ARCHIVUM MATHEMATICUM (BRNO)
Tomus 29 (1993), 227 – 234

ON TRANSFORMATIONS OF FUNCTIONAL–DIFFERENTIAL EQUATIONS

Jan Čermák

Abstract. The paper contains applications of Schröder's equation to differential equations with a deviating argument. There are derived conditions under which a considered equation with a deviating argument intersecting the identity $y = x$ can be transformed into an equation with a deviation of the form $\tau(x) = \lambda x$. Moreover, if the investigated equation is linear and homogeneous, we introduce a special form for such an equation. This special form may serve as a canonical form suitable for the investigation of oscillatory and asymptotic properties of the considered equation.

1. Introduction and notation

In this article the transformations of functional-differential equations with one deviating argument are studied. These transformations are supposed to be global, i.e., they are defined on the whole definition intervals of corresponding equations.

The case where the deviating argument of a considered equation is a sufficiently smooth function with a positive derivative and which does not intersect the identity $y = x$ in its domain has been already solved in [7]. It has been proved the possibility of converting any such equation into an equation with a constant deviation.

Here we investigate the case where the deviating argument is a sufficiently smooth function with a nonzero derivative and having just one fixed point in its domain. If the initial set of such an equation consists of this fixed point only some authors call such a case singular (see [1]). Moreover, in accordance with [8] we show that under mild assumptions any linear homogeneous differential equation with such a deviation can be globally transformed into an equation having a certain special form.

We introduce the following notation. Let I be an interval of any type with endpoints a, b where $a < b$, b may be infinite. An interval I is called a submodulus (resp.overmodulus) interval for a function f if $f(I) \subset I$ (resp.$f(I) \supset I$). Further, denote by $\sigma_k[f]$ the set of fixed points of order k of the function f in I, i.e., the set of $x \in I$ which fulfil $f^k(x) = x$ and $f^i(x) \neq x$ for $i = 1, 2, \dots, k-1$. Here f^n

1991 *Mathematics Subject Classification*: 34K05; Secondary 34K15.
Key words and phrases: Functional-differential equation, singular case, transformation, canonical form.
Received March 15, 1993.

denotes the $n-$th iterate of f (for $n > 0$) and $(-n)-$th iterate of the inverse f^{-1} (for $n < 0$); we put $f^0 = id$.

2. The case of a nonlinear equation

Consider a functional-differential equation of the form

$$F(x, y(x), y(\tau(x)), y'(x), y'(\tau(x)), \dots, y^{(n)}(x), y^{(n)}(\tau(x))) = 0 \qquad \text{on } I. \tag{1}$$

It was shown in [7] that supposing $\tau \in C^n(I),\ \tau'(x) > 0$ on I and $\tau(x) \neq x$ in I one can transform equation (1) by a change of the independent variable $t = \varphi(x) \in C^n(I),\ \varphi'(x) > 0$ on I, into an equation with a constant deviation

$$G(t, z(t), z(t+c), z'(t), z'(t+c), \dots, z^{(n)}(t), z^{(n)}(t+c)) = 0 \qquad \text{on } \varphi(I),$$

where $\operatorname{sign} c = \operatorname{sign}(\tau(x) - x)$.

Since the transformation function $\varphi(x)$ is a solution of Abel's equation

$$\varphi(\tau(x)) = \varphi(x) + c, \qquad x \in I$$

it is obvious that I must not contain any fixed point of τ. For the next we deal with the problem whether we can transform equation (1) with a deviating argument intersecting the identity in its domain into another suitable form. Functional equations play an important role in this investigation.

First we prove the following theorem.

Theorem 1. *Assume that I is a submodulus or overmodulus for f and let $f \in C^n(I),\ n \geq 2,\ f'(x) \neq 0$ on I and $\sigma_1[f] = \{p\},\ \sigma_2[f] = \emptyset$. If $|f'(p)| \neq 1$ then there exists a unique one-parameter family of C^n solutions of Schrőder's equation*

$$\varphi(f(x)) = \lambda\varphi(x), \qquad x \in I \tag{2}$$

where $\lambda = f'(p)$. All these solutions are defined on the interval $I \cup f(I)$ and have a positive derivative here.

Proof. First suppose $f \in C^n(I)$, $n \geq 2$, $f'(x) \geq 0$ on I, $f(I) \subset I$. It is well-known (see, e.g.[4]) that there exists a unique solution of (2)

$$\varphi(x) = c \lim_{n\to\infty} \frac{f^n(x) - p}{\lambda^n}$$

satisfying $\varphi'(p) = c$. If $c > 0$ then this formula yields one-parameter family of C^n functions increasing in I. We show that the assumption $f'(x) > 0$ on I implies $\varphi'(x) > 0$ on I.

φ' is the continuous solution of the equation

$$\varphi'(f(x)) = \frac{\lambda}{f'(x)}\varphi'(x), \qquad x \in I. \tag{3}$$

Admit that $\varphi'(x_0) = 0$ for some $x_0 \in I$. Then $\varphi'(f^n(x_0)) = 0$ for all positive integers n. On the other hand

$$\lim_{n\to\infty} \varphi'(f^n(x_0)) = \varphi'(\lim_{n\to\infty} f^n(x_0)) = \varphi'(p) > 0$$

what is impossible.

Further, let $f \in C^n(I)$, $n \geq 2$, $f'(x) > 0$ on I and $f(I) \supset I$. Then rewrite equation (2) to the form

$$\varphi(f^{-1}(x)) = \lambda^{-1}\varphi(x), \qquad x \in f(I).$$

According to the previous part of the proof we can see that the formula

$$\varphi(x) = c \lim_{n\to\infty} \lambda^n (f^{-n}(x) - p)$$

yields the unique one-parameter C^n solutions with a positive derivative in $f(I)$.

Finally suppose $f \in C^n(I)$, $n \geq 2$ and $f'(x) < 0$ on I. Assume the case $f(I) \subset I$ occurs. Then every C^n solution of (2) satisfies also the equation

$$\varphi(f^2(x)) = \lambda^2\varphi(x), \qquad x \in I.$$

Conversely, if φ^* is a solution of the previous equation in an interval I^* with endpoints a and $p \in I^*$ (which exists and it is determined uniquely up to multiplicative constant) then it is easy to verify that the function

$$\varphi(x) = \begin{cases} \varphi^*(x), & \text{for } x \in I^*, \\ \lambda\varphi^*(f^{-1}(x)), & \text{for } x \in f(I^*) \end{cases}$$

is a solution of (2) in $I^* \cup f(I^*)$. Since $I^* \cup f(I^*) \subset I$ this solution need not be defined in the whole I. If we define

$$\varphi(x) = \lambda^{-1}\varphi^*(f(x))$$

for $x \in I - I^* \cup f(I^*)$ we continue this C^n solution onto I.

The case $I \subset f(I)$ is only a trivial modification of the previous one. The statement is proved. $\square$

Remark. It is obvious that if we consider f with a positive derivative in I then I is a submodulus or overmodulus interval for I.

The assumption $f \in C^1(I)$ instead of $f \in C^2(I)$ need not be sufficient for the existence of a solution of (2) with required properties. More precisely, this solution need not belong to the class $C^1(I)$ nor to be an increasing function in I. However, it can be shown (see [10]) that by adding some reasonable requirements on $f \in C^1(I)$ we obtain C^1 solutions of (2) with a positive derivative in I.

Further, remark that provided $\lambda \neq f'(p)$ we cannot get any solution with a positive derivative in I because then equation (3) implies $\varphi'(p) = 0$.

Finally, it is easy to see that the case $|f'(p)| = 1$ has to be excluded from our considerations because assuming this we would obtain Schrőder's equation having constant functions as the only monotonic C^1 solutions in I.

Now we can formulate the statement concerning the transformation of (1).

Theorem 2. *Consider equation (1) and suppose that I is a submodulus or overmodulus interval for $\tau \in C^n(I)$ (if $n \geq 2$) or $\tau \in C^2(I)$ (if $n = 1$). Further, let $\tau'(x) \neq 0$ on I, $\sigma_1[\tau] = \{p\}$, $\sigma_2[\tau] = \emptyset$ and $|\tau'(p)| \neq 1$. If (1) has a solution then can be transformed by a change of the independent variable $t = \varphi(x) \in C^n(I)$, $\varphi'(x) > 0$ on I into a functional-differential equation of the form*

$$H(t, z(t), z(\lambda t), z'(t), z'(\lambda t), \ldots, z^{(n)}(t), z^{(n)}(\lambda t)) = 0 \qquad \text{on } J,$$

where $\lambda = \tau'(p)$ and $J = \varphi(I)$.

Moreover, the interval J with endpoints c, d has the following properties:

(i) if $\lim_{x\to a+} \tau(x) = a \in I$ (resp. $\lim_{x\to b-} \tau(x) = b \in I$) then $c = 0$ (resp.$d = 0$)

(ii) if $\lim_{x\to a+} \tau(x) = a \notin I$ (resp. $\lim_{x\to b-} \tau(x) = b \notin I$) then $c = -\infty$ (resp.$d = \infty$)

Proof. Let $y(x)$ be a solution of equation (1) in I. By a change of the independent variable $t = \varphi(x)$ we get the function $z(t) = z(\varphi(x)) = y(x)$ as a solution of an equation obtained from (1) by the above transformation. Since we seek the transformation function φ satisfying $z(\varphi(\tau(x))) = z(\lambda t)$, this function can be obtained as a solution of Schrőder's equation

$$\varphi(\tau(x)) = \lambda\varphi(x), \qquad x \in I.$$

Due to Theorem 1 this equation has the one-parameter family of C^n solutions with a positive derivative in I if the multiplicative parameter is positive. Further, notice that any $k-$th derivative of y at $\tau(x)$ ($k = 1, \ldots, n$) can be expressed in the form of a linear combination

$$y^{(k)}(\tau(x)) = a_k(t)z^{(k)}(\lambda t) + a_{k-1}z^{(k-1)}(\lambda t) + \cdots + a_0(t)z(\lambda t),$$

where $a_i(t)$ are suitable functions changing with respect to the degree k.

It remains to show that the interval J has the above given form. The property (i) is trivial because $\varphi(p) = 0$. As for the property (ii) concerns we prove, e.g., the case of transform right endpoint. It is obvious that τ has to be an increasing function. Let $x_0 \in (p, b)$. Because of the assumptions the iterates $\tau^n(x_0)$ exist for all integers n and

$$\lim_{n\to\gamma\infty} \tau^n(x_0) = b,$$

where $\gamma = \text{sign}(\tau(x_0) - x_0)$. Since $\varphi(\tau^n(x_0)) = \lambda^n\varphi(x_0)$ for all integers n and $\varphi(x_0) > 0$ we get

$$d = \lim_{x\to b+} \varphi(x) = \infty.$$

Quite analogously it can be proved the case of transform left endpoint. □

3. Canonical form of a linear homogeneous equation

In the next part we deal with the question of the transformation of a linear homogeneous functional-differential equation of the n-th order

$$y^{(n)}(x)+\sum_{i=0}^{n-1}p_i(x)y^{(i)}(x)+\sum_{i=0}^{n-1}q_i(x)y^{(i)}(\tau(x))=0, \qquad x\in[a,b) \tag{4}$$

where the initial set $E_a=\{a\}$, into an equation of the same type the solutions of which are defined on some interval J. We wish to obtain an equation more suitable not only by the form of the deviation but also by the form of its coefficients. For the sake of simplicity we restrict our considerations to equations with delay only, i.e., such that $\tau(x)\le x$ for $x\in[a,b)$. Remark that the next results are valid for some other types of deviation, e.g., for equations with an advanced argument.

In [8] there were derived the conditions under which equation (4) can be transformed into a linear homogeneous functional-differential equation of the n-th order with a constant deviation and with the vanishing coefficient at the $(n-1)$-th derivative of an unknown function. Similarly we can prove

Theorem 3. *Let $I:=[a,b)$ and consider equation (4), where $p_i,\ q_i\in C^0(I)$ $(i=0,1,\dots,n-1)$, $p_{n-1}\in C^{n-1}(I)$, $\tau\in C^{n+1}(I)$, $\tau(a)=a$, $\tau(x)<x$ for $x>a$, $\tau'(x)>0$ on I and $\tau'(a)\neq 1$. Then the singular case occurs and this equation can be transformed into an equation of the form*

$$z^{(n)}(t)+\sum_{i=0}^{n-2}r_i(t)z^{(i)}(t)+\sum_{i=0}^{n-1}s_i(t)z^{(i)}(\lambda t)=0, \qquad t\in J$$

where $\lambda=\tau'(a)$ and J has the same properties as in Theorem 1.

Proof. It is easy to see that $E_a=\{a\}$. The most general pointwise transformation which converts equation (4) into an equation of the same type on some interval J with the delay $\mu(t)=\lambda t$ has the form (see [11])

$$z(t)=g(t)y(h(t)),$$

where h is a C^n-diffeomorphism of J onto I, $h'(t)>0$ on J, $\tau(h(t))=h(\lambda t)$ on J and $g\in C^n(J)$, $g(t)\neq 0$ on J.

Denote by φ an increasing solution of Schrőder's equation (2) (where $f{=}\tau$) and put $J:=\varphi(I)$, $h:=\varphi^{-1}$ on J. Under the assumptions put on τ we get with respect to Theorem 1 that the function h satisfies all the required properties and, moreover, h is a C^{n+1}-diffeomorphism between J and I.

Further, as for the introducing g concerns, it can be followed step by step the method used in [8]. There it was derived that putting

$$g(t):=\exp\{(\frac{1}{n})\int_{a^*}^{h(t)}p_{n-1}(s)\mathrm{d}s\}(h'(t))^{\frac{1-n}{2}}, \qquad t\in J$$

where $a^* \in I$ we get the zero coefficient of $z^{(n-1)}(t)$. Obviously $g \in C^n(J)$ and $g(t) \neq 0$ on J. $\square$

Remark. It follows from the form of g that if equation (4) of the first order is considered then it is enough to require h to be C^1-diffeomorphism of J on I, hence φ to be C^1-diffeomorphism of I on J. However, as it was mentioned in Remark after Theorem 1 the assumption $\tau \in C^1(I)$ does not ensure this requirement.

Consequence. *Consider equation*

$$y'(x) + p(x)y(x) + q(x)y(\tau(x)) = 0 \qquad \text{on } I = [a, \infty)$$

where $p,\ q \in C^0(I),\ \tau \in C^2(I),\ \tau(a) = a,\ \tau(x) < x$ *for* $x > a,\ \lim_{x\to\infty} \tau(x) = \infty,\ \tau'(x) > 0$ *on* I *and* $\tau'(a) \neq 1$. *Then this equation can be globally converted into an equation*

$$z'(t) + s(t)z(\lambda t) = 0 \qquad \text{on } [0, \infty) \tag{5}$$

with $\lambda = \tau'(a)$ *and* $s(t) = \exp\{\int_{\tau(h(t))}^{h(t)} p(s)ds\}q(h(t))h'(t)$ *on* $[0, \infty)$, *where* $h = \varphi^{-1},\ \varphi(x) = \lim_{n\to\infty} \lambda^{-n}(\tau^n(x) - a)$ *in* I.

Proof. We show that s has the given form. Indeed,

$$z'(t) = g'(t)y(h(t)) + g(t)y'(h(t))h'(t),$$

hence it holds

$$y'(h(t)) + s(t)\frac{g(\lambda t)}{h'(t)g(t)}y(\tau(h(t))) = 0 \qquad \text{on } [0, \infty),$$

where $g(t) = \exp(\int_{a^*}^{h(t)} p(s)\mathrm{d}s)$. Therefore

$$s(t) = \frac{g(t)}{g(\lambda t)}q(h(t))h'(t).$$

Now the statement follows immediately. $\square$

Example. Consider

$$y'(x) + py(x) + qy(x^\alpha) = 0 \qquad \text{on } (1, \infty), \tag{6}$$

where $\alpha > 0,\ \alpha \neq 1,\ p,\ q \in \mathbb{R}$. Due to [7] this equation can be transformed into an equation with a constant deviation. The transformation function is a solution of Abel's equation

$$\varphi(x^\alpha) = \varphi(x) + c \qquad \text{on } (1, \infty),$$

where $\operatorname{sign} c = \operatorname{sign} \ln \alpha$. If we put $c = \ln \alpha$ then $\varphi(x) = \ln \ln x$ is one of solutions of this equation. Moreover, according to [8] it is possible to convert equation (6) into an equation

$$z'(t) + a(t)z(t + \ln \alpha) = 0 \qquad \text{on } (-\infty, \infty),$$

$a(t) = q\exp(t+\exp(t)+p\exp(\exp(t))-p\exp(\alpha\exp(t)))$, which is a suitable canonical form for equation (6). We have used the above mentioned transformation, where $h(t) = \varphi^{-1}(t) = \exp(\exp(t))$, $g(t) = \exp(p\int_2^{h(t)} \mathrm{d}s)$.

However, if we consider

$$y'(x) + py(x) + qy(x^\alpha) = 0 \qquad \text{on } [1, \infty), \tag{6'}$$

where $\alpha > 0$, $\alpha \neq 1$, $p, q \in \mathbb{R}$ then the above transformation cannot be done. On the other hand, $\tau(x) = x^\alpha$ fulfils all the assumptions of the previous statement, hence we can transform (6') into the form given there. One can read the corresponding Schrőder's equation as

$$\varphi(x^{\alpha)} = \alpha\varphi(x) \qquad \text{on } [1, \infty).$$

Since $\tau^n(x) = x^{\alpha^n}$ for all integers n we get C^∞ solutions of this functional equation on $[1, \infty)$ in the form $\varphi(x) = c \ln x$. Put $c = 1$. Using the considered pointwise transformation, where $h(t) = \varphi^{-1}(t) = \exp(t)$, $g(t) = \exp(p\int_1^{h(t)} \mathrm{d}s)$ we can convert equation (6') into an equation

$$z'(t) + s(t)z(\alpha t) = 0 \qquad \text{on } [0, \infty),$$

$s(t) = q\exp(t + p\exp(t) - p\exp(\alpha t))$, which may serve as a canonical form for equation (6').

Remark that equation (6') as well as equation (6) is the equation with the delay (resp.with the advanced argument) if and only if $\alpha < 1$ (resp. $\alpha > 1$).

4. Final remarks

We have seen that the existence of fixed points of a deviating argument plays a very important role in the investigation of transformations of functional-differential equations. If a deviating argument contains none or one fixed point in its domain then this deviation can be transformed under mild assumptions again into a deviation with none or one fixed point in its domain but having more suitable form, e.g., $\tau(x) = x - 1$ or $\tau(x) = \lambda x$. Generally, the considered global transformation preserves a number of fixed points of any deviating argument. Indeed, if we wish to convert an equation with a deviating argument τ_1 into an equation with a deviating argument τ_2 then studied transformation function φ can be obtained as a solution of the functional equation

$$\varphi(\tau_1(x)) = \tau_2(\varphi(x)).$$

Supposing τ_1 has n fixed points in its domain we see that τ_2 has to have the same number of fixed points as well, namely $\varphi(p_1), \dots, \varphi(p_n)$.

Some types of canonical forms for certain classes of linear homogeneous functional-differential equations were introduced in [8] and in the previous part. Because of the form of the used global transformation which preserves the distribution of zeros of solutions it is sufficient to restrict the investigation of oscillatory or nonoscillatory properties of a considered equation to the investigation of its canonical form. Similarly it is possible to study the asymptotic behavior of solutions of some linear homogeneous equation. The global transformation of any such equation into its canonical form is required to be invariant with respect to the investigated property. The equation (5) and its modifications have been extensively studied (see, e.g., [9], [5] or [3]) as well as the equation of the form (6) or (6') (see [2]). From this point of view we can make some results obtained in these areas more general.

References

[1] Eĺsgolc, L. E., *Vvedenije v teoriju differencialnych uravnenij s otklonjajuščimsa argumentom*, Nauka, Moscow, 1964. (Russian)

[2] Heard, M. L., *Asymptotic behavior of solutions of the functional differential equation* $x'(t) = ax(t) + bx(t^\alpha)$, $\alpha > 1$, J.Math.Anal.Appl. **44** (1973), 745–757.

[3] Kato, T., McLeod, J. B., *The functional-differential equation* $y'(x) = ay(\lambda x) + by(x)$, Bull. Amer. Math. Soc. **77** (1971), 891–937.

[4] Kuczma, M., *Functional Equations in a Single Variable*, Polish Scient.Publ., Warszawa, 1968.

[5] Lade, G. S., Lakshmikantham, V., Zhang, B. G., *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker, New York, 1983.

[6] Lim, E.-B., *Asymptotic behavior of solutions of the functional differential equation* $x'(t) = Ax(\lambda t) + Bx(t)$, $\lambda > 0$, J.Math.Anal.Appl. **55** (1976), 794–806.

[7] Neuman, F., *On transformations of differential equations and systems with deviating argument*, Czechoslovak Math.J. **31(106)** (1981), 87-90.

[8] Neuman, F., *Transformation and canonical forms of functional-differential equation*, Proc. Roy.
Soc.Edinburgh **115A** (1990), 349-357.

[9] Pandofi, L., *Some observations on the asymptotic behaviors of the solutions of the equation* $x'(t) = A(t)x(\lambda t) + B(t)x(t)$, $\lambda > 0$, J.Math.Anal.Appl. **67** (1979), 483–489.

[10] Szekeres, G., *Regular iteration of real and complex functions*, Acta Math. **100** (1958), 203–258.

[11] Tryhuk, V., *The most general transformation of homogeneous retarded linear differential equations of the n-th order*, Math.Slovaka **33** (1983), 15–21.

Jan Čermák
Department of Mathematics
Technical University of Brno
Technická 2
616 69 Brno, CZECH REPUBLIC