^\ell \subset \Omega^{\ell+1} \subset \dots \subset \Omega = \cup \Omega^\ell \quad (\cap \Omega^\ell = 0)$$

by submodules satisfying $\boldsymbol{\ell}(\Omega^\ell) < \infty$,

$$\mathcal{L}_\partial \Omega^\ell \subset \Omega^{\ell+1} \text{ (all } \ell), \; \Omega^\ell + \mathcal{L}_\partial \Omega^\ell = \Omega^{\ell+1} \quad (\ell \text{ large enough}). \tag{5}$$

Then Ω is called a *diffiety*. In practice, it is sufficient to determine appropriate forms $\omega^1, \dots, \omega^c \in \Omega$ such that the family of all forms of the kind $\mathcal{L}_\partial^k \omega^j$ ($j = 1, \dots, c$; $k = 0, 1, \dots$) generates Ω and put $\Omega^\ell = 0$ ($\ell < 0$), $\Omega^\ell =$ the module generated by all $\mathcal{L}_\partial^k \omega^j$ where $k \le \ell$.

3. Normal filtrations. On the graded module Grad $\Omega^* = \oplus\Omega^{\ell+1}/\Omega^\ell$, the operator $\mathcal{L}_\partial$ induces a homomorphism ∂ defined by

$$\partial[\omega] = [\mathcal{L}_\partial\omega] \in \Omega^{\ell+1}/\Omega^\ell \quad (\omega \in \Omega^\ell, [\omega] \in \Omega^\ell/\Omega^{\ell-1});$$

the square brackets denote the classes. We speak of a *normal filtration* Ω^* if (besides (5)) also $\Omega^\ell \equiv 0$ $(\ell < -1)$ and

$$\mathcal{L}_\partial\Omega^{-1} \subset \Omega^{-1},\ \partial : \Omega^\ell/\Omega^{\ell-1} \to \Omega^{\ell+1}/\Omega^\ell \text{ is injective if } \ell \geq 0 \tag{6}$$

Such a filtration can be obtained after an appropriate change of lower order terms of any filtration Ω^*. In fact, clearly $\partial : \Omega^\ell/\Omega^{\ell-1} \to \Omega^{\ell+1}/\Omega^\ell$ is bijective for all ℓ large enough, say, for $\ell \geq L$. One may then denote $\bar\Omega^\ell \equiv \Omega^\ell$ $(\ell \geq L)$ and inductively put

$$\bar\Omega^{k-1} = \text{ kernel of the composition } \bar\Omega^k \to \bar\Omega^{k+1} \to \bar\Omega^{k+1}/\bar\Omega^k, \tag{7}$$

where $\mathcal{L}_\partial$ is composed with factorization. Clearly $\bar\Omega^L \supset \bar\Omega^{L-1} \supset \dots$ hence $\bar\Omega^K = \bar\Omega^{K-1} = \dots$ for a certain $K \leq L$. It follows $\mathcal{L}_\partial\bar\Omega^K = \bar\Omega^K$ and $\partial : \bar\Omega^\ell/\bar\Omega^{\ell-1} \to \bar\Omega^{\ell+1}/\bar\Omega^\ell$ is injective if $\ell > K$. Thus a mere shift of indices for the dashed filtration yields the desired result.

Let Ω^* be a normal filtration. Let $\omega^j_{(r)}$ $(j = 1, \dots, j_r$; we admit $j_r = 0)$ be such forms that their classes provide a basis of $\Omega^r/(\Omega^{r-1} + \mathcal{L}_\partial\Omega^{r-1})$. Then, for every $s \geq 0$, the classes of forms $\mathcal{L}^k_\partial\omega^j_{(r)}$ with $k + r = s$, $j = 1, \dots, j_r$ provide a basis of Ω^s/Ω^{s-1}. So, denoting

$$\omega^i_{k+r} = \mathcal{L}^k_\partial\omega^j_{(r)} \in \Omega^{k+r} \quad (i = j_0 + \dots + j_r + j; j = 1, \dots, j_r),$$

we have $\mathcal{L}_\partial\omega^i_s \equiv \omega^i_{s+1}$ and the classes of forms ω^j_s with a fixed s provide a basis of Ω^s/Ω^{s-1}. All ω^j_s provide a basis of Ω/Ω^{-1}.

As the term Ω^{-1} of a normal filtration is concerned, it is plain that it consists of all forms $\omega \in \Omega$ such that the family $\mathcal{L}^k_\partial\omega$ $(k = 0, 1, \dots)$ is contained in a finite-dimensional submodule of Ω (e.g., in Ω^{-1}). It follows that $\Omega^{-1} = \mathcal{R}(\Omega)$ is independent of the choice of the filtration. It may be proved that Ω^{-1} is *completely integrable*, i.e., it has a basis consisting of total differentials. (See [5] for a conceptual proof but the direct approach using the above basis ω^i_s is quite easy and therefore omitted here. It is to be noted that $\mathcal{R}(\Omega)$ can be related to the concept of "accessible points" of the optimal regulation theory.)

The above forms $\omega^j_{(r)}$ will be called the *initial* ones. Their total number $\mu(\Omega) = \Sigma j_r$ (a finite sum, cf. (5)) is independent of the choice of the filtration. In fact, if $\Omega^*, \bar\Omega^*$ are two filtrations of Ω satisfying (5), then $\Omega^\ell \subset \bar\Omega^K \subset \Omega^L$ for appropriate $K = K(\ell)$, $L = L(\ell)$. For ℓ large enough, we obtain $\Omega^{\ell+k} \subset \bar\Omega^{K+k} \subset \Omega^{L+k}$ by applying $\mathcal{L}^k_\partial$. If we deal with normal filtrations, then $\boldsymbol{\ell}(\Omega^\ell) \equiv \mu\ell + m$, $\boldsymbol{\ell}(\bar\Omega^\ell) \equiv \bar\mu\ell + \bar m$ where μ $(\bar\mu)$ is the number of initial forms for the filtration $\Omega^*(\bar\Omega^*)$ and $m, \bar m$ are constants. It follows

$$\mu(\ell + k) + m \leq \bar\mu(K + k) + \bar m \leq \mu(L + k) + m \quad (k = 0, 1, \dots)$$

which implies $\mu = \bar\mu$. (Since every filtration Ω^* satisfying (5) is identical with a normal one for all higher order terms, clearly $\boldsymbol{\ell}(\Omega^\ell) = \mu(\Omega)\ell +$ const. is valid, too.)

4. A particular case. If $\mu(\Omega)=1$, we have only one initial form $\omega^1_{(0)}$ uniquely determined up to a nonvanishing factor from $\mathcal{F}$ and a summand from $\mathcal{R}(\Omega)$. This form has a typical property (in the family of all forms of Ω) namely that the forms $\omega^1_k \equiv \mathcal{L}^k_\partial \omega^1_{(0)}$ $(k=0,1,\dots)$ provide a basis of $\Omega/\mathcal{R}(\Omega)$. It follows that $\omega^1_{(0)}$ does not depend on the choice of the normal filtration or, in equivalent terms, the *normal filtration Ω^* of Ω is unique*. (Consequently, for the case $\mu=1$, the normal filtrations of various diffieties are changed one into the other by the equivalence transformations. The fact was stated in [3] without proof and expressed by the phrase "the order of derivatives is preserved".)

5. Example. The diffieties may be regarded as abstract substitutes for the infinitely prolonged underdetermined systems of ordinary differential equations when a definite choice of dependent and independent variables is not appointed. Instead of (a simple but lengthy) discussion of this principle (which can be carried over to partial differential equations, cf. [4,10] and the next Part III), we shall present some illustrative examples.

Denoting rather by $x,z,y_0,y_1,\dots$ the coordinates in $\mathbf{R}^\infty$, we introduce the submodule $\Omega\subset\Phi$ generated by the forms $\zeta = dz - g\,dx$ $(g=g(x,z,y_0,\dots,y_k)\in\mathcal{F}$ is given), $\vartheta_s \equiv dy_s - y_{s+1}dx$ $(s=0,1,\dots)$. Clearly $\Omega^\perp$ consists of all multiples of $\partial = \partial/\partial x + g\partial/\partial z + \Sigma y_{s+1}\partial/\partial y_s$. Since

$$\mathcal{L}_\partial\zeta = dg - \partial g\,dx = g_z\zeta + \Sigma g_s\vartheta_s, \quad \mathcal{L}_\partial\vartheta_s \equiv \vartheta_{s+1} \tag{8}$$

$(g_s \equiv \partial g/\partial y_s)$, we have a diffiety with the filtration consisting of the terms

$$\Omega^\ell \equiv \{\vartheta_0,\dots,\vartheta_{\ell-1}\}\ (\ell\le k),\ \Omega^\ell \equiv \{\zeta,\vartheta_0,\dots,\vartheta_{\ell-1}\}\ (\ell>k).$$

The diffiety Ω (or better, the Pfaff's system $\omega\equiv 0$ $(\omega\in\Omega)$) represents the infinite prolongation of the equation $dz/dx = g(x,z,y,\dots,d^ky/dx^k)$ in the obvious sense: the variables y_s stand for the derivatives d^sy/dx^s and the derivatives $d^sz/dx^s \equiv \partial^{s-1}g$ $(s\ge 1)$ need not be adjoint to the coordinates since they are expressed by other variables.

Assume $k\ge 1$ and $g_k\ne 0$. Then the classes $[\zeta],[\vartheta_{k-1}]\in\Omega^k/\Omega^{k-1}$ satisfy $\partial[\zeta] = g_k\cdot[\vartheta_k]$, $\partial[\vartheta_{k-1}] = [\vartheta_k]$ (cf. (8)), hence $\partial[\zeta - g_k\zeta_{k-1}] = 0\in\Omega^{k+1}/\Omega^k$ and Ω^* is not a normal filtration. But one may put $\bar\Omega^\ell\equiv\Omega^\ell$ $(\ell>k)$ and apply (7) to obtain the desired terms $\bar\Omega^k,\bar\Omega^{k-1},\dots$ of the normal improvement $\bar\Omega^*$. After some calculations (cf. [3, Section 7]), the final result is as follows. If we denote

$$\begin{aligned}&b_i \equiv g_{i+1} + (g_z-\partial)g_{i+2} + \dots + (g_z-\partial)^{k-i-1}g_k \quad (i=k-1,\dots,-1),\\ &\zeta_0 = \zeta - b_{k-1}\vartheta_{k-1} - \dots - b_0\vartheta_0,\end{aligned}$$

then $\mathcal{L}_\partial\zeta_0 = g_z\zeta_0 + b_{-1}\vartheta_0$ and two subcases are to be distinguished. If $b_{-1}\ne 0$, then we may put

$$\bar\Omega^\ell\equiv 0\ (\ell<0),\ \bar\Omega^0 = \{\zeta_0\},\ \bar\Omega^\ell = \{\zeta_0,\vartheta_0,\dots,\vartheta_{\ell-1}\}\ (\ell\ge 1) \tag{9}$$

with the initial form $\omega^1_{(0)} = \zeta_0$. If $b_{-1} = 0$, we may put

$$\bar{\Omega}^\ell \equiv 0(\ell < -1), \bar{\Omega}^{-1} = \{\zeta_0\}, \bar{\Omega}^\ell = \{\zeta_0, \vartheta_0, \ldots, \vartheta_\ell\}(\ell \geq 0) \tag{10}$$

with the initial form $\omega^1_{(0)} = \vartheta_0$. (In the latter case clearly $\partial \rfloor d\zeta_0 = \mathcal{L}_\partial \zeta_0 = g_z \zeta_0$ implies $d\zeta_0 \cong g_z dx \wedge \zeta_0$ (modulo all ϑ_s) and analysing the identity $d^2\zeta_0 = 0$, one can even obtain $d\zeta_0 \cong 0$ (modulo ζ_0). It follows by Frobenius theorem that ζ_0 is a multiple of total differential.) Since $\mu(\Omega) = 1$, other normal filtrations do not exist.

6. Example. Denoting by x, y^i_s ($i = 1, \ldots, m$; $s = 0, 1, \ldots$) the coordinates in $\mathbf{R}^\infty$, we introduce the submodule $\Omega \subset \Phi$ generated by all contact forms $\vartheta^i_s \equiv dy^i_s - y^i_{s+1}dx$. Clearly $\Omega^\perp$ consists of all multiples of the vector field $\partial = \partial/\partial x + \Sigma y^i_{s+1}\partial/\partial y^i_s$. The submodules $\Omega^\ell \equiv 0$ ($\ell < 0$), Ω^ℓ ($\ell \geq 0$) generated by all forms $\mathcal{L}^k_\partial \vartheta^i = \vartheta^i_k$ ($k \leq \ell$) provide a normal filtration of Ω. This diffiety Ω represents the empty system of differential equations for the functions $y^1(x), \ldots, y^m(x)$. It is well-known as the one dimensional "infinite order jet space", cf. [10]. However, *an important remark is to be pointed out:* neither the choice of coordinates x, y^i_s, nor the choice of the basis ϑ^j_s, nor the above mentioned normal filtration Ω^* are of intrinsical sense from our point of view. Only the submodule $\Omega \subset \Phi$ is the true given object.

7. Variational problems. Returning to general theory and the notation of Sections 1-4, let $\lambda \in \Phi$ be a given form (the *Lagrange density*). We introduce the (*constrained*) *variational integral*

$$\int p^*\lambda \rightarrow \text{ extremum }, p^*\omega \equiv 0 \quad (\omega \in \Omega), \tag{11}$$

where $p : t^i \equiv t^i(\tau)$, $a \leq \tau \leq b$, is ranging over the family of curves in the underlying space (and the diffiety Ω realizes the constraints). Following some reasonable arguments [5,7], such a curve p is called an *extremal* if $p^*\omega \equiv 0$ ($\omega \in \Omega$) and moreover

$$p^* Z \rfloor d(\lambda + \bar{\omega}) = 0 \tag{12}$$

for an appropriate form $\bar{\omega} \in \Omega$ and all vector fields Z. In principle, this form $\bar{\omega}$ may depend on the choice of p. In practice, it can be selected from a certain finite-dimensional submodule of Ω, that is, it depends on some auxiliary variables (the *phase variables* in the terminology of [7]). In this approach, (12) is equivalent to the common Euler-Lagrange system *with* Lagrange multiplies. Beyond all expectation, the auxility variables can be completely eliminated.

8. Theorem. *To a given $\lambda \in \Phi$, there is a universal form $\bar{\omega} \in \Omega$ such that (12) is valid for all extremals p.* (Then $\lambda + \bar{\omega}$ may be called the *Poincaré-Cartan form* and $e^j_{(r)}$ appearing in (14) may be regarded as *Euler-Lagrange operators*, cf. [5].)

Proof. Let Ω^* be a fixed normal filtration of Ω. Recall the relevant initial forms $\omega^j_{(r)}$, the special basis ω^i_s satisfying $\mathcal{L}_\partial \omega^i_s \equiv \omega^i_{s+1}$ of $\Omega/\mathcal{R}(\Omega)$, and choose a basis $dh^1, \dots, dh^c$ of $\mathcal{R}(\Omega)$. Let $x \in \mathcal{F}$ satisfies $dx \notin \Omega$. We may normalize the vector $\partial \in \Omega^\perp$ by $\partial x = 1$ and then $df \cong \partial f \, dx$ (modulo Ω) for every $f \in \Omega$.

Assume that a curve p is *nearly* an extremal in the sense that (12) is satisfied in the following weakened sense: $p^*\omega \equiv (\omega \in \Omega)$ and there is a form $\bar\omega \in \Omega$ such that

$$d(\lambda + \bar\omega) \cong \Sigma a^i_s \omega^i_s \wedge dx \qquad (\text{modulo } \mathcal{R}(\Omega) \text{ and } \Omega \wedge \Omega) \tag{13}$$

with $p^* a^i_s \equiv 0$. (Roughly saying, (12) is satisfied modulo $\mathcal{R}(\Omega)$.) Assuming $p^*dx \neq 0$ (the other case is trivial), clearly $0 = p^* da^i_s = p^*\partial a^i_s \cdot p^*dx$, $p^*\partial a^i_s = 0$ and thus $p^*\partial^k a^i_s \equiv 0$ for all k. Consequently, if $a^j_r \omega^j_r$ is a particular summand in (12) with the form $\omega^j_r = \mathcal{L}_\partial \omega^j_{r-1}$ *not* an initial one, then $d\omega^j_{r-1} \cong dx \wedge \omega^j_r$ (modulo $\Omega \wedge \Omega$) and thus

$$p^* Z \rfloor d(a^j_r \omega^j_{r-1}) = p^* Z \rfloor (\partial a^j_r \cdot dx \wedge \omega^j_{r-1} + a^j_r dx \wedge \omega^j_r) = 0\,.$$

It follows that the original form $\bar\omega$ in (12) can be replaced by $\bar\omega + a^j_r \omega^j_{r-1}$. Then (12) remains true but the summand $a^j_r \omega^j_r$ in (13) turns into a lower order term $\partial a^j_r \omega^j_{r-1}$. Repeatedly applying this reduction, such modified form $\bar\omega$ appears that only the initial form survive in the resulting relation (13):

$$d(\lambda + \bar\omega) \cong \Sigma e^j_{(r)} \omega^j_{(r)} \wedge dx \quad (\text{modulo } \mathcal{R}(\Omega) \text{ and } \Omega \wedge \Omega). \tag{14}$$

Recall that $p^* e^j_{(r)} \equiv 0$ for our extremal p (since we have made a mere change of notation: $e^j_{(r)}$ stand for the previous a^j_r). But the point lies in the (easily verifiable) fact that the form $\bar\omega$ satisfying the congruence (14) is unique modulo $\mathcal{R}(\Omega)$. So it follows that $p^* e^j_r \equiv 0$ for *all* curves which are nearly extremals.

At last, we shall prove that a nearly extremal p is in reality the *true* extremal. For this aim, let $\bar\omega = \Sigma b^i_s \omega^i_s + \Sigma b^j dh^j$ satisfies (14), where $b^i_s \in \mathcal{F}$ are uniquely determined but $b^j \in \mathcal{F}$ may be (as yet) arbitrary. Let $d\lambda \cong \Sigma c^j dh^j \wedge dx$ (modulo all ω^i_s). Since (14) with $p^* e^j_{(r)} \equiv 0$ and $p^*\omega \equiv 0$ ($\omega \in \Omega$, especially $p^* dh^j \equiv 0$) are valid, the original requirement (12) simplifies into

$$p^* Z \rfloor d(\lambda + \bar\omega) = p^* \Sigma (c^j + \partial b^j) Z h^j \cdot p^* dx = 0.$$

But $p^*(c^j + \partial b^j) \equiv 0$ can be always satisfied by a choice proper of b^j. This concludes the proof. $\square$

9. Example. Returning to Section 5, let $\lambda = f\,dx$ ($f = f(x, z, y_0, \dots, y_m)dx$) and assume $\bar\omega = c\zeta_0 + c^0\vartheta_0 + \dots + c^n\vartheta_n$. Then

$$\begin{aligned} d(\lambda + \bar\omega) = (df - c(g_z\zeta_0 + b_{-1}\vartheta_0) - \Sigma c^i \vartheta_{i+1}) \wedge dx + \\ + dc \wedge \zeta_0 + dc^i \wedge \vartheta_i\,, \end{aligned}$$

where df, dc, dc^i may be developed by using the general formula

$$dh = \partial h\, dx + h_x\zeta_0 + (h_0 + b_0)\vartheta_0 + \cdots + (h_{k-1} + b_{k-1})\vartheta_{k-1} + h_k\vartheta_k + \ldots$$

(here $h_s \equiv \partial h/\partial y_s$). Assuming $b_{-1} \neq 0$, the Poincaré-Cartan form is determined by the recurrent formula

$$c^i = 0\ (i \geq n = \max(m, k)),\ \ c^{n-1} = f_{n-1} + b_{n-1},$$
$$c^i \equiv f_{i+1} + b_{i+1} - \partial c^{i+1}\ (i = n-2, \ldots, 0), c = (f_0 + b_0 - \partial c^0)/b_{-1}$$

and the Euler-Lagrange operator is $e = f_z - cg_z - \partial c$ (cf. [3, Section 7]). On the contrary, assuming $b_{-1} = 0$, the Poincaré-Cartan form is determined by the same recurrences but without the last formula for c (which remains quite arbitrary). The Euler-Lagrange operator is $e = f_0 + b_0 - \partial c^0$.

A test example for equivalence

10. A particular problem. Before passing to more difficult problems, we should like to demonstrate various aspects of equivalence on a relatively simple example. So we shall discuss the constrained variational integral

$$\int f\, dx + g\, dy \to \text{ extremum },\ dz = p\, dx + q\, dy, \tag{15}$$

where f, g, p, q are functions of x, y, z. Choosing x for independent variable, the symmetry is lost and the integral can be equivalently expressed by

$$\int (f + g\, dy/dx)dx \to \text{ extremum },\ dz/dx = p + q\, dy/dx.$$

Turning to diffieties, we introduce the variables $x, z, y = y_0, y_1, y_2, \ldots,$ and the module Ω generated by the forms $\zeta = dz - p\, dx - q\, dy$, $\eta_s \equiv dy_s - y_{s+1}dx$ ($s = 0, 1, \ldots$). Moreover we have $\lambda = f\, dx + g\, dy$. Clearly $\partial = \partial/\partial x + (p + qy_1)\partial/\partial z + \Sigma y_{s+1}\partial/\partial y_s \in \Omega^\perp$. The formula

$$dh = \partial h\, dx + h_z(\zeta + q\eta_0) + \Sigma h_s\eta_s \tag{16}$$

easily yields

$$d\zeta = dx \wedge ((p_z + q_z y_1)\zeta + a\eta_0) + q_z\eta_0 \wedge \zeta \tag{17}$$

where $a = p_z q - q_z p + p_0 - q_z$. Two cases are to be distinguished. If $a \neq 0$ then the normal filtration is

$$\Omega^\ell \equiv 0\ (\ell < 0),\ \Omega^\ell \equiv \{\zeta, \eta_o, \ldots, \eta_{\ell-1}\}\ (\ell \geq 0)$$

with $\omega^1_{(0)} = \zeta$ the initial form. If $a = 0$, then we put

$$\Omega^\ell \equiv 0(\ell < -1), \Omega^{-1} = \{\zeta\}, \Omega^\ell \equiv \{\zeta, \eta_0, \dots, \eta_\ell\}(\ell \geq 0)$$

with the initial form $\omega^1_{(0)} = \eta_0$. In the latter case, clearly $d\zeta \cong 0$ (modulo ζ) hence ζ is a multiple of a total differential dh ($h = h(x, y, z)$) and can be even replaced by dh in the latter filtration.

Since $\mu(\Omega) = 1$, the normal filtrations are unique and we have the intrinsical families of forms

$$\omega_s = a^0_s \zeta + a^1_s \eta_0 + \cdots + a^s_s \eta_{s-1} \quad (a^s_s \neq 0) \tag{18}$$

with varying coefficients $a^i_s \in \mathcal{F}$, for every $s = 0, 1, \dots$. We have moreover the intrisical family of forms

$$\xi = f\, dz + g\, dy + c^0\zeta + c^1\eta_0 + \cdots + c^n\eta_{n-1} \tag{19}$$

with varying $c^i \in \mathcal{F}$ of undetermined length n. Assuming either $f \neq 0$ or $g \neq 0$, the vector field $D = \partial/(f + gy_1)$ defined by the properties $D \in \Omega^\perp$, $\xi(D) = 1$ is intrinsical, too.

Turn to the specifications. If $n \geq 1$, then the obvious congruence $d\xi \cong c^n dx \wedge \eta_n$ (modulo $\omega_0, \dots, \omega_{n-1}$ and $\Omega \wedge \Omega$) clearly permits to assume $\bar{c}^n = 0$. Continuing in this way, we obtain the intrinsical specifications $\bar{c}^1 = \bar{c}^2 = \cdots = 0$. Then the use of (16), (17) yields

$$d\xi = (e\zeta + e'\eta_0) \wedge dx + b\eta_0 \wedge \zeta \quad (b = q_z - g_z + (q\frac{\partial}{\partial z} + \frac{\partial}{\partial y_0})c^0), \tag{20}$$

where

$$e = f_z + g_z y_1 - c^0(p_z + q_z y_1) - \partial c^0, \quad e' = f_z q + f_0 - g_z p - g_x - c^0 a.$$

At this place, two cases are to be distinguished.

11. Continuation. if $a \neq 0$. One can then introduce the intrinsical requirement $d\xi \cong 0$ (modulo $\omega_0, \Omega \wedge \Omega$), that is, $e' = 0$. This yields the specification $\bar{c}^0 = (f_z q + f_0 - g_z p - g_x)/a$, the Poincaré-Cartan form

$$\bar{\xi} = f\, dx + g\, dy + \bar{c}^0\zeta = (f + gy_1)dx + \bar{c}^0\zeta + g\eta_0,$$

and the Euler-Lagrange operator $\bar{e} = f_z + g_z y_1 - \bar{c}^0(p_z + q_z y_1) - \partial\bar{c}^0$. With the use of (18) we have

$$d\bar{\xi} = \frac{\bar{e}}{(f + gy_1)a^0_0}\omega_0 \wedge \bar{\xi} + I\omega_1 \wedge \omega_0, \; I = \left(\frac{\bar{e}g}{f + gy_1} + b\right)/a^1_1 a^0_0,$$

and assuming $\bar{e} \neq 0$ (the subcase $\bar{e} = 0$ is delayed for a moment), we may intrinsically specify $\bar{a}_0^0 = \bar{e}/(f + gy_1)$. This determines $\bar{\omega}_0$ and thus the remaining specifications $\bar{\omega}_s \equiv \mathcal{L}_D^s \bar{\omega}_0$. The latter formula can be expressed by the recurrence

$$\bar{a}_{s+1}^0 \equiv (p_z + q_z y_1 + \partial)\bar{a}_s^0/(f + gy_1), \bar{a}_{s+1}^1 \equiv (a\bar{a}_s^0 + \partial\bar{a}_s^1)/(f + gy_1),$$
$$\bar{a}_{s+1}^j \equiv (\bar{a}_s^{j-1} + \partial\bar{a}_s^j)/(f + gy_1) \ (j \geq 2)$$

(use (16), (17)). The Frenet coframe $\bar{\xi}, \bar{\omega}_0, \bar{\omega}_1, \dots$ is determined.

Reduction to the finite-dimensional space of variables x, y, z (that is, to the classical setting of the equivalence problem, cf. [3]) can be achieved by the use of the function I (or better: by the specification $\bar{I}$ with $\bar{a}_0^0, \bar{a}_1^1$ inserted). Alas, a complete discussion of a large number of subcases which may in principle happen seems to be not appropriate here. To outline the most essential step, we mention the formula

$$\bar{I} = (u + vy_1)(f + gy_1)^2/a\bar{e}^2, \quad v = gq_z(1 - \bar{c}^0)$$

where $u = u(x, y, z)$ is a certain function (not explicitly stated here). Assuming $v \neq 0$ (and moreover $vf \neq ug$ to ensure the finiteness of $\bar{a}_0^0$, see below), the intrinsical requirement $\bar{I} = 0$ permits to employ the reduction $y_1 = -u/v$ (hence $\bar{a}_0^0 = \bar{e}v/(vf - ug)$) and thus $y_{1+k} \equiv -D^k(u/v)$ for all $k \geq 0$. (The latter equations determine a three-dimensional submanifold with coordinates x, y, z and restrictions of $\bar{\xi}, \bar{\omega}_0, \bar{\omega}_1$ on it determine the sought classical transcription of the equivalence problem.) The equality $v = 0$ is realized in three subcases $g = 0$, $q_z = 0$, $\bar{c}^0 = 1$. Then we may advantageously use the requirement $\bar{I} = \pm 1$. If $\bar{e}$ is depending on y_1 (i.e., if $\partial\bar{e}/\partial y_1 = -(g + \bar{c}^0 q)_z - \partial\bar{c}^0/\partial y_0 \neq 0$), we obtain a quadratic equation for y_1, quite analogously as in [3, Section 2].

12. Continuation. if $a = 0$. We know that then ζ can be replaced by a complete differential dh (a multiple od ζ). Since $h_z \neq 0$, we may even assume $h = z$ (i.e., $p = q = 0$) by a mere change of variables. If $e' = f_0 - g_x \neq 0$ (the subcase $e' = 0$ is much easier and may be omitted), the form $f\,dx + g\,dy$ can be transformed into $x\,dx$ by an appropriate change of variables $x \to u(x, y)$, $y \to v(x, y, z)$, $z \to z$. It follows that all these variational problems are equivalent.

On this occasion, let us briefly mention the case $a \neq 0$ but $\bar{e} = 0$. One can then see that $d\bar{\xi} = 0$ (look at $d^2\bar{\xi} = 0$), hence $\bar{\xi} = dh$ for an appropriate function $h = h(x, y, z)$. After a change of variables, we may assume $\zeta = dz - y\,dx$. After an additional change of independent variable, we may even assume $h = x$. So it follows that all variational problems of this kind are mutually equivalent, too.

13. The divergence equivalence. for the variational integral (15) is based on the same intrinsical families (18) as above, however, instead of the family (19) we may employ only its exterior differential $d\xi$. It follows that at the beginning, the specification procedure runs exactly as in Section 10 and we obtain the formula (20). Assuming moreover $a \neq 0$ (the case $a = 0$ is trivial, cf. Section 12), we can even use the same specification $\bar{c}^0$ ensuring $e' = 0$ (cf. Section 11) so that we obtain the intrinsical 2-form

$$\overline{d\xi} = \bar{e}\zeta \wedge dx + b\eta_0 \wedge \zeta = \omega_0 \wedge (\bar{e}\,dx - b\eta_0)/a_0^0$$

with $a_0^0 \in \mathcal{F}$ variable. It follows that the family of forms

$$\varepsilon = (\bar{e}\, dx + c\zeta - b\eta_0)/a_0^0 \quad (c, a_0^0 \in \mathcal{F} \text{ are varying})$$

is of the intrinsical nature. It may be used for a convenient substitute for the previous form ξ in the sought Frenet coframe and in the following procedure of specification. So (assuming $\bar{e} \neq 0$) one can see that

$$d\omega_0 = da_0^0 \wedge \zeta + a_0^0 d\zeta \cong \frac{a(a_0^0)^2}{\bar{e} a_1^1} \varepsilon \wedge \omega_1 \text{ (modulo } \varepsilon \wedge \omega_0, \Omega \wedge \Omega)$$

and we may introduce the intrinsical relation $a_1^1 = a(a_0^0)^2/\bar{e}$. Quite analogously

$$d\omega_1 \cong a_1^1 d\eta_0 = a_1^1 dx \wedge \eta_1 = \frac{a_1^1}{\bar{e}}(a_0^0 \varepsilon + \frac{b}{a_1^1}\omega_1) \wedge \frac{\omega_2}{a_2^2}$$

and we may introduce the intrinsical relations

$$a_1^1 a_0^0 / \bar{e} a_2^2 = 1 \quad b/\bar{e} a_2^2 = 1.$$

Altogether $b/\bar{e} = a_2^2 = a_1^1 a_0^0/\bar{e} = a(a_0^0)^3/\bar{e}^2$ whence $\bar{a}_0^0 = (b\bar{e}/a)^{1/3}$. We have tacitly supposed $b \neq 0$. (One can observe that the unpleasant case $b = 0$ is a highly degenerate one. In this case $d\bar{\xi} = \zeta \wedge \bar{e}\, dx$, hence $0 = d^2\bar{\xi} = d\bar{e} \wedge \zeta \wedge dx + \bar{e} d\zeta \wedge dx$, which implies in particular that $\bar{e}$ is not depending on y_1. We shall not deal with it.)

At this stage, we know the intrinsical vector field $D = \bar{a}_0^0/\bar{e} \cdot \partial$ (determined by the requirements $D \in \Omega^\perp$, $\varepsilon(D) = 1$), the intrinsical form $\bar{\omega}_0 = \bar{a}_0^0 \zeta$, and thus the intrinsical sequence $\bar{\omega}_s \equiv \mathcal{L}_D^s \bar{\omega}_o$ $(s = 0, 1, \ldots)$.

The (as yet variable) coefficient c appearing in the form ε can be specified by looking at the differential $d\bar{\omega}_0 = d\bar{a}_0^0 \wedge \zeta + \bar{a}_0^0 d\zeta =$

$$\bar{a}_0^0((d \ln \bar{a}_0^0 + (p_z + q_z y_1)dx + q_z \eta_0) \wedge \zeta - a\eta_0 \wedge dx) =$$
$$= \cdots + \frac{1}{\bar{a}_1^1}\left\{\left(\frac{b}{\bar{e}}\partial + \frac{\partial}{\partial y_0} - \frac{\bar{a}_2^1}{\bar{a}_2^2}\frac{\partial}{\partial y_1}\right) \ln \bar{a}_0^0 + (p_z + q_z y_1)\frac{b}{\bar{e}} + q_z + \frac{ac}{\bar{e}}\right\} \bar{\omega}_1 \wedge \bar{\omega}_0.$$

The intrinsical assumption $\{\ldots\} = 0$ gives the sought specification $\bar{c}$, (hence $\bar{\varepsilon} = (\bar{e}\, dx + \bar{c}\zeta - \beta\eta_0)/\bar{a}_0^0$) and the Frenet coframe $\bar{\varepsilon}, \bar{\omega}_0, \bar{\omega}_1, \ldots$

14. Subordinated equivalence. Assume, for instance, that the foliation $dx = dy = 0$ is taken for an additional intrinsical object to the original variational integral (15). Then the family $\eta = u\, dx + v\, dy$ with $u, v \in \mathcal{F}$ variable is intrinsical. It follows that the form $\lambda = f\, dx + g\, dy$ (the unique common form of the families ξ and η) is intrinsical. Continuing in this direction, the form $\bar{c}^0\zeta = \bar{\xi} - \lambda$ is intrinsical, hence (by looking at $\bar{\omega}_0 = \bar{a}_0^0\zeta$) the function $\bar{c}^0/\bar{a}_0^0$ is a new invariant. Quite analogously, the form $\bar{a}_1^1\vartheta_0$ is intrinsical (this is the only form of the family η

satisfying $\bar{\omega}_1 - \eta \cong 0$ modulo ω_0), hence $\bar{a}_1^0 \zeta = 4\omega_1 - \bar{a}_1^1 \vartheta_0$ is intrinsical and $\bar{c}^0/\bar{a}_1^0$ is again an invariant (we assume $\bar{a}_1^0 \neq 0$, the other case is easier).

Conversely, let $\bar{c}^0/\bar{a}_0^0$ and $\bar{c}^0/\bar{a}_1^0$ be taken for additional invariants for the equivalences to the integral (15). One can then see (by reverse run of the above arguments) that both λ and $\bar{a}_1^1 \vartheta_0$ are intrinsical forms, hence the system $dx = dy = 0$ (equivalent to $\lambda = \vartheta_0 = 0$) is of the intrinsical nature.

So we have seen that the subordinated problem differ from the original one by a mere presence of additional invariants (and invariant forms which may be used to simplify the Frenet coframe).

Equivalence of spatial problems

15. The classical variational integral. We leave the case $\mu(\Omega) = 1$ (where the existence of unique normal filtrations makes the calculation a mere matter of patience) and turn to $\mu(\Omega) = 2$ where the things became substantially more complicated. (For all such equivalence problems in which a finite-dimensional intrinsical subspace is not given in advance, as far no finite solution algorithm is known.) We shall begin with the variational integral

$$\int f(x, y, z, dy/dx, dz/dx)dx \to \text{ extremum} \tag{21}$$

without any further constraints and under the classical assumption of *contact equivalence*, that is, we assume that the space of variables $x, y = y_0$, $z = z_0$, $y_1 = dy/dx$, $z_1 = dz/dx$ is of the intrinsical nature. In other terms, we shall look only for equivalences which preserve the order of derivatives.

Passing to diffieties, we introduce the variables x, y_s, z_s $(s = 0, 1, \dots)$, the diffiety Ω generated by the contact forms $\eta_s \equiv dy_s - y_{s+1}dx$, $\zeta_s \equiv dz_s - z_{s+1}dx$, and the Lagrange density $\lambda = f\,dx$ $(f = f(x, y_0, z_0, y_1, z_1))$. Clearly $\partial = \partial/\partial x + \Sigma y_{s+1}\partial/\partial y_s + \Sigma z_{s+1}\partial/\partial z_s \in \Omega^{\perp}$ satisfies $\mathcal{L}_{\partial}\eta_s \equiv \eta_{s+1}$, $\mathcal{L}_{\partial}\zeta_s \equiv \zeta_{s+1}$ so that the modules $\Omega^{\ell} \equiv \{\eta_0, \zeta_0, \dots, \eta_\ell, \zeta_\ell\}$ provide a normal filtration. From the point of view of our equivalence problem, this filtration is of the intrinsical nature and thus the families of forms

$$\omega_s \equiv a_s^0 \eta_0 + b_s^0 \zeta_0 + \cdots + a_s^s \eta_s + b_s^s \zeta_s \quad (|a_s^s| + |b_s^s| \neq 0) \tag{22}$$

with $a_s^i, b_s^i \in \mathcal{F}$ variable functions are intrinsical for every $s = 0, 1, \dots$. (Clearly ω_0 is intrinsical by definition and ω_{s+1} can be characterized in terms $\omega_0, \dots, \omega_s$ by the property $d\omega_s \cong 0$ (modulo $\omega_0, \dots, \omega_{s+1}$).) Moreover, we have the intrinsical family

$$\xi = f\,dx + a^0\eta_0 + b^0\zeta_0 + \cdots + a^n\eta_n + b^n\zeta_n \tag{23}$$

with varying $a^i, b^i \in \mathcal{F}$ and undetermined n, hence the intrinsical vector field $D = \partial/f$.

If $n \geq 1$, then the congruence $d\xi \cong dx \wedge (a^n \eta_{n+1} + b^n \zeta_{n+1})$ (modulo $\Omega^{n-1}, \Omega \wedge \Omega$) leads to the intrinsical specification $\bar{a}^n = \bar{b}^n = 0$. So we may assume $a^i = b^i \equiv 0$ for all $i \geq 1$, and it follows

$$d\xi \cong (e\eta_0 + e'\zeta_0 + \left(\frac{\partial f}{\partial y_1} - a^0\right)\eta_1 + \left(\frac{\partial f}{\partial z_1} - b^0\right)\zeta_1) \wedge dx \quad (\text{modulo } \Omega \wedge \Omega),$$

where $e = \partial f/\partial y_0 - \partial a^0$, $e' = \partial f/\partial z_0 - \partial b^0$. This permits to specify $\bar{a}^0 = \partial f/\partial y_1$, $\bar{b} = \partial f/\partial z_1$ to ensure the intrinsical requirement $d\xi \cong 0$ (modulo $\Omega^0, \Omega \wedge \Omega$). We obtain the Poincaré-Cartan form $\bar{\xi} = f\,dx + \bar{a}^0\eta_0 + \bar{b}^0\zeta_0$ satisfying

$$d\bar{\xi} = (\bar{e}\eta_0 + \bar{e}'\zeta_0) \wedge dx + \delta\bar{a}^0 \wedge \eta_0 + \delta\bar{b}_0 \wedge \zeta_0 \tag{24}$$

where

$$\bar{e} = \partial f/\partial y_0 - \partial(\partial f/\partial y_1), \quad \bar{e}' = \partial f/\partial z_0 - \partial(\partial f/\partial z_1)$$

are the Euler-Lagrange operators and $\delta = \Sigma(\partial/\partial y_s) \cdot \eta_s + \Sigma(\partial/\partial z_s) \cdot \zeta_s$ is the "truncated differential". It follows that

$$\pi_0 = \mathcal{L}_D\bar{\xi} = D\rfloor d\bar{\xi} = \frac{1}{f}(\bar{e}\eta_0 + \bar{e}'\zeta_0), \;\; \pi_s \equiv \mathcal{L}_D^s\pi_0$$

are intrinsical forms. However, as yet we do not have a coframe.

We shall search for other intrinsical forms. Abbreviating the notation by $A = \bar{e}$, $B = \bar{e}'$, $L = \partial^2 f/\partial y_1^2$, $M = \partial^2 f/\partial y_1 \partial z_1$, $N = \partial^2 f/\partial z_1^2$ and assuming $A \neq 0$, $B \neq 0$, we shall employ the above family ω_0 (cf. (22)) in the modified transcription $\omega_0 = a\frac{A}{f}\eta_0 + b\frac{B}{f}\zeta_0$ (thus $a = a_0^0 f/A$, $b = b_0^0 f/B$ are variable functions) and choose $\omega_1 = \mathcal{L}_D\omega_0 \cong (aA/f^2)\eta_1 + (bB/f^2)\zeta_1$ (modulo η_0, ζ_0). We shall suppose that ω_0, π_0 are linearly independent (i.e., $a - b \neq 0$). Then

$$\frac{A}{f}\eta_0 = \frac{\omega_0 - b\pi_0}{a - b}, \;\; \frac{B}{f}\zeta_0 = \frac{\omega_0 - a\pi_0}{b - a}, \;\; \frac{A}{f^2}\eta_1 \cong \frac{\omega_1 - b\pi_1}{a - b}, \;\; \frac{B}{f^2}\zeta_1 \cong \frac{\omega_1 - a\pi_1}{b - a}.$$

Inserting these $\eta_0, \ldots, \zeta_1$ into the right hand side of (24), the coefficients of the products $\omega_1 \wedge \omega_0$, $\omega_1 \wedge \pi_0$, $\pi_1 \wedge \omega_0$, $\pi_1 \wedge \pi_0$ are of the intrinsical nature. One can see that they are respectively equal to

$$U = L/A^2 - 2M/AB + N/B^2, \;\; V = -bL/A^2 + (a+b)M/AB - aN/B^2,$$
$$V, W = b^2L/A^2 - 2abM/AB + a^2N/B^2$$

multiplied by the function $f^3/(a-b)^2$. Suppose $U \neq 0$. (One can verify by direct computation that this is equivalent to the regularity $LN \neq M^2$. We shall not discuss the non-regular case here since it deserves a separate article.) Then we may introduce the intrinsical requirements $Uf^3/(a-b)^2 = \pm 1$ (i.e., $a - b = (\pm Uf^3)^{1/2}$) and $V = 0$ which leads to the specifications

$$\bar{a} = \left(\frac{L}{A} - \frac{M}{B}\right)/AU^2, \quad \bar{b} = \left(\frac{M}{A} - \frac{N}{B}\right)/BU^2.$$

So we have the intrinsical forms $\bar{\omega}_s \equiv \mathcal{L}_D^s(\bar{a}\frac{A}{f}\eta_0 + \bar{b}\frac{B}{f}\zeta_0)$ and thus the Frenet coframe $\bar{\xi}, \pi_0, \bar{\omega}_0, \pi_1, \bar{\omega}_1, \dots$. The remaining function W (specified to $\bar{W}$) gives the invariant

$$\bar{W}f^3/(\bar{a}-\bar{b})^2 = \pm\bar{W}/U = \pm(LN - M^2)/A^2B^2U^4.$$

Other invariants can be quite automatically derived by the common methods.

16. The divergence equivalence for the variational integral (21) will be discussed by the use of two independent intrinsical families (22) and an intrinsical 2-form, the exterior differential of the Poincaré-Cartan form

$$d\bar{\xi} = (A\eta_0 + B\zeta_0)\wedge dx + C\eta_0\wedge\zeta_0 + (L\eta_1 + M\zeta_1)\wedge\eta_0 + (M\eta_1 + N\zeta_1)\wedge\zeta_0$$

(here $C = \partial^2 f/\partial z_1\partial y_0 - \partial^2 f/\partial y_1\partial z_0$), see formula (24). We through retain some abbreviations of the preceding Sections. Recall once more that the form $\bar{\xi}$ is *not* of the intrinsical nature for the divergence equivalences. The previous role of $\bar{\xi}$ undertakes the 2-form $d\bar{\xi}$.

Passing to more detail, the filtration Ω^* of Section 15 is again regarded for an intrinsical object (i.e., we deal with the divergence equivalences preserving the order of derivatives). Therefore the families of forms

$$\omega_0 = p\eta_0 + q\zeta_0,\ \vartheta_0 = r\eta_0 + s\zeta_0 \quad (u = \begin{vmatrix} p & s \\ q & r \end{vmatrix} \neq 0),$$

with variable functions $p, q, r, s \in \mathcal{F}$ are of the intrinsical nature (they constitute the most general basis of Ω^0). By using the inverse formulae,

$$\eta_0 = S\omega_0 - Q\vartheta_0,\ \zeta_0 = -R\omega_0 + P\vartheta_0 \quad (P = p/u, \dots, S = s/u), \tag{25}$$

one can easily verify that

$$d\bar{\xi} = \omega_0\wedge\pi + \vartheta_0\wedge\vartheta + C\eta_0\wedge\zeta_0 \tag{26}$$

where

$$\pi = \begin{vmatrix} A & B \\ R & S \end{vmatrix} dx + \begin{vmatrix} L & M \\ R & S \end{vmatrix}\eta_1 + \begin{vmatrix} M & N \\ R & S \end{vmatrix}\zeta_1,$$
$$\vartheta = \begin{vmatrix} P & Q \\ A & B \end{vmatrix} dx + \begin{vmatrix} P & Q \\ L & M \end{vmatrix}\eta_1 + \begin{vmatrix} P & Q \\ M & N \end{vmatrix}\zeta_1.$$

The families of forms π, ϑ (with $P, Q, R, S \in \mathcal{F}$ variable) are of the intrinsical nature modulo Ω^0. Assuming $|A| + |B| \neq 0$, we may introduce the intrinsical requirement $\vartheta \in \Omega$, that is,

$$P = vA,\ Q = vB \qquad (\text{thus } \begin{vmatrix} A & B \\ R & S \end{vmatrix} = \frac{1}{v}\begin{vmatrix} P & Q \\ R & S \end{vmatrix} = 1/uv), \tag{27}$$

where $v \in \mathcal{F}$ is a variable function. Then $d\vartheta_0 \cong dx \wedge (r\eta_1 + s\zeta_1)$, that is,

$$d\vartheta_0 \cong uv\pi \wedge u(R\eta_1 + S\zeta_1) \quad (\text{modulo } \Omega^0, \Omega \wedge \Omega), \tag{28}$$

in virtue of (25,27). It follows that the family of forms $u^2v(R\eta_1 + A\zeta_1)$ (where $u, v \in \mathcal{F}$ are variable) is intrinsical modulo Ω^0. We shall identify it with ϑ ((27) is assumed) which implies the specifications

$$\bar{R} = \begin{vmatrix} A & B \\ L & M \end{vmatrix} / u^2, \quad \bar{S} = \begin{vmatrix} A & B \\ M & N \end{vmatrix} / u^2 \tag{29}$$

(of a weakened sense since $u \in \mathcal{F}$ is variable). This identification is correct if the formula

$$u = 1/\begin{vmatrix} P & Q \\ R & S \end{vmatrix} = 1/v\begin{vmatrix} A & B \\ R & S \end{vmatrix} \tag{30}$$

makes a sense after the substitution $R \to \bar{R}$, $S \to \bar{S}$. That means, we must suppose $A\bar{S} \neq B\bar{R}$. (To this point, see Section 17.)

On the other hand, the family of operators $D = uv\partial$ (defined by the properties $D \in \Omega^\perp$, $\pi(D) = 1$) is of the intrinsical nature and so are the families of forms

$$\omega_1 = \mathcal{L}_D\omega_0 \cong uv(p\eta_1 + q\zeta_1) = u^2v^2(A\eta_1 + B\zeta_1),$$
$$\vartheta_1 = \mathcal{L}_D\vartheta_0 \cong uv(r\eta_1 + s\zeta_1) = u^2v(\bar{R}\eta_1 + \bar{S}\zeta_1)$$

(both modulo Ω^0). It follows in particular

$$\omega_1 \wedge \vartheta_1 \cong u^3v^2\eta_1 \wedge \zeta_1 \quad (\text{modulo } \Omega^0).$$

Using the latter formulae, one can see that

$$\begin{aligned} d\omega_0 &\cong dx \wedge (p\eta_1 + q\zeta_1) \\ &= u^2v^2(\pi - \begin{vmatrix} L & M \\ \bar{R} & \bar{S} \end{vmatrix}\eta_1 - \begin{vmatrix} M & N \\ \bar{R} & \bar{S} \end{vmatrix}\zeta_1) \wedge (A\eta_1 + B\zeta_1) \\ &= \pi \wedge \omega_1 - u^2v^2 \cdot \frac{W}{u^2} \cdot \frac{1}{u^3v^2}\omega_1 \wedge \vartheta_1, \end{aligned}$$

where W is a quite definite (in general nonvanishing) function:

$$W = \begin{vmatrix} L & M \\ \begin{vmatrix} A & B \\ L & M \end{vmatrix} & \begin{vmatrix} A & B \\ M & N \end{vmatrix} \end{vmatrix} B - \begin{vmatrix} M & N \\ \begin{vmatrix} A & B \\ L & M \end{vmatrix} & \begin{vmatrix} A & B \\ M & N \end{vmatrix} \end{vmatrix} A.$$

The coefficient of $\omega_1 \wedge \vartheta_1$ is of the intrinsical nature and assuming $W \neq 0$, it may be equated to 1 which yields the specification $\bar{u} = (W)^{1/3}$, hence also the specification

$$\bar{v} = \bar{u} / \begin{vmatrix} A & B \\ \begin{vmatrix} A & B \\ L & M \end{vmatrix} & \begin{vmatrix} A & B \\ M & N \end{vmatrix} \end{vmatrix}$$

following from (27). (Quite analogously, by looking at the differential

$$d\vartheta_0 \cong dx \wedge (r\eta_1 + s\zeta_1) = \pi \wedge \vartheta_1 - u^2 v \cdot \frac{w}{u^4} \cdot \frac{1}{u^3 v^2}\omega_1 \wedge \vartheta_1$$

where

$$w = \begin{vmatrix} L & M \\ \begin{vmatrix} A & B \\ L & M \end{vmatrix} & \begin{vmatrix} A & B \\ M & N \end{vmatrix} \end{vmatrix} \cdot \begin{vmatrix} A & B \\ M & N \end{vmatrix} - \begin{vmatrix} M & N \\ \begin{vmatrix} A & B \\ L & M \end{vmatrix} & \begin{vmatrix} A & B \\ M & N \end{vmatrix} \end{vmatrix} \cdot \begin{vmatrix} A & B \\ L & M \end{vmatrix},$$

we obtain the invariant $w/\bar{u}^5\bar{v}$, the coefficient of $\omega_1 \wedge \vartheta_1$.)

Altogether taken, we already know the specifications $\bar{P} = \bar{v}A$, $\bar{Q} = \bar{v}B$ and complete specifications $\bar{R}, \bar{S}$ defined by (29) with $\bar{u}$ instead of u. So we have the relevant intrinsical forms $\bar{\omega}_0, \bar{\vartheta}_0$, the intrinsical operator $\mathcal{L}_D$ (here $D = \bar{u}\bar{v}\partial$) and thus the intrinsical chains $\bar{\omega}_s \equiv \mathcal{L}_D^s \bar{\omega}_s$, $\bar{\vartheta}_s \equiv \mathcal{L}_D^s \bar{\vartheta}_0$. One can also observe that $\bar{\vartheta}_1 \cong \vartheta$ (modulo Ω^0).

Since $\bar{\pi}$ is intrinsical modulo Ω^0 (and thus modulo Ω), we have the intrinsical family of forms $\zeta = \bar{\pi} + \omega$ ($\omega \in \Omega$ is variable), that is,

$$\zeta = \begin{vmatrix} A & B \\ \bar{R} & \bar{S} \end{vmatrix} dx + A^0\eta_0 + B^0\zeta^0 + \cdots + A^n\eta_n + B^n\zeta^n$$

where $A^0, \ldots, B^n \in \mathcal{F}$ are varying functions and n is undetermined. One can then see that there are unique specifications $\bar{A}^0, \ldots, \bar{B}^n$ such that $D \rfloor d\bar{\zeta} \in \Omega^0$ for the relevant specified $\bar{\zeta}$. (In other terms, we have the intrinsically related variational integral

$$\int \begin{vmatrix} A & B \\ \bar{R} & \bar{S} \end{vmatrix} dx \to \text{ extremum}$$

and $\bar{\zeta}$ is the corresponding Poincaré-Cartan form.) At this stage, the Frenet coframe consisting of $\bar{\zeta}$ and all forms $\bar{\omega}_s, \bar{\vartheta}_s$ is determined.

17. Remark. The identity $A\bar{S} = B\bar{R}$ means that the form $\mathcal{L}_{g\partial}\omega_0 \cong gv(A\eta_1 + B\zeta_1)$ is proportional to ϑ (modulo Ω^0), for any function $g \in \mathcal{F}$. The proportionality turns into the equality if we specify

$$\bar{g} = \begin{vmatrix} A & B \\ L & M \end{vmatrix} / A = \begin{vmatrix} A & B \\ M & N \end{vmatrix} / B.$$

This provides the intrinsical vector field $D = \bar{g}\partial$ and thus the intrinsical family of differential forms $\zeta = dx/\bar{g} + A^0\eta_0 + B^0\zeta_0 + \cdots + A^n\eta_n + B^n\zeta_n$ (with variable coefficients). It follows that we deal with the common equivalence problem for the variational integral $\int dz/\bar{g} \to$ extremum (endowed moreover with the additional intrinsical 2-form $d\xi$).

Nonstandard equivalences

18. Setting the problem. Passing to the equivalence transformations which may change the order of derivatives, we enter an extensive and rather unusual realm. Since we should like to explain the ideas and methods as clearly as possible, only a very particular problem will be investigated: to determine whether a *given* second order variational integral

$$\int g(x, y_0, z_0, \dots, y_2, z_2)dx \to \text{ extremum} \tag{31}$$

can be obtained by transformation of an *unknown* in advance first order variational integral (21). The notation of Sections 15-17 is preserved so that the sought equivalence transformation can be symbolically written as

$$x \to \tilde{x}, y_s \to \tilde{y}_s, z_s \to \tilde{z}_s \quad (s = 0, 1, \dots). \tag{32}$$

where $\tilde{x}, \tilde{y}_s, \tilde{z}_s \in \mathcal{F}$ are certain functions (which will be determined together with the integral (21) in the course of the following calculations). The sought transformation (32) should preserve the diffiety Ω and should carry the (as yet unknown) differential form $f\,dx$ (cf. (21)) into the well-known form $g\,dx$ (cf. (31)) modulo a summand from Ω. It is moreover necessary to ensure the invertibility of (32) in the infinite-dimensional underlying space of variables x, y_s, z_s $(s = 0, 1, \dots)$ and we shall assume that this is guaranted if the differentials $d\tilde{x}$, $d\tilde{y}_s$, $d\tilde{z}_s$ $(s = 0, 1, \dots)$ can be taken for a basis of the module Φ of all differential 1-forms (and refer to [4] for more details).

Let us made the above requirements on the sought equivalence (32) explicit. First of all, the equivalence (32) should be an *automorphism* of Ω: the forms

$$\tilde{\eta}_s = d\tilde{y}_s - \tilde{y}_{s+1}d\tilde{x},\ \tilde{\zeta}_s = d\tilde{z}_s - \tilde{z}_{s+1}d\tilde{x} \quad (s = 0, 1, \dots) \tag{33}$$

(transforms of η_s, ζ_s) should constitute a basis of Ω. One can then observe that the latter condition together with the assumption $d\tilde{x} \neq 0$ ensures the invertibility of (32) since then the forms (33) and $d\tilde{x}$ (and thus all differentials $d\tilde{x}, d\tilde{y}_s, d\tilde{z}_s$) constitute a basis of Φ. One can also observe that

$$\partial\tilde{x}d\tilde{y}_s - \partial\tilde{y}_s d\tilde{x},\ \partial\tilde{x}d\tilde{z}_s - \partial\tilde{z}_s d\tilde{x} \in \Omega \quad (s = 0, 1, \dots)$$

and it follows that the forms (33) are lying in Ω if and only if

$$\tilde{y}_{s+1} \equiv \partial\tilde{y}_s/\partial\tilde{x}, \quad \tilde{z}_{s+1} \equiv \partial\tilde{z}_s/\partial\tilde{x} \tag{34}$$

(the *prolongation* formulae). So the knowledge of the initial terms $x \to \tilde{x}$, $y_0 \to \tilde{y}_0$, $z_0 \to \tilde{z}_0$ of (32) is quite enough.

At second, let us look at the variational integral (21) or better, at a general variational integral

$$\int f(x, y_0, z_0, \dots, y_n, z_n)dx \to \text{ extremum} \tag{35}$$

of exactly n-th order. Recalling the common filtration $\Omega^0 \subset \Omega^1 \subset \dots$, one can easily see that the relevant Poincaré-Cartan form $\bar\xi = f\,dx + \bar\omega$ is defined by the congruences

$$\bar\xi \cong f\,dx \quad (\text{modulo } \Omega), \quad \partial\rfloor d\bar\xi \cong 0 \ (\text{modulo } \Omega^0)$$

(the first one is trivial, the second one is identical with (14) since $\mathcal{R}(\Omega) = 0$ and Ω^0 consists of initial forms). One can then observe that the first congruence can be strenghtened as $\bar\xi \cong f\,dx$ (modulo Ω^{n-1}) and the indice $n-1$ cannot be diminished here. In our case $n = 1$. The equivalence (32) carries the filtration $\Omega^0 \subset \Omega^1 \subset \dots$ into the filtration $\tilde\Omega^0 \subset \tilde\Omega^1 \subset \dots$ (where $\tilde\Omega^\ell = \{\tilde\eta^0, \tilde\zeta^0, \dots, \tilde\eta^\ell, \tilde\zeta^\ell\}$), the form $f\,dx$ into the form

$$\tilde f\,d\tilde x = \tilde f(\partial\tilde x\,dx + \delta\tilde x) \cong g\,dx \quad (\text{modulo } \Omega)$$

(hence $\tilde f = g/\partial\tilde x$ and thus $\tilde f\,d\tilde x = g\,dx + \tilde f\,\delta\tilde x$), and the above form $\bar\xi$ into the Poincaré-Cartan form $\tilde\xi$ (a simplified notation for $(\bar\xi)^\sim$) for the integral (31). Altogether taken, the congruences

$$\tilde\xi \cong g\,dx + \tilde f\,\delta\tilde x, \quad \partial\rfloor d\tilde\xi \cong 0 \quad (\text{modulo } \tilde\Omega^0) \tag{36}$$

define the form $\tilde\xi$ and ensure the order $n = 1$ of the integral (35). This problem will be not resolved in full generality here.

19. Explicit calculations. For technical reasons, the original problem will be investigated under the additional assumption $\tilde\Omega^0 \subset \Omega^1$. One can observe that this happens if and only if the functions $\tilde x$, $\tilde y_0$, $\tilde z_0$ are depending only on the coordinates x, y_0, y_1, z_0, z_1 (i.e., we suppose that the order of derivatives may increase on 1 at most). We are going to determine the module $\tilde\Omega^0$.

If $\tilde\Omega^0 = \Omega^0$ then we deal with a common point equivalence (32). The functions $\tilde x, \tilde y_0, \tilde z_0$ are depending only on x, y_0, z_0 (as follows from the Lie-Bäcklund theorem, cf. Section 21), the order of derivatives is not changed and (31) is in reality a mere first order integral. Omitting this trivial subcase, we may suppose $\tilde\Omega^0 = \{\eta, \zeta\}$ where both η and ζ are not lying in Ω^0. Since clearly $\tilde\Omega^{\ell+1} = \tilde\Omega^\ell + \mathcal{L}_\partial\Omega^\ell$ ($\ell \geq 0$) and $\cup\tilde\Omega^\ell = \Omega$, the family of all forms of the kind $\mathcal{L}_\partial^\ell\eta$, $\mathcal{L}_\partial^\ell\zeta$ should generate the module Ω (the *main principle*). Owing to the latter principle, one can see that η, ζ cannot be linearly independent modulo Ω^0. (Proof: assuming $\eta = \eta_1 + \dots$, $\zeta = \zeta_1 + \dots$ where $\dots \in \Omega^0$, then $\mathcal{L}_\partial^\ell\eta = \eta_\ell + \dots$, $\mathcal{L}_\partial^\ell\zeta = \zeta_\ell + \dots$ cannot generate any form from Ω^0.) So we may assume $\eta \in \Omega^0$, $\zeta \notin \Omega^0$ without any loss of generality, and even

$$\eta = a\eta_0 + \zeta_0, \quad \zeta = b\eta_0 + c\eta_1 + \zeta_1 \tag{37}$$

with appropriate $a, b, c \in \mathcal{F}$. But applying the main principle, it follows $a = c$, $b \neq \partial a$. (Proof: $\mathcal{L}_\partial\eta = \partial a\eta_0 + a\eta_1 + \zeta_1$ and ζ cannot be linearly independent modulo Ω^0 which implies $a = c$. Then $\mathcal{L}_\partial\eta - \zeta$ must be independent of η, hence $b \neq \partial a$.) Summarizing the achievement, we have determined all submodules $\tilde\Omega^0 \subset \Omega^1$ which give rise to the filtration $\tilde\Omega^0 \subset \tilde\Omega^1 = \tilde\Omega^0 + \mathcal{L}_\partial\tilde\Omega^0 \subset \dots$ of the diffiety Ω from the "algebraic" point of view. (There are some additional conditions of deeper nature, see belov.)

Let us turn to the form $\tilde{\xi}$, that is, to the congruences (36). For technical reasons, we shall deal with a mere *fiber equivalences* where $\tilde{x} = \tilde{x}(x)$, hence $\delta\tilde{x} = \Sigma\partial\tilde{x}/\partial y^s \cdot \eta_s + \Sigma\partial\tilde{x}/\partial z^s \cdot \zeta_s = 0$ is vanishing and (36) means that

$$\tilde{\xi} = g\,dx + p\eta + q\zeta, \quad \partial\rfloor\tilde{\xi} = r\eta + s\zeta$$

where $p, q, r, s \in \mathcal{F}$ are certain unknown functions. Inserting (37) with $c = a$ into the latter equations, six conditions

$$\frac{\partial g}{\partial z_2} = q, \frac{\partial g}{\partial y_2} = aq, \frac{\partial g}{\partial z_1} = p + \partial q + s, \frac{\partial g}{\partial y_1} = ap + \partial(aq) + as$$
$$\frac{\partial g}{\partial z_0} = \partial p + r, \frac{\partial g}{\partial y_0} = \partial(ap) + \partial(bq) + ar + bs$$

for the unknowns p, q, r, s, a, b appear. They uniquely determine q and a, yield the compatibility condition

$$\frac{\partial g}{\partial y_1} = \left(\frac{\partial g}{\partial z_1} + \frac{\partial g}{\partial y_2}\partial\right)\left(\frac{\partial g}{\partial y_2}/\frac{\partial g}{\partial z_2}\right) \tag{38}$$

for the function g, and permit to express p, r, s in terms of b. We state only the most complicated formula

$$p = \frac{1}{\partial a - b}\left(\frac{\partial g}{\partial y_0} - \frac{\partial g}{\partial z_0}\frac{\partial g}{\partial y_2}/\frac{\partial g}{\partial z_2} - \frac{\partial g}{\partial z_1}b - \frac{\partial g}{\partial z_2}\partial b\right),$$

the remaining for a, r, s are quite clear.

At last, let us pass to the most interesting and nontrivial property of the module $\tilde{\Omega}^0$: being a transformation of the module Ω^0, there exist a rather special basis $\tilde{\eta}_0, \tilde{\zeta}_0$ in $\tilde{\Omega}^0$. Now recall the following result: a module $\Xi = \{\eta, \zeta\}$ of 1-forms admits an alternative basis of the kind $\Xi = \{dv - v'du, dw - w'du\}$ where u, v, w, v', w' are appropriate functions if and only if

$$d\eta \cong \pi \wedge \eta', \ d\zeta \cong \pi \wedge \zeta' \quad (\text{modulo } \Xi) \tag{39}$$

for appropriate forms π, η', ζ' such that the module $\{\pi, \eta, \zeta\}$ (is completely integrable, i.e.,) has a basis consisting of total differentials. (See [2, p.939 formula IV'] but for convenience of reader, we outline a brief proof. In the trivial direction, assuming $\Xi = \{\eta, \zeta\} = \{dv - v'du, dw - w'du\}$, then clearly (39) with $\pi = du$ and $\{\pi, \eta, \zeta\} = \{du, dv, dw\}$ are valid. In the opposite direction, assuming (39) and $\{\pi, \eta, \zeta\} = \{du, dv, dw\}$, one can take $\pi = du$ in (39) without any loss of generality. Then, by applying the Frobenius theorem on (39) with $u = \text{const.}$ kept fixed, one can conclude that $\{\eta, \zeta\} = \{dv, dw\}$ modulo du which is the desired result.)

In our case $\Xi = \tilde{\Omega}^0$ and it may be taken $u = \tilde{x}$, $v = \tilde{y}_0$, $w = \tilde{z}_0$, of course. Using (37), we have

$$d\eta \cong dx \wedge (\partial a - b)\eta_0 + \delta a \wedge \eta_0, \tag{40}$$

$$d\zeta \cong dx \wedge (\partial b\eta_0 + (\partial a + b)\eta_1 + a\eta_2 + \zeta_2) + \delta b \wedge \eta_0 + \delta a \wedge \eta_1 \tag{41}$$

and this should be represented like (39). But clearly $\{\pi, \eta, \zeta\} = \{du, dv, dw\} = \{d\tilde{x}, d\tilde{y}_0, d\tilde{z}_0\}$ where $\tilde{x} = \tilde{x}(x)$ and so we may take $\pi = d\tilde{x}$ or better, $\pi = dx$. Then (40, 41) imply

$$\delta a \wedge \eta_0 \cong \delta b \wedge \eta_0 + \delta a \wedge \eta_1 \cong 0 \quad (\text{modulo } \tilde{\Omega}^0),$$

and using the development $\delta = \Sigma(\partial/\partial y_s) \cdot \eta_s + \Sigma(\partial/\partial z_s) \cdot \zeta_s \cong$

$$\left(\frac{\partial}{\partial y_0} - a\frac{\partial}{\partial z_0} - b\frac{\partial}{\partial z_1}\right) \cdot \eta_0 + \left(\frac{\partial}{\partial y_1} - a\frac{\partial}{\partial z_1}\right) \cdot \eta_1 + \left(\frac{\partial}{\partial y_1}\right) \cdot \eta_2 + \left(\frac{\partial}{\partial z_2}\right) \cdot \zeta_2 + \dots,$$

modulo ζ, η, the latter congruences proves to be equivalent to the system

$$\frac{\partial a}{\partial y_1} = a\frac{\partial a}{\partial z_1}, \ \frac{\partial a}{\partial y_s} = \frac{\partial a}{\partial z_s} = 0 \quad (s \geq 2), \tag{42}$$

$$\frac{\partial b}{\partial y_1} - a\frac{\partial b}{\partial z_1} + \frac{\partial a}{\partial z_1}b = \frac{\partial a}{\partial y_s} - a\frac{\partial a}{\partial z_0}, \ \frac{\partial b}{\partial y_s} = \frac{\partial b}{\partial z_s} = 0 \quad (s \geq 2). \tag{43}$$

This concludes the calculations.

20. Summary of results. *Assume that the compatibility conditions (38) and (42) with $a = (\partial g/\partial y_2)/(\partial g/\partial z_2)$ for the function g are satisfied. We may choose a function $b = b(x, y_0, z_0, y_1, z_1)$ satisfying (43). Then the forms η, ζ (see (37) with $c = a$) and thus the module $\tilde{\Omega}^0 = \{\eta, \zeta\}$ (and also all $\tilde{\Omega}^\ell$, $\ell \geq 1$) are known. By applying the Frobenius theorem on the module $\{\pi, \eta, \zeta\}$ where $\pi = dx$, we obtain $\{\pi, \eta, \zeta\} = \{du, dv, dw\}$ where $u = x$. It follows $\tilde{\Omega}^0 = \{dv - v'du, dw - w'du\}$ for appropriate $v', w' \in \mathcal{F}$ (explicitly $v' = \partial v/\partial u = \partial v$, $dw = \partial w$) and we may put $\tilde{x} = u = x$, $\tilde{y}_0 = v$, $\tilde{z}_0 = w$ which completely determines the sought equivalence transformation (32).*

21. Theorem. *Every automorphism of Ω which preserves Ω^0 is a prolonged point transformation.*

Proof. The mentioned automorphism preserves moreover the module Ω^1 since $\tilde{\Omega}^1 = \tilde{\Omega}^0 + \mathcal{L}_\partial \tilde{\Omega}^0 = \Omega^0 + \mathcal{L}_\partial \Omega^0 = \Omega^1$. It is an automorphism of both Ω^0 and Ω^1, hence

$$\begin{aligned}
\tilde{\eta}_0 &= A\eta_0 + B\zeta_0 \\
\tilde{\zeta}_0 &= C\eta_0 + D\zeta_0 \\
\tilde{\eta}_1 &= \dots \qquad\qquad + P\eta_1 + Q\zeta_1 \\
\tilde{\zeta}_1 &= \dots \qquad\qquad + R\eta_1 + S\zeta_1
\end{aligned}$$

with a regular matrix. Moreover, the module $\{dx, dy_0, dz_0, dy_1, dz_1\}$ is preserved, too. (This follows from the fact that the family of functions $x, y_0, \ldots, z_1$ is intrinsically related to Ω^0: it is the minimal family such that there is a basis of Ω^0 expressible in terms of it.) So the above system may be completed by

$$d\tilde{x} = \cdots + U\eta_1 + V\zeta_1 + W\,dx.$$

Then the congruence

$$d\tilde{x} \wedge \tilde{\eta}_1 = d\tilde{\eta}_0 = dA \wedge \eta_0 + dB \wedge \zeta_0 + dx \wedge (A\eta_1 + B\zeta_1) \cong 0$$

modulo dx, η_0, ζ_0 implies $PV = UQ$. Quite analogously $PV = US$. Since $PS \neq QR$, we obtain $V = U = 0$. It follows

$$d\tilde{x} \cong 0, d\tilde{y}_0 = \tilde{\eta}_0 + \tilde{y}_1 d\tilde{x} \cong 0, d\tilde{z}_0 \cong 0 \quad (\text{modulo } dx, \eta_0, \zeta_0).$$

and thus $\cong 0$ modulo dx, dy_0, dz_0. This concludes the proof. □

It is to be noted that the proof of the latter Theorem is not easily available in current literature and that the original Bäcklund argument seems to be not quite correct. (See [1, p. 47]: the osculating curve C need not behave continuously if P_i' converge to P.

Miscellanery

22. On a Hilbert problem. The Poincaré-Cartan form $\bar{\xi} = \lambda + \bar{\omega}$ to the constrained variational integral (11) depends on the choice of the normal filtration Ω^* and even on the choice of the initial forms $\omega^j_{(r)}$ appearing in the definition formula (14). However, it may be proved that the restriction of $\bar{\xi}$ on the subspace $\mathbf{E} \subset \mathbf{R}^\infty$ which consists of all points that satisfy the infinitely prolonged Euler-Lagrange system $\partial^k e^j_{(r)} \equiv 0$ (all possible k, j, r) is unique modulo $\mathcal{R}(\Omega)$, cf. [5]. Assume $\mathcal{R}(\Omega) = 0$ from now on, for simplicity, then the restriction of $\bar{\xi}$ on $\mathbf{E}$ is a well-determined form. Since the *extremals* are such curves which satisfy the Pfaff's system $\omega \equiv 0$ $(\omega \in \Omega)$ and the Euler-Lagrange system, it follows that if we deal with extremals, the restriction of $\bar{\xi}$ to $\mathbf{E}$ is quite enough. In this way, it is possible to carry over to the constrained variational problems (11) most of the important concepts of the classical calculus of variations (e.g., variational formulae, E. Noether's theory, integral invariants, geodetics fields, Hamilton-Jacobi equation, and so on) without any essential change. Even the singular variational problems with extremals depending on functions can be included without much trouble.

For instance, a multi-parameter family of extremals may be called a *field* if $d\bar{\xi} = 0$ on the submanifold $\mathbf{F} \subset \mathbf{E}$ covered by the extremals of the mentioned family. Then the (uniquely determined) restriction of $\bar{\xi}$ to $\mathbf{F}$ is a generalization of the famous *Hilbert invariant integral* and easily leads to generalized *Weierstrass theory* for all contrained variational problems. May be, this is the way to the solution of the 23rd Hilbert problem which was suggestively explained (see [8]) but never explicitly formulated: *to investigate the field theory of constrained variational problems.*

23. Empty Euler-Lagrange system. Assuming $\mathcal{R}(\Omega) = 0$, we shall be interested in the case $\mathbf{E} = \mathbf{R}^\infty$. So assume that $e^j_{(r)} \equiv 0$ are identically vanishing in (14). It follows $d\bar{\xi} = \Sigma a^{ij}_{rs}\omega^i_r \wedge \omega^j_s$ in terms of the special basis defined in Section 3. The latter double sum may be taken only over $r \le s$ and $i < j$ if $r = s$. We may assume that $\partial x = 1$ for appropriate $\partial \in \Omega^\perp$, $x \in \mathcal{F}$. Then $\mathcal{L}_\partial \omega^i_s \equiv \omega^i_{s+1}$ means that $d\omega^i_s \cong dx \wedge \omega^i_{s+1}$ (modulo $\Omega \wedge \Omega$). Inserting this into the trivial identity $d^2\bar{\xi} = 0$, one can derive that $a^{ij}_{rs} \equiv 0$ by (a decreasing) double induction on s and j. It follows $d\bar{\xi} = 0$, $\bar{\xi} = dh$ for an appropriate function $h \in \mathcal{F}$. Hence $\lambda \cong \bar{\xi} \cong dh \cong \partial h\, dx$ (modulo Ω) is a generalized divergence.

The divergence equivalence problem is concerned with the study of the differential $d\bar{\xi}$ (not of $\bar{\xi}$), that is, the above result can be interpreted by saying that we deal with families of variational integrals with the same Euler-Lagrange operators. So even the (rather weak) divergence equivalence problem is strongly subordinated to the related equivalence of the corresponding systems of Euler-Lagrange equations: it may well happen that Euler-Lagrange system of two variational integrals are equivalent but the relevant Euler-Lagrange operators differ.

24. Correction. We should like to mention once more the divergence equivalence problem of the variational integral

$$\int f(x, y_0, \dots, y_n)dx \to \text{ extremum }, y_s \equiv d^s y/dx^s,$$

with general m (which was not quite correctly treated in [3, Section 5] in the particular case $m = 2$). We deal with the space of variables $x, y_0, y_1, \dots$, the diffiety Ω generated by the contact forms $\vartheta_s \equiv dy_s - y_{s+1}dx$, the vector field $\partial = \partial/\partial x + \Sigma s_{s+1}\partial/\partial y_s$, and the Lagrange density $\lambda = f\, dx$. Intrinsical objects for the divergence problem are the families

$$\omega_i = a^0_0\vartheta_0 + \dots + a^i_i\vartheta_i \quad (a^i_i \ne 0;\ i = 0, 1, \dots)$$

and the differential $d\bar{\xi}$ of the Poincaré-Cartan form

$$\bar{\xi} = f\, dx + \bar{a}^0\vartheta_0 + \dots + \bar{a}^{m-1}\vartheta_{m-1} \quad (\bar{a}^{m-1} = f_m, \bar{a}^i = f_{i+1} - \partial\bar{a}^{i+1}),$$

see [3, (2,5)].

First of all, if $\partial^2 f/\partial y^2_m = 0$ (the *singular case*), then $f = A + By_m$ where A, B does not depend on y_m, and we may introduce the lower-order variational integral

$$\int (f - \partial g)dx \to \text{ extremum } \quad (g = \int B dy_{m-1})$$

instead of the original one (its Poincaré-Cartan form is $\bar{\xi} - dg$ so that the differential $d\bar{\xi}$ is retained). Repeatedly applying this reduction, we may assume that $\partial^2 f/\partial y^2_m \ne 0$.

Passing to calculation of the Frenet coframe, we begin with

$$\begin{aligned} d\bar{\xi} &= E\vartheta_0 \wedge dx + \Sigma\delta\bar{a}^i \wedge \vartheta_i \\ &\cong \omega_0 \wedge (Edx + \cdots + f_{mm}\vartheta_{2m-1})/a_0^0 \text{ (modulo } \Omega \wedge \Omega) \end{aligned}$$

where $E = \Sigma(-\partial)^i f_i$ is the Euler-Lagrange operator (we denote $f_i \equiv \partial f/\partial y_i$ and analogously for higher derivatives, see [3, (6)]). It follows that

$$\zeta = (Edx + \cdots + f_{mm}\vartheta_{2m-1})/a_0^0$$

with $a_0^0 \in \mathcal{F}$ variable is an intrinsical family of forms. Then

$$d\omega_0 \cong a_0^0 dx \wedge \vartheta_1 \cong a_0^0(a_0^0\zeta - f_{mm}\omega_{2m-1}/a_{2m-1}^{2m-1}) \wedge \omega_1/a_1^1 E$$

(modulo $\omega_0, \omega_2\wedge\omega_1, \ldots, \omega_{2m-2}\wedge\omega_1$) and we may introduce the relations $(a_0^0)^2/a_1^1 E = 1$, $a_0^0 f_{mm}/a_{2m-1}^{2m-1}a_1^1 E = 1$. Continuing more easily

$$d\omega_1 \cong a_1^1 dx \wedge \vartheta_2 \cong a_1^1 a_0^0 \zeta \wedge \omega_2/a_2^2 E \quad \text{(modulo } \Omega \wedge \Omega)$$

we introduce $a_1^1 a_0^0/a_2^2 E = 1$. Then $d\omega_2$ yields $a_2^2 a_0^0/a_3^3 E = 1$, and so on. Altogether taken,

$$\begin{gathered} a_1^1 = (a_0^0)^2/E, a_2^2 = a_1^1 a_0^0 E = (a_0^0)^3/E^2, \ldots \\ a_{2m-1}^{2m-1} = (a_0^0)^{2m}/E^{2m-1} = f_{mm}/a_0^0, \bar{a}_0^0 = (f_{mm}E^{2m-1})^{1/(2m+1)}. \end{gathered}$$

One can observe that after this specification of a_0^0, we may introduce the intrinsical vector field $D = \bar{a}_0^0\partial/E$ and then the intrinsical forms $\bar{\omega}_s \equiv \mathcal{L}_D^s\bar{\omega}_0$ where $\bar{\omega}_0 = \bar{a}_0^0\vartheta_0$. The specification $\bar{\zeta}$ is not yet completely known since the form ζ is in reality determined only modulo $\vartheta_0 : \zeta = \zeta' + b\vartheta_0$ where ζ' is known but $b \in \mathcal{F}$ is a variable function. It is, however possible to use the congruence $d\zeta \cong \zeta \wedge \Sigma b^i\bar{\omega}_i$ (modulo $\Omega \wedge \Omega$) just in the same manner as in [3, Section 5]. The sought Frenet coframe is constituted by $\bar{\zeta}, \bar{\omega}_0, \bar{\omega}_1 \ldots$.

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