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s–WEAKLY REGULAR GROUP RINGS

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Abstract. In this note we obtain a necessary and sufficient condition for a ring to be s-weakly regular

(i) When R is a ring with identity and without divisors of zero
(ii) When R is a ring without divisors of zero. Further it is proved in a s-weakly regular ring with identity and without units every element is a zero divisor.

Following Gupta a ring R is s-weakly regular if for each $a \in A$, $a \in aAa^2A$. We in this note obtain conditions for a group ring to be s-weakly regular. For more about s-weakly regular rings please refer [1].

Example 1. Let $Z_2 = (0,1)$ be a field of characteristic 2 and $G = \langle g | g^3 = 1\rangle$ be a cyclic group. $Z_2G = \{0, 1, g, g^2, 1+g, 1+g^2, g+g^2, 1+g+g^2\}$ is the group ring of G over Z_2. Clearly Z_2G is s-weakly regular.

Every group ring is not s-weakly regular; by the following example.

Example 2. Let $G = \langle g | g^2 = 1\rangle$ and $Z_2 = (0,1)$. $1+g \in Z_2G$; but $1+g \notin 1+gZ_2G$. $(1+g)^2 Z_2G = \{0\}$. Hence Z_2G is not s-weakly regular.

Proposition 1. *Let $Z_2 = (0,1)$ and $G = \langle g | g^{2n} = 1\rangle$. The group ring Z_2G is not s-weakly regular.*

Proof. Take $a = 1 + g + \cdots + g^{2n-1}$ in Z_2G clearly $a \notin aZ_2Ga^2Z_2G$. Hence Z_2G is not s-weakly regular. □

Proposition 2. *Let $Z_2 = (0,1)$ and G be any group such that it has an element of order n where n is an even integer. Then the group ring Z_2G is not s-weakly regular.*

Proof. Let $g \in G$ with $g^n = 1$; clearly $a = 1 + g + \cdots + g^{n-1} \in Z_2G$ where $a \notin aZ_2Ga^2Z_2G$. Hence the result. □

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Theorem 3. *Let $Z_2 = (0,1)$ and S_n be the symmetric group of degree n. The group ring Z_2S_n is not s-weakly regular.*

Proof. Take $\alpha = 1 + \begin{pmatrix} 1 & 2 & 3 & \dots & i & \dots & j & \dots & n \\ 1 & 2 & 3 & \dots & j & \dots & i & \dots & n \end{pmatrix}$ in Z_2S_n.
Clearly $\alpha \notin \alpha Z_2S_n\, \alpha^2 Z_2 S_n$. Hence Z_2S_n is not s-weakly regular. □

Theorem 4. *Let $Z_p = (0, 1, \dots, p-1)$ be a field of characteristic p and G any group having an element of order p. Then the group ring Z_pG is not s-weakly regular.*

Proof. Take $\alpha = 1 + g + \cdots + g^{p-1}$ in Z_pG where $g \in G$ with $g^p = 1$. Clearly $\alpha \notin \alpha A \alpha^2 A$ as $\alpha^2 = 0$. Thus Z_pG is not s-weakly regular. □

Problem. Let $Z_p = (0, 1, \dots, p-1)$, p a prime and G be a group having no elements of order p. (1) Is Z_pG s-weakly regular? (11) If G has elements of finite orders say P_i, $i = 1, 2, \dots,$ such that $(p, P_i) = 1$, $i = 1, 2, \dots,$ is Z_pG s-weakly regular?.

Theorem 5. *Let R be a ring with identity. If R is a ring in which $a^3 = a$ for every $a \in R$ then R is a s-weakly regular ring.*

Proof. Obvious; as for every $a \in R$ we have $a \in aRa^2R$. (ie $a = a \cdot 1 \cdot a^2 \cdot 1$). □

Theorem 6. *Let R be a ring without identity. If for every $a \in R$; $a^5 = a$ then R is a s-weakly regular ring.*

Proof. Obvious; as for every $a \in R$ take $a = a \cdot a \cdot a^2 a = a^5 \in aRa^2R$. Hence the theorem. □

Theorem 7. *Let R be a finite ring without identity and without nilpotent elements then the ring R is a s-weakly regular ring.*

Proof. We have for every $a \in R$ $a \in aRa^2R$ as $a^n = a$ for some n, as R is a finite ring and as R has no nilpotent elements. □

Theorem 8. *Let R be a ring with identity and without divisors of zero. The ring R is s-weakly regular if and only if $a^2 = 1$ or $ba^2 \cdot c = 1$ for every $a \in R$.*

Proof. Given R is a ring with identity; which has no proper divisors of zero. Now let us assume R is s-weakly regular; to prove $a^2 = 1$ or $ba^2c = 1$ for every $a \in R$. Given R is s-weakly regular, hence $a \in aRa^2R$ for every $a \in R$. Thus $a = aba^2c$ for every $a \in R$; if $b = c = 1$; then we have $a = a^3$ i.e. $a(1 - a^2) = 0$ but R has no zero divisors; hence $a^2 = 1$. If $b \neq 1$, $c \neq 1$; then $a = aba^2c$ i.e. $a(1 - ba^2c) = 0$ since R has no zero divisors $1 = ba^2c$. □

Conversely if $1 = ba^2c$ or $a^2 = 1$ for every $a \in R$ we get immediately R to be s-weakly regular using the fact R has no zero divisors.

Theorem 9. *Let R be a ring without identity and without divisors of zero. R is s-weakly regular if and only if for every $a \in R$ there exists $b, c \in R$ with $a = aba^2c$.*

Proof. Given R is a ring without identity and without divisors of zero. Let R be a s-weakly regular; to prove $a = aba^2c$ for every $a \in R$. Given R is s-weakly regular hence for every $a \in R$ we have $a \in aRa^2R$; thus $a = aba^2c$ for some b, c.□

Conversely if $a = aba^2c$ for every $a \in R$; we have obviously R to be s-weakly regular as given R has no identity and zero divisors.

Theorem 10. *Let R be a s-weakly regular ring with 1 and without units. Then every element of R is a zero divisors.*

Proof. Given $1 \in R$, R is s-weakly regular and R has no units. To prove in R every element is a zero divisor. For every $a \in R$ we have $a \in aRa^2R$; hence $a = a \cdot 1 \cdot a^2 \cdot 1$ or $a = aba^2c$. In both cases we have $a(1 - a^2) = 0$ or $a(1 - ba^2c) = 0$; as we are given R has no units. Hence the result. □

References

[1] Gupta, V., *A generalization of strongly regular rings*, Acta. Math. Hungar **43** (1984), No 1-2, 57-61.

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