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EINSTEIN–LIKE SEMI–SYMMETRIC SPACES

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Abstract. One proves that semi-symmetric spaces with a Codazzi or Killing Ricci tensor are locally symmetric. Some applications of this result are given.

1. Introduction

A manifold (M, g) is said to be *semi-symmetric* ([S], [Sz1]) if its curvature tensor R satisfies the condition $R_{XY} \cdot R = 0$ for all vector fields X and Y on M, where R_{XY} acts as a derivation on R. This condition means that, at each point p of M, R_p is the same as the curvature tensor of a symmetric space. This symmetric space may change with the point. This class of spaces was first studied by E. Cartan ([Ca]) as a direct generalization of the class of symmetric spaces. Other authors in the field are A. Lichnerowicz, R. S. Couty and N. S. Sinjukov, who first used the name "semi-symmetric space" for manifolds satisfying the above curvature condition ([S]).

In 1982, Z. I. Szabó gave the full local classification of semi-symmetric spaces. He proved ([Sz, Theorem 4.5]) that a semi-symmetric space is locally a de Rham product of irreducible semi-symmetric spaces, namely symmetric spaces, two-dimensional surfaces, semi-symmetric Riemannian manifolds foliated by Euclidean spaces of codimension two, and six types of cones (real elliptic, real hyperbolic, real Euclidean and three types of Kählerian cones). We remark that he describes the class of semi-symmetric manifolds foliated by Euclidean spaces of codimension two only *implicitly*, i.e., he does not give explicit expressions for the metric of such spaces, but only an integrable system of partial differential equations and the exact number of solutions which follows from the Cauchy-Kowalewski theorem. The global classification is treated in [Sz2].

In [K2], O. Kowalski studies the class of foliated semi-symmetric spaces in dimension three. He solves the partial differential equations to obtain *explicit* expressions for the metrics of such spaces. Applying his method also to the higher-dimensional case, O. Kowalski, L. Vanhecke and the present author were able

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to give explicit expressions for the metrics of a special subclass of the foliated semi-symmetric spaces, namely those for which the scalar curvature is constant along the Euclidean leaves ([BKV]). In particular, this led to the classification of semi-symmetric spaces which are curvature homogeneous, or equivalently, have constant scalar curvature. The more general case will be treated in a forthcoming study ([Bo]).

In this note, we consider semi-symmetric spaces satisfying additional conditions on the Ricci tensor. More precisely, we will prove the

Main theorem. *A semi-symmetric manifold (M, g) whose Ricci tensor ρ is cyclic-parallel (i.e., $(D_X\rho)_{XX} = 0$), or is a Codazzi tensor (i.e., $(D_X\rho)_{YZ} = (D_Y\rho)_{XZ}$), is locally symmetric.*

Manifolds with cyclic-parallel or Codazzi tensor ρ are called *Einstein-like* in [G]. (See also [B, Chapter 16].) The proof of the main theorem is based on the classification theorem by Z. I. Szabó and on the explicit expression for the metric of a foliated semi-symmetric space with constant scalar curvature. In Section 3, we derive some immediate consequences for semi-symmetric spaces with volume-preserving geodesic symmetries and for semi-symmetric $\mathfrak{C}$-spaces ([BV]). Also, from the proof of the main theorem, we show that a locally homogeneous semi-symmetric space is locally symmetric.

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2. Proof of the main theorem

All manifolds, vector fields and differential forms are assumed to be C^∞. We denote the Riemann curvature tensor by R, the Ricci tensor by ρ, the scalar curvature by τ and the Levi Civita connection by D.

Let (M, g) be a (connected) semi-symmetric space whose Ricci tensor is either cyclic-parallel or a Codazzi tensor. Then, each factor in Szabó's structure theorem satisfies the same condition. Next, both conditions imply that the scalar curvature of each factor is constant (see, for example, [D'AN], [G]). As the scalar curvature of any type of cone in Szabó's structure theorem is not constant, the local decomposition of (M, g) does not contain factors of cone type. Moreover, a two-dimensional surface with constant scalar curvature is locally isometric to a two-dimensional symmetric space. Hence, (M, g) is locally a product of symmetric spaces and of Riemannian manifolds foliated by Euclidean spaces of codimension two with constant scalar curvature.

We consider now a space of this last type. Let (M^{n+1}, g) be an irreducible semi-symmetric manifold foliated by Euclidean spaces of codimension two with constant scalar curvature. In [BKV], O. Kowalski, L. Vanhecke and the present author proved that the metric of (M^{n+1}, g) is given explicitly, on an open and dense subset of M, by the following expressions for an orthonormal coframe in a special adapted coordinate system $(w, x^1, \dots, x^n)$:

$$
\begin{aligned}
\omega^0 &= f(w, x^1)\, dw, \\
\omega^i &= dx^i + \sum_{j=1}^{n} D^i_j(w) x^j\, dw, \qquad i = 1, \dots, n,
\end{aligned}
\tag{2.1}
$$

where $D^i_j(w) + D^j_i(w) = 0$ and $f''_{x^1x^1} + kf = 0$, for some non-zero constant k. The scalar curvature is given by $\tau = 2k$. Using these expressions, we show that the condition on the Ricci tensor implies that $\|DR\|^2 = 0$ on this open and dense subset, and hence everywhere. So, M will be locally symmetric.

Using the standard formulas (see [KN]), we easily obtain the connection forms ω^i_j:

$$
\begin{aligned}
\omega^0_1 &= f^{-1} f'_{x^1}\, \omega^0, \\
\omega^0_j &= 0, & j \geq 2, \\
\omega^i_j &= f^{-1} D^i_j(w)\, \omega^0, & i, j \geq 1,
\end{aligned}
$$

where $\omega^i_j + \omega^j_i = 0$. Then the Riemann curvature tensor and the Ricci tensor are given by

$$
\begin{aligned}
R &= 2\tau\, \omega^0 \wedge \omega^1 \otimes \omega^0 \wedge \omega^1, \\
\rho &= \frac{\tau}{2}(\omega^0 \otimes \omega^0 + \omega^1 \otimes \omega^1).
\end{aligned}
$$

By the standard formulas $D_X \omega^i = \sum \omega^i_j(X)\, \omega^j$, we then obtain

$$
\begin{aligned}
D_X \omega^0 &= \omega^0(X) f^{-1} f'_{x^1}\, \omega^1, \\
D_X \omega^1 &= -\omega^0(X) f^{-1} \,(f'_{x^1}\, \omega^0 - \sum_{j=2}^{n} D^1_j\, \omega^j).
\end{aligned}
$$

Hence, the covariant derivative of ρ is given by

$$
D_X \rho = \frac{\tau}{2} f^{-1} \omega^0(X) \sum_{j=2}^{n} D^1_j (\omega^1 \otimes \omega^j + \omega^j \otimes \omega^1).
\tag{2.2}
$$

Now, if ρ is cyclic-parallel or if it is a Codazzi tensor, then D^1_j must vanish for $j = 2, \dots, n$. So, we have

$$
\begin{aligned}
D_X \omega^0 &= \omega^0(X) f^{-1} f'_{x^1}\, \omega^1, \\
D_X \omega^1 &= -\omega^0(X) f^{-1} f'_{x^1}\, \omega^0.
\end{aligned}
$$

But then, obviously, $D_X R = 0$ for all vector fields X on M. Hence, (M^{n+1}, g) is locally symmetric. □

Note that the main theorem yields that semi-symmetric Einstein and semi-symmetric Ricci parallel spaces are locally symmetric. This result also follows at once from [Sz1].

3. Applications

Before giving some applications of the main theorem, we give a result which follows from its proof. We mention first that each homogeneous semi-symmetric space is locally symmetric ([Sz2]). From the formulas in the previous section we derive the following stronger result:

Proposition 3.1. *A locally homogeneous semi-symmetric space is locally symmetric.*

Proof. A locally homogeneous space always has constant scalar curvature. As in the previous section, it is then sufficient for our purpose to consider only spaces (M^{n+1}, g) with an orthonormal coframe of the form (2.1). From (2.2), it then follows that

$$\|D\rho\|^2 = \frac{\tau^2}{2} f^{-2} \sum_{j=2}^{n} (D_j^1)^2. \tag{3.1}$$

This is a global constant for a locally homogeneous space. In particular, the right-hand side of (3.1) does not depend on the variable x^1. Because $f \neq 0$ depends explicitly on x^1 and D_j^1 are independent of x^1, the only possibility is that $\sum_{j=2}^{n} (D_j^1)^2 = 0$, i.e., D_j^1 must vanish for $j = 2, \ldots, n$. But then, the proof of the main theorem shows that (M^{n+1}, g) is locally symmetric. $\square$

A first application of the main theorem itself concerns *spaces with volume-preserving geodesic symmetries*. These spaces were introduced by J. E. D'Atri and H. K. Nickerson in [D'AN]. Typical examples are spaces which are locally isometric to naturally reductive homogeneous spaces, homogeneous spaces such that all their geodesics are orbits of one-parameter subgroups of isometries, commutative spaces, probabilistic commutative spaces, generalized Heisenberg groups and harmonic spaces (see [K1], [V1], [V2]). All these examples are locally homogeneous spaces and it is still an open problem whether or not all spaces with volume-preserving geodesic symmetries are indeed locally homogeneous. Here we have

Corollary 3.2. *A semi-symmetric space with volume-preserving geodesic symmetries is locally symmetric.*

Proof. The curvature tensor of a space with volume-preserving geodesic symmetries satisfies an infinite number of curvature conditions, the so-called odd Ledger conditions (see [V1]). The first of this list expresses that its Ricci tensor must be cyclic-parallel. The main theorem then gives the result. $\square$

A second application deals with *$\mathfrak{C}$-spaces*. These spaces are introduced as generalizations of symmetric spaces by J. Berndt and L. Vanhecke in [BV]. They are characterized by the property that the eigenvalues of the Jacobi operators $R_\gamma = R_{\cdot\gamma'}\gamma'$ are constant along every geodesic γ. Several examples are given in [BV], [BPV] and it turns out that again all the known examples are locally homogeneous. Now we get

Corollary 3.3. *A semi-symmetric $\mathfrak{C}$-space is locally symmetric.*

Proof. It is shown in [BV] that the Ricci tensor of a $\mathfrak{C}$-space is a Killing tensor, or equivalently, is cyclic-parallel. Hence, the main theorem implies the result. □

Remark. This last result was obtained independently and by a different approach by J. T. Cho ([Ch]). Our method shows the usefulness of the explicit description of the metrics obtained in [BKV]. Finally, we derive a result concerning *globally Osserman spaces*. These spaces are characterized by the property that the eigenvalues of the Jacobi operator $R_X := R_{\cdot X}X$ are independent of the choice of unit vector $X \in T_pM$ and of the choice of $p \in M$. Osserman conjectured that all these spaces are locally isometric to a two-point homogeneous space ([O]). From the above characterizing property, it follows at once that a globally Osserman space is a $\mathfrak{C}$-space. As an immediate consequence of Corollary 3.3, we then get

Corollary 3.4. *A semi-symmetric globally Osserman space is locally isometric to a two-point homogeneous space.*

Proof. From the previous corollary we know that a semi-symmetric globally Osserman space is locally symmetric. It then follows from [GSV, Lemma 2.3] that it is locally isometric to a two-point homogeneous space. □

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