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NATURAL TRANSFORMATIONS TRANSFORMING VECTOR FIELDS INTO AFFINORS ON THE EXTENDED R–TH ORDER TANGENT BUNDLES

Włodzimierz M. Mikulski

Abstract. A classification of natural transformations transforming vector fields on n-manifolds into affinors on the extended r-th order tangent bundle over n-manifolds is given, provided $n \geq 3$.

0. The extended r-th order tangent bundle E^rM over an n-dimensional manifold M is defined as dual vector bundle $E^rM = (J^r(M, \mathbf{R}))^*$. The r-th order tangent bundle $T^rM = (J^r(M, \mathbf{R})_0)^*$ over M is a vector subbundle of E^rM and we have a natural decomposition $E^rM = T^rM \times \mathbf{R}$. For $r = 1$ we obtain the time-dependent tangent bundle $E^1M = TM \times \mathbf{R}$.

In this paper we determined all natural transformations transforming vector fields on n-dimensional manifolds into affinors (i.e. tensor fields of type (1.1)) on E^r. In item 6 we defined geometrically $2(r + 2)$ natural transformations transforming vector fields on n-dimensional manifolds into affinors on E^r. Then we prove that all natural transformations transforming vector fields on n-manifolds into affinors on E^r are their linear combinations, the coefficients of which are arbitrary smooth functions on $\mathbf{R}$, provided $n \geq 3$. Any natural affinor on E^r in the sense of J. Gancarzewicz and I. Kolář, c.f. [1], determines a constant natural transformation transforming vector fields into affinors on E^r. Hence this paper is a generalization of [1].

In items 1 — 4 we cite some definitions and propositions. In item 5 we introduce the definition of natural transformations transforming vector fields on n-dimensional manifolds into affinors on E^r. The main result (Theorem 6.1) is formulated in item 6. In item 7 we make some preparations to prove the main theorem. The proof of Theorem 6.1 is given in item 8.

All manifolds and maps are assumed to be of class C^∞. If M is a manifold, we denote the vector space of all vector fields on M by $\mathcal{X}(M)$. We denote the category of all n-dimensional manifolds and their embeddings by $\mathcal{M}_n$.

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1. Let M be a manifold. The vector bundle $\pi : E^rM = (J^r(M,\mathbf{R}))^* \to M$, where $J^r(M,\mathbf{R})$ is the vector bundle of r-jets of mappings $M \to \mathbf{R}$, is called *r-th order extended tangent bundle of* M. The target map $\beta : J^r(M,\mathbf{R}) \to \mathbf{R}$ is a vector bundle epimorphism of $J^r(M,\mathbf{R})$ onto the 1-dimensional vector bundle $M\times\mathbf{R}$ which admits a splitting defined by the r-jets of the constant function on M. Hence $ker\beta = J^r(M,\mathbf{R})_0$ is a vector subbundle of $J^r(M,\mathbf{R})$ such that $J^r(M,\mathbf{R}) = ker\beta \times \mathbf{R}$. The vector bundle $T^rM = (\ker\beta)^*$ is called *r-th order tangent bundle* over M. This is a vector subbundle of E^rM and we have a natural decomposition $E^rM = T^rM \times \mathbf{R}$, provided we have used the canonical identification of $\mathbf{R}$ with $\mathbf{R}^*$. Every smooth map $f : M \to N$ induces a linear map

$$J^r_{f(x)}(N,\mathbf{R}) \ni j^r_{f(x)}\varphi \to j^r_x(\varphi\circ f) \in J^r_x(M,\mathbf{R}),$$

$x \in M$, $\varphi : N \to \mathbf{R}$. The transposed linear map $E^r_xM \to E^r_{f(x)}N$ determines a vector bundle homomorphism $E^rf : E^rM \to E^rN$ covering f. One verifies easily that the rule $M \to E^rM$, $f \to E^rf$ is a bundle functor on the category of all manifolds in the sense of [2]. Since $E^rf(T^rM) \subset T^rN$ for every $f : M \to N$ and pullbacks of constant functions are constant functions, we have $E^rf = T^rf \times id_{\mathbf{R}}$ under the decomposition $E^rM = T^rM \times \mathbf{R}$.

2. An affinor on a manifold M is a tensor field of type (1.1) on M, i.e. a section $M \to (T\otimes T^*)(M)$ which is also interpreted as a vector bundle homomorphism $TM \to TM$ covering the identity on M. Let $\mathcal{F}$ be a natural bundle over $n-$dimensional manifolds, see e.g. [6]. Let us recall that a *natural affinor on $\mathcal{F}$* in the sense of [1] is a system of affinors Q_M on $\mathcal{F}M$, for every n-manifold M, satisfying the condition

$$(T(\mathcal{F}f)\otimes T^*(\mathcal{F}f^{-1}))\circ Q_M = Q_N \circ \mathcal{F}f$$

for every embedding $f : M \to N$.

In [1], the authors defined the following four natural affinors on $E^r|\mathcal{M}_n$.

I. Let $\delta_M : T(T^rM) \to T(T^rM)$ be the identity map. By means of the decomposition $T(E^rM) = T(T^rM)\times T\mathbf{R}$, $\delta = \{\delta_M\}$ induces a natural affinor $\tilde\delta = \{\tilde\delta_M\}$ on $E^r|\mathcal{M}_n$.

II. Analogously, the identity affinor $\delta^{\mathbf{R}} : T\mathbf{R} \to T\mathbf{R}$ on $\mathbf{R}$ induces a natural affinor $\tilde\delta^{\mathbf{R}}$ on $E^r|\mathcal{M}_n$. Let us observe that $\tilde\delta + \tilde\delta^{\mathbf{R}}$ is the identity affinor on $E^r|\mathcal{M}_n$.

III. Let $y \in T^rM$ and $x = \pi(y) \in M$. There is the natural linear isomorphism $\psi_y : V_y(T^rM) \to T^r_xM$ between the vertical space $V_y(T^rM) = T_y(T^r_xM)$ and the fiber T^r_xM of T^rM over x. The jet projection $\beta_1 : J^r(M,\mathbf{R})_0 \to J^1(M,\mathbf{R})_0$ induces an inclusion $i_M : TM = T^1M \to T^rM$. Now we define a linear map $V_{M,y} : T_y(T^rM) \to T_y(T^rM)$ as the composition

$$T_y(T^rM) \xrightarrow{T_y\pi} T_{\pi(y)}M \xrightarrow{i_M} T^r_{\pi(y)}M \xrightarrow{\psi_y^{-1}} V_y(T^rM) \subset T_y(T^rM).$$

Let $V_M : T(T^rM) \to T(T^rM)$ be defined by $V_M|T_y(T^rM) = V_{M,y}$ for any $y \in T^rM$. The system $V = \{V_M\}$ is a natural affinor on $T^r|\mathcal{M}_n$ which induces a natural affinor $\tilde{V}$ on $E^r|\mathcal{M}_n$.

IV. Let L_M be the Liouville vector field on T^rM, i.e. the vector field determined by the fibre homotheties. This is a natural vector field on T^rM. Then the system $L \otimes dt = \{L_M \otimes dt\}$ is a natural affinor on $E^r|\mathcal{M}_n$, where t is the canonical coordinate on $\mathbf{R}$.

Next, the authors proved the following proposition.

Proposition 2.1. ([1]) *All natural affinors on $E^r|\mathcal{M}_n$ are linear combinations of $\tilde{\delta}, \tilde{\delta}^{\mathbf{R}}, \tilde{V}$ and $L \otimes dt$, the coefficients of which are arbitrary smooth functions on $\mathbf{R}$.*

3. Let $\mathcal{F}$ be a natural bundle over n-manifolds. Let us recall that a *natural transformation transforming vector fields on n-manifolds into vector fields on $\mathcal{F}$* in the sense of [5] is a system of functions

$$\mathcal{D}_M : \mathcal{X}(M) \to \mathcal{X}(\mathcal{F}M),$$

for every n-manifold M, satisfying the following two conditions:

(a) (Naturality condition) for any two n-manifolds M, N, two vector fields $X \in \mathcal{X}(M), Y \in \mathcal{X}(N)$ and any embedding $f : M \to N$ the assumption $Tf \circ X = Y \circ f$ implies

$$T(\mathcal{F}f) \circ \mathcal{D}_M(X) = \mathcal{D}_N(Y) \circ \mathcal{F}f,$$

(b) (Regularity condition) if U is a manifold and $X : U \times M \to TM$ is a C^∞ map such that $X_t : M \to TM$, $X_t(y) = X(t, y)$, is a vector field on M for every $t \in U$, then the mapping

$$U \times \mathcal{F}M \ni (t, w) \to \mathcal{D}_M(X_t)(w) \in T(\mathcal{F}M)$$

is of class C^∞.

In [5], we have the following classification of natural transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$, provided $n \geq 2$.

I. For $s = 1, 2, ..., r$ the s-iterated differentiation $X \circ X \circ ... \circ X(f)(x)$ of $f : M \to \mathbf{R}$, $f(x) = 0$, with respect to $X \in \mathcal{X}(M)$ gives a linear map $J^r_x(M, \mathbf{R})_0 \to \mathbf{R}$, i.e. an element $\overset{(s)}{D}_M(X)(x) \in T^r_xM$. Hence we have a section $\overset{(s)}{D}_M(X)$ of T^rM. This section (using the fibre translations) one can extend to a vertical vector field $\overset{(s)}{D}{}^V_M(X)$ on T^rM. Of course, the family $\overset{(s)}{D}{}^V$ of functions

$$\mathcal{X}(M) \ni X \to \overset{(s)}{D}{}^V_M(X) \in \mathcal{X}(T^rM),$$

$M \in \mathcal{M}_n$, is a natural transformation transforming vector fields on n-ma-nifolds into vector fields on $T^r|\mathcal{M}_n$.

II. On T^rM we have the Liouville vector field $L_M \in \mathcal{X}(T^rM)$ defined by the fibre homotheties. Of course the family L of constant functions L_M of $\mathcal{X}(M)$, $M \in \mathcal{M}_n$, is a natural transformation transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$.

III. On T^r we have also the complete lifting of vector fields defined by

$$T^r(X) = \frac{\partial}{\partial t}|_0 T^r(exp\ tX),$$

where $exp\ tX$ is the flow of X on M. This is also a natural transformation transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$.

In [5], we proved the following proposition.

Proposition 3.1. ([5]) *All natural transformations transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ are linear combinations of $\overset{(1)}{D}{}^V, ..., \overset{(r)}{D}{}^V$, L and T^r, the coefficients of which are arbitrary real numbers, provided $n \geq 2$.*

4. Let $\mathcal{F}$ be a natural bundle over $n-$manifolds. Let us recall that a *natural transformation transforming vector fields on n-manifolds into functions on* $\mathcal{F}$ is a system of functions

$$\mathcal{L}_M : \mathcal{X}(M) \to C^\infty(\mathcal{F}M),$$

for every n-manifold M, such that for any two n-manifolds M, N, two vector fields $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ and any embedding $f : M \to N$ the assumption $Tf \circ X = Y \circ f$ implies

$$\mathcal{L}_M(X) = \mathcal{L}_N(Y) \circ \mathcal{F}f.$$

We have the following proposition.

Proposition 4.1. ([4]) *Let $\mathcal{F}|\mathcal{M}_n$ be the restriction of a bundle functor (defined on all manifolds and all maps) to $\mathcal{M}_n$, $n \geq 2$. Let $\mathcal{L} = \{\mathcal{L}_M\}$ be a natural transformation transforming vector fields on n-manifolds to functions on $\mathcal{F}|\mathcal{M}_n$. Then there exists a map $h : \mathcal{F}\mathbf{R}^0 \to \mathbf{R}$ such that $\mathcal{L}_M(X) = h \circ \mathcal{F}q_M$ for any $M \in \mathcal{M}_n$ and any $X \in \mathcal{X}(M)$, where $q_M : M \to \mathbf{R}^0 = \{0\}$ is the map. In particular, $\mathcal{L}_M = const$ on $\mathcal{X}(M)$.*

5. Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds. A *natural transformation transforming vector fields on n-manifolds into affinors on* $\mathcal{F}$ is a system of affinors $Q_M(X)$ on $\mathcal{F}M$, for every n-manifold M and every vector field $X \in \mathcal{X}(M)$, satisfying the following two conditions:

(a) (Naturality condition) for every embedding $f : M \to N$ of two n-manifolds and every vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(N)$ the assumption $Tf \circ X = Y \circ f$ implies

$$(T(\mathcal{F}f) \otimes T^*(\mathcal{F}f^{-1})) \circ Q_M(X) = Q_N(Y) \circ \mathcal{F}f,$$

(b) (Regularity condition) if U is a manifold and $X : U \times M \to TM$ is a C^∞ map such that $X_t : M \to TM$, $X_t(y) = X(t, y)$, is a vector field on M for every $t \in U$, then the mapping

$$U \times T(\mathcal{F}M) \ni (t, w) \to Q_M(X_t)(w) \in T(\mathcal{F}M)$$

is of class C^∞.

Since any non-vanishing vector field is (locally) $\frac{\partial}{\partial x^1}$ with respect to some coordinate system, then (by the naturality condition) we get the following lemma. (The proof is similar to the proof of Lemma 2.1 in [5].)

Lemma 5.1. *Let Q^1, Q^2 be two natural transformations transforming vector fields on n-manifolds into affinors on $\mathcal{F}$ such that*

$$Q^1_{\mathbf{R}^n}(\partial_1)|T_v(\mathcal{F}\mathbf{R}^n) = Q^2_{\mathbf{R}^n}(\partial_1)|T_v(\mathcal{F}\mathbf{R}^n)$$

for any $v \in \mathcal{F}_0\mathbf{R}^n$, where $\partial_1 = \frac{\partial}{\partial x^1}$ is the canonical vector field on $\mathbf{R}^n$. Then $Q^1 = Q^2$.

If $\{Q_M\}$ is a natural affinor on $\mathcal{F}$, then $\tilde{Q}_M(X) = Q_M$, $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, is a natural transformation transforming vector fields on n-manifolds into affinors on $\mathcal{F}$. Conversely, if $Q_M(X)$, $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, is a natural transformation transforming vector fields on n-manifolds into affinors on $\mathcal{F}$, then $Q_M(0_M)$, $M \in \mathcal{M}_n$, is a natural affinor on $\mathcal{F}$, where $0_M \in \mathcal{X}(M)$ is the 0 vector field.

Our problem is to find all natural transformations transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$.

6. First we define $2(r+2)$ natural transformations transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$.

I. The natural affinors $\tilde{\delta}, \tilde{\delta}^{\mathbf{R}}$ described in item 2 are natural transformations transforming vector fields on $n-$manifolds into affinors on $E^r|\mathcal{M}_n$.

II. Let $\mathcal{D} \in \{L, T^r, \overset{(s)}{D}{}^V, s = 1, ..., r\}$ be a natural transformation transforming vector fields on $n-$manifolds into vector fields on $T^r|\mathcal{M}_n$, see item 3. Then the system $\mathcal{D} \otimes dt = \{\mathcal{D}_M(X) \otimes dt\}$ is a natural transformation transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$, where t is the canonical coordinate on $\mathbf{R}$.

III. Let $s = 0, 1, ..., r-1$. Let $X \in \mathcal{X}(M)$. Let $y \in T^rM$ and $x = \pi(y) \in M$. There is the natural isomorphism $\psi_y : V_y(T^rM) \to T^r_xM$ between the vertical space $V_y(T^rM) = T_y(T^r_xM)$ and the fiber T^r_xM of T^rM over x. For any $v \in T_xM$, we have the (naturally dependent on x, v, X) linear map $\overset{s}{i}_{M,x,X}(v) : J^r_x(M, \mathbf{R})_0 \to \mathbf{R}$ given by

$$\overset{s}{i}_{M,x,X}(v)(j^r_x\gamma) = v(X^{(s)}(\gamma)),$$

where $X^{(s)} = X \circ ... \circ X$, s-times. Hence we have the (naturally dependent on x and X) linear map $\overset{s}{i}_{M,x,X} : T_xM \to T^r_xM$. (We see that $\overset{0}{i}_{M,x,X} = i_M|T_xM$, where $i_M : TM \to T^r M$ is the natural inclusion defined in item 2.) Now, we define a linear map $\overset{(s)}{Q}_{y,M}(X) : T_y(T^r M) \to T_y(T^r M)$ as the composition

$$T_y(T^r M) \xrightarrow{T_y \pi} T_x M \xrightarrow{\overset{s}{i}_{M,x,X}} T^r_x M \xrightarrow{\psi_y^{-1}} V_y(T^r M) \subset T_y(T^r M).$$

Let $\overset{(s)}{Q}_M(X) : T(T^r M) \to T(T^r M)$ be defined by $\overset{(s)}{Q}_M(X)|T_y(T^r M) = \overset{(s)}{Q}_{y,M}(X)$ for any $y \in T^r M$. The system $\overset{(s)}{Q} = \{\overset{(s)}{Q}_M(X)\}$ is a natural transformation transforming vector fields on n-manifolds into affinors on $T^r|\mathcal{M}_n$ which induces the natural transformation $\overset{(s)}{Q}{}^+$ transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$. Thus $\overset{(s)}{Q}{}^+_M(X)(v,w) = (\overset{(s)}{Q}_M(X)(v), 0) \in T_{(y,\tau)}E^r M$ for every $(v,w) \in T_{(y,\tau)}E^r M = T_y T^r M \times T_\tau \mathbf{R}$, $(y,\tau) \in E^r M = T^r M \times \mathbf{R}$, $M \in \mathcal{M}_n$ and $X \in \mathcal{X}(M)$. Of course, $\overset{(0)}{Q}{}^+ = \tilde{V}$ (see item 2).

We remark that if $f : \mathbf{R} \to \mathbf{R}$ is a mapping and Q a natural transformation transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$, then fQ is a natural transformation transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$ given by

$$(fQ)_M(X)(v,w) = f(\tau) Q_M(X)(v,w)$$

for any $(v,w) \in T_{(y,\tau)}E^r M = T_y T^r M \times T_\tau \mathbf{R}$, $(y,\tau) \in E^r M = T^r M \times \mathbf{R}$, $M \in \mathcal{M}_n$ and $X \in \mathcal{X}(M)$.

The main result of this paper is the following theorem.

Theorem 6.1. *All natural transformations transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$ are linear combinations of $\tilde{\delta}$, $\tilde{\delta}^{\mathbf{R}}$, $L \otimes dt$, $T^r \otimes dt$, $\overset{(s)}{D}{}^V \otimes dt$, $s = 1, \dots, r$, and $\overset{(s)}{Q}{}^+$, $s = 0, 1, \dots, r-1$, the coefficients of which are arbitrary smooth functions on $\mathbf{R}$, provided $n \geq 3$.*

Since any natural transformation transforming vector fields on n-manifolds into affinors on $T^r|\mathcal{M}_n$ induces the natural transformation transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$ (constant with respect to the coordinate on $\mathbf{R}$), we have the following corollary of Theorem 6.1.

Corollary 6.1. *All natural transformations transforming vector fields on n-manifolds into affinors on $T^r|\mathcal{M}_n$ are linear combinations of δ (see item 2) and $\overset{(s)}{Q}$, $s = 0, 1, \dots, r-1$, the coefficients of which are arbitrary real numbers, provided $n \geq 3$.*

The proof of Theorem 6.1 will occupy the rest of the paper.

7. We start with the proof of the following technical proposition.

Proposition 7.1. *Let Q be a natural transformation transforming vector fields on $n-$manifolds into affinors on $T^r|\mathcal{M}_n$, $n \geq 3$. Suppose that*

$$Q_{\mathbf{R}^n}(0_{\mathbf{R}^n}) = 0, \tag{7.1}$$

$$Q_{\mathbf{R}^n}(X)(V_0(T^r\mathbf{R}^n)) \subset \{0\} \tag{7.2}$$

for any $X \in \mathcal{X}(\mathbf{R}^n)$, where $0_{\mathbf{R}^n} \in \mathcal{X}(\mathbf{R}^n)$ is the 0 vector field and $V_0(T^r\mathbf{R}^n)$ denotes the vertical space of $T^r\mathbf{R}^n$ at $0 \in T^r_0\mathbf{R}^n$. Then there exist real numbers $\lambda_1, ..., \lambda_{r-1}$ such that $Q = \lambda_1 \overset{(1)}{Q} + ... + \lambda_{r-1} \overset{(r-1)}{Q}$.

To prove this proposition we need some preparations.

Throughout the whole of this item we shall keep the following notation. Let $q = card(S)$, where

$$S = \{\alpha = (\alpha_1, ..., \alpha_n) \in (\mathbf{N} \cup \{0\})^n : 1 \leq |\alpha| = \alpha_1 + ... + \alpha_n \leq r\}.$$

For every $\alpha = (\alpha_1, ..., \alpha_n) \in S$ let $x^\alpha : \mathbf{R}^n \to \mathbf{R}$ be given by $x^\alpha(y^1, ..., y^n) = (y^1)^{\alpha_1}...(y^n)^{\alpha_n}$. Let $X^\alpha : \mathbf{R}^q \to \mathbf{R}$ $(\alpha \in S)$ be the projection onto α-th factor. By Ω we denote the linear isomorphism

$$\Omega : T^r_0\mathbf{R}^n = (J^r_0(\mathbf{R}^n, \mathbf{R})_0)^* \to \mathbf{R}^q, \qquad \Omega(w) = (w(j^r_0 x^\alpha); \alpha \in S).$$

Given $l \in \mathbf{N}$ and $i = 1, ..., n$ let $\varphi^i_l : \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$\varphi^i_l(y) = y + (y^n)^l e_i,$$

where $y = (y^1, ..., y^n) \in \mathbf{R}^n$ and $e_i = (0, ..., 1, ..., 0) \in \mathbf{R}^n$, 1 in i-th position. In [5], we proved the following lemma.

Lemma 7.1. (Lemma 5.1 in [5]) *Let $h : \mathbf{R}^q \to \mathbf{R}^m$, $m \in \mathbf{N}$, be a polynomial in the X^α, $\alpha \in S$, such that*

$$\frac{\partial}{\partial X^\beta} h = 0 \text{ and } \frac{\partial}{\partial X^\beta}(h \circ \Omega \circ T^r_0\varphi^i_l \circ \Omega^{-1}) = 0$$

for all $\beta \in S$ with $|\beta| = r$ and all integers $l \geq 2$ and $i = 1, ..., n$. Then $h = const$.

Using this lemma we prove the following one.

Lemma 7.2. *Let Q be as in Proposition 7.1. Then*

$$Q_{\mathbf{R}^n}(t\partial_1)(V_w(T^r\mathbf{R}^n)) = \{0\}, \quad \text{and} \tag{7.3}$$

$$Q_{\mathbf{R}^n}(t\partial_1)(T^r(s\partial_2)(w)) \in V_w(T^r\mathbf{R}^n) \tag{7.4}$$

for any $t, s \in \mathbf{R}$ and $w \in T^r_0\mathbf{R}^n$, where T^rX is the complete lift of X to $T^r\mathbf{R}^n$.

Proof. For every $t \in \mathbf{R}$ we define $F_t : \mathbf{R}^q \times \mathbf{R}^q \to \mathbf{R}^n$ to be the composition

$$\mathbf{R}^q \times \mathbf{R}^q \xrightarrow{\Omega^{-1}\times\Omega^{-1}} T^r_0\mathbf{R}^n \times T^r_0\mathbf{R}^n$$

$$\xrightarrow{J} (VT^r)_0\mathbf{R}^n \xrightarrow{Q_{\mathbf{R}^n}(t\partial_1)} (TT^r)_0\mathbf{R}^n \xrightarrow{T\pi} T_0\mathbf{R}^n = \mathbf{R}^n,$$

where J is the diffeomorphism given by $J(w,u) = (\psi_w)^{-1}(u)$ $(= \frac{\partial}{\partial \tau}|_{\tau=0}(w+\tau u))$. Then the map $F : \mathbf{R} \times \mathbf{R}^q \times \mathbf{R}^q \to \mathbf{R}^n$, $F(t,.) = F_t$, $t \in \mathbf{R}$, is of class C^∞, because of the regularity condition. From the naturality condition with respect to the homotheties $\tau id_{\mathbf{R}^n}$, $\tau \in \mathbf{R} - \{0\}$, it follows that

$$F(\tau t, \tau^{|\alpha|}Y^\alpha, \tau^{|\beta|}Z^\beta; \alpha, \beta \in S) = \tau F(t, Y^\alpha, Z^\beta; \alpha, \beta \in S)$$

for all $\tau \in \mathbf{R} - \{0\}, t \in \mathbf{R}$ and $(Y^\alpha; \alpha \in S), (Z^\beta; \beta \in S) \in \mathbf{R}^q$. By the homogeneous function theorem, c.f. [3], F is linear with respect to t, Y^{e_j}, Z^{e_k}, for $j,k = 1,...,n$ and it is independent of the Y^α, Z^β with $|\alpha| > 1$ and $|\beta| > 1$. By (7.1), $F(0, Y^{e_j}, Z^{e_k}; j,k = 1,...,n) = 0$. Since $Q_{\mathbf{R}^n}(t\partial_1)$ is an affinor, $F(t,0,0) = 0$. Therefore $F = 0$. Hence

$$Q_{\mathbf{R}^n}(t\partial_1)(V_w(T^r\mathbf{R}^n)) \subset V_w(T^r\mathbf{R}^n)$$

for all $t \in \mathbf{R}$ and $w \in T^r_0\mathbf{R}^n$.

For any $t \in \mathbf{R}$ and $w \in T^r_0\mathbf{R}^n$ let $\tilde{H}(t,w) = (\tilde{H}^\alpha_\beta(t,w))\alpha, \beta \in S$ be the matrix of the linear map

$$Q_{\mathbf{R}^n}(t\partial_1)|V_w(T^r\mathbf{R}^n) : V_w(T^r\mathbf{R}^n) \to V_w(T^r\mathbf{R}^n)$$

with respect to the basis $((\Omega^{-1})_* \frac{\partial}{\partial X^\gamma})(w)$, $\gamma \in S$.

We see that the formula (7.3) will be proved after proving that $\tilde{H}(t,w) = 0$ for all $t \in \mathbf{R}$ and $w \in T^r_0\mathbf{R}^n$. Consider the map $H : \mathbf{R} \times \mathbf{R}^q \to gl(q) = \mathbf{R}^q \otimes (\mathbf{R}^q)^* = \mathbf{R}^{q^2}$

$$H(t, Y^\beta; \beta \in S) = \tilde{H}(t, \Omega^{-1}(Y^\beta; \beta \in S)).$$

H is of class C^∞, because of the regularity condition. By the naturality condition with respect to the homotheties, we obtain that

$$H^\beta_\alpha(\tau t, \tau^{|\gamma|}Y^\gamma; \gamma \in S) = \tau^{|\alpha|-|\beta|}H^\beta_\alpha(t, Y^\gamma; \gamma \in S)$$

for any $\tau \in \mathbf{R} - \{0\}, t \in \mathbf{R}$, $(Y^\gamma; \gamma \in S) \in \mathbf{R}^q$ and $\alpha, \beta \in S$, where $H = (H^\beta_\alpha; \alpha, \beta \in S)$. Since $|\alpha| - |\beta| \le r$ for all $\alpha, \beta \in S$ and $H(0,.) = 0$ (because of the formula (7.1)), then (by the homogeneous function theorem) $H(t,.) : \mathbf{R}^q \to \mathbf{R}^q$ is a polynomial (in the X^α, $\alpha \in S$) and

$$\frac{\partial}{\partial X^\gamma}(H(t,.)) = 0$$

for any $t \in \mathbf{R}$ and any $\gamma \in S$ with $|\gamma| = r$. Since $n \ge 2$, then φ^i_l preserves ∂_1, and then (by the naturality condition)

$$H(t,.) \circ \Omega \circ T^r_0\varphi^i_l \circ \Omega^{-1} = ((\Omega \circ T^r_0\varphi^i_l \circ \Omega^{-1}) \otimes (\Omega \circ T^r_0(\varphi^i_l)^{-1} \circ \Omega^{-1})^*) \circ H(t,.)$$

for all $t \in \mathbf{R}$, $i = 1,...,n$ and $l \in \mathbf{N}$. Therefore $H(t,.) = const$ for any $t \in \mathbf{R}$, because of Lemma 7.1. On the other hand, from (7.2) we get that $H(t,0) = 0$ for any $t \in \mathbf{R}$. Hence $H = 0$. The formula (7.3) is proved.

It remains to prove the formula (7.4). Let us consider the map $G : \mathbf{R}\times\mathbf{R}\times\mathbf{R}^q \to \mathbf{R}^n$ given by

$$\begin{aligned} &G(t, s, Y^\alpha; \alpha \in S) \\ &= T\pi \circ Q_{\mathbf{R}^n}(t\partial_1)(T^r(s\partial_2)(\Omega^{-1}(Y^\alpha; \alpha \in S))) \in T_0\mathbf{R}^n = \mathbf{R}^n. \end{aligned}$$

Using the same arguments as for F, we deduce that $G = 0$, as well. □

We are now in position to prove Proposition 7.1. We define the map $K : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^q \to \mathbf{R}^q$ by

$$\begin{aligned} &K(t, s, Y^\alpha; \alpha \in S) \\ &= \Omega \circ \psi_{\Omega^{-1}(Y^\alpha;\alpha\in S)} \circ Q_{\mathbf{R}^n}(t\partial_1)(T^r(s\partial_2)(\Omega^{-1}(Y^\alpha; \alpha \in S))), \end{aligned}$$

where $\psi_w : V_w(T^r\mathbf{R}^n) \to T^r_0\mathbf{R}^n$ is the isomorphism. By the formula (7.4) K is well-defined. Using similar arguments as for H we see that $K(t, s, .) : \mathbf{R}^q \to \mathbf{R}^q$ is a polynomial (in the $X^\gamma, \gamma \in S$) and

$$\frac{\partial}{\partial X^\beta}(K(t, s, .)) = 0$$

for all $\beta \in S$ with $|\beta| = r$ and all $t, s \in \mathbf{R}$. Since $n \geq 3$, then φ^i_l preserves ∂_1 and ∂_2, and then (by the naturality condition)

$$K(t, s, .) \circ \Omega \circ T^r_0\varphi^i_l \circ \Omega^{-1} = \Omega \circ T^r_0\varphi^i_l \circ \Omega^{-1} \circ K(t, s, .)$$

for all $i = 1, ..., n, l \in \mathbf{N}$ and all $t, s \in \mathbf{R}$. Then $K(t, s, .) = const$ for every $t, s \in \mathbf{R}$, because of Lemma 7.1.

By the naturality condition with respect to the homotheties $\mathbf{R}^n \ni (y^1, ..., y^n) \to (y^1, y^2, \tau y^3, ..., \tau y^n) \in \mathbf{R}^n,\ \tau \in \mathbf{R} - \{0\}$, it follows that

$$K^\alpha(t, s) = \tau^{\alpha_3+\dots+\alpha_n} K^\alpha(t, s)$$

for all $\alpha = (\alpha_1, ..., \alpha_n) \in S$, $t, s \in \mathbf{R}$ and $\tau \in \mathbf{R} - \{0\}$, where $K = (K^\alpha; \alpha \in S)$. Therefore $K^\alpha = 0$ for all $\alpha \in S$ with $\alpha_3 + ... + \alpha_n \neq 0$.

Since $Q_{\mathbf{R}^n}(t\partial_1)$ is an affinor, then K is linear with respect to s. Then using the homotheties $\mathbf{R}^n \ni (y^1, ..., y^n) \to (y^1, \tau y^2, y^3, ..., y^n) \in \mathbf{R}^n, \tau \in \mathbf{R} - \{0\}$, we get (similarly as above) that $K^\alpha = 0$ for all $\alpha = (\alpha_1, \alpha_2, 0) \in S$ with $\alpha_2 \neq 1$.

Similarly, using $\tau id_{\mathbf{R}^n}$ and (7.1) we get $K^{(0,1,0)} = 0$.

On the other hand by the definition of $\overset{(s)}{Q}$ it is easy to verify that

$$\Omega \circ \psi_{\Omega^{-1}(Y^\beta;\beta\in S)} \circ \overset{(s)}{Q}_{\mathbf{R}^n}(\partial_1)(T^r\partial_2(\Omega^{-1}(Y^\beta; \beta \in S))) = (\frac{1}{s!}\delta^{(s,1,0)}_\alpha; \alpha \in S) \tag{7.5}$$

for any $(s, 1, 0) \in S$ and $(Y^\beta; \beta \in S) \in \mathbf{R}^q$, where δ^α_β is the Kronecker delta. Therefore

$$Q_{\mathbf{R}^n}(\partial_1)(T^r\partial_2(w)) = \sum_{s=1}^{r-1} \lambda_s \overset{(s)}{Q}_{\mathbf{R}^n}(\partial_1)(T^r\partial_2(w)) \tag{7.6}$$

for any $w \in T^r_0 \mathbf{R}^n$, where $\lambda_s = s! K^{(s,1,0)}(1,1) \in \mathbf{R}$.

If $(\mu^1, ..., \mu^n), e_1, e_2 \in \mathbf{R}^n$ are linearly independent, then there exists a linear isomorphism $\varphi_\mu : \mathbf{R}^n \to \mathbf{R}^n$ such that φ_μ preserves ∂_1 and

$$T\varphi_\mu \circ \partial_2 = \sum_{j=1}^{n} \mu^j \partial_j \circ \varphi_\mu .$$

Hence by the naturality condition with respect to φ_μ it follows from (7.6) that

$$Q_{\mathbf{R}^n}(\partial_1)(T^r(\sum_{j=1}^{n} \mu^j \partial_j)(w)) = \sum_{s=1}^{r-1} \lambda_s \overset{(s)}{Q}_{\mathbf{R}^n}(\partial_1)(T^r(\sum_{j=1}^{n} \mu^j \partial_j)(w))$$

for any $w \in T^r_0 \mathbf{R}^n$. Then from the formula (7.3) and the similar formula for $\overset{(s)}{Q}$, $s = 0, ..., r-1$, it follows that

$$Q_{\mathbf{R}^n}(\partial_1)|T_w(T^r\mathbf{R}^n) = \sum_{s=1}^{r-1} \lambda_s \overset{(s)}{Q}_{\mathbf{R}^n}(\partial_1)|T_w(T^r\mathbf{R}^n)$$

for any $w \in T^r_0 \mathbf{R}^n$. The proposition is proved, because of Lemma 5.1. □

8. Now, we prove Theorem 6.1. Let us consider $Q = \{Q_M(X)\}$ a natural transformation transforming vector fields on n-manifolds into affinors on $E^r|\mathcal{M}_n$. Let $Q^0 = \{Q_M(0_M)\}$ be the natural affinor on $E^r|\mathcal{M}_n$, where $0_M \in \mathcal{X}(M)$ is the 0-vector field. Using Proposition 2.1 and replacing Q by $Q - Q^0$ one can assume that

$$Q_M(0_M) = 0 : T(E^r M) \to T(E^r M), \text{ for } M \in M_n. \tag{8.1}$$

Using the isomorphisms $T_{(w,\tau)}E^r M = T_w(T^r M) \times T_\tau \mathbf{R} = T_w(T^r M) \times \mathbf{R}$, where $(w, \tau) \in E^r M = T^r M \times \mathbf{R}$, one can define a natural transformation $\mathcal{L}_M : \mathcal{X}(M) \to C^\infty(T(E^r M))$, $M \in \mathcal{M}_n$, transforming vector fields on n-manifolds into functions on $TE^r|\mathcal{M}_n$ by

$$\mathcal{L}_M(X)(v) = \text{ the } \mathbf{R}- \text{ component of } Q_M(X)(v),$$

where $X \in \mathcal{X}(M)$, $v \in T_{(w,\tau)}(E^r M)$, $(w, \tau) \in E^r M$. Then by Proposition 4.1 and (8.1)

$$\{\mathcal{L}_M\} = 0. \tag{8.2}$$

Therefore for any $\tau \in \mathbf{R}$ we have a natural transformation $\mathcal{D}^\tau_M : \mathcal{X}(M) \to \mathcal{X}(T^r M)$, $M \in \mathcal{M}_n$, transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ given by

$$\mathcal{D}^\tau_M(X)(w) = Q_M(X)(\frac{\partial}{\partial t}(w, \tau)) \in T_w(T^r M) \times \{0\} = T_w(T^r M),$$

where $(w,\tau)\in E^rM=T^rM\times\mathbf{R}$, $M\in\mathcal{M}_n$, $X\in\mathcal{X}(M)$ and $\frac{\partial}{\partial t}$ is the canonical vector field on $T^rM\times\mathbf{R}$, $\frac{\partial}{\partial t}(w,\tau)=\frac{\partial}{\partial t}|_{t=0}[t\to(w,\tau+t)]$. Since $Q_M(0_M)=0$, $\mathcal{D}^\tau_M(0_M)=0$. Then (by Proposition 3.1), there exist the functions $a,c_1,...,c_r:\mathbf{R}\to\mathbf{R}$ (not necessarily smooth) such that

$$\mathcal{D}^\tau_M(X)(w)=a(\tau)T^r_MX(w)+c_1(\tau)\overset{(1)}{D}{}^V_M(X)(w)+...+c_r(\tau)\overset{(r)}{D}{}^V_M(X)(w).$$

Since $T^r_{\mathbf{R}^n}(\partial_1)(0)$, $\overset{(s)}{D}{}^V_{R^n}(\partial_1)(0)$, where $s=1,...,r, 0\in T_0\mathbf{R}^n$, are linearly independent, then $a,c_1,...,c_r$ are of class C^∞. Therefore replacing Q by $Q-aT^r\otimes dt-c_1\overset{(1)}{D}{}^V\otimes dt-...-c_r\overset{(r)}{D}{}^V\otimes dt$ one can assume that

$$Q_M(X)(\frac{\partial}{\partial t})=0,\ \text{for any } M\in\mathcal{M}_n,\ X\in\mathcal{X}(M). \tag{8.3}$$

For any $M\in\mathcal{M}_n$, $X\in\mathcal{X}(M)$, $x\in M$, $w\in T^r_xM$ and $\tau\in\mathbf{R}$ we denote the composition of linear mappings

$$T^r_xM\overset{\psi_0}{=}V_0(T^rM)$$
$$=V_0(T^rM)\times\{0\}\xrightarrow{Q_M(X)}T_0(T^rM)\times\{0\}\xrightarrow{T\pi}T_xM\xrightarrow{i_M}T^r_xM\overset{\psi_w}{=}V_w(T^rM),$$

where $\{0\}\subset T_\tau\mathbf{R}$ and $0\in T^r_xM$, by $\beta^\tau_{M,X,w}:T^r_xM\to V_w(T^rM)$. (This composition is well-defined because of (8.2).) For any $\tau\in\mathbf{R}$ we define a natural transformation $\mathcal{E}^\tau_M:\mathcal{X}(M)\to\mathcal{X}(T^rM)$, $M\in\mathcal{M}_n$, transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ by

$$\mathcal{E}^\tau_M(X)(w)=\beta^\tau_{M,X,w}(w),$$

where $M\in\mathcal{M}_n$, $X\in\mathcal{X}(M)$, $x\in M$ and $w\in T^r_xM$. Since $\mathcal{E}^\tau$ is of vertical type and $\mathcal{E}^\tau_{\mathbf{R}^n}(\partial_1)(0)=0$ $(0\in T^r_0\mathbf{R}^n)$ and $\mathcal{E}^\tau_M(0_M)=0$ (for $Q_M(0_M)=0$) and $\overset{(s)}{D}{}^V_{R^n}(\partial_1)(0), s=1,...,r\,(0\in T_0\mathbf{R}^n)$, are linearly independent, then (by Proposition 3.1) $\mathcal{E}^\tau=0$. Therefore

$$Q_M(X)(V_{(0,\tau)}(E^rM))\subset V_{(0,\tau)}(E^rM)$$

for any $M\in\mathcal{M}_n$, $X\in\mathcal{X}(M)$, $x\in M$ and $\tau\in\mathbf{R}$, where $0\in T^r_xM$. Therefore for any $M\in\mathcal{M}_n$, $X\in\mathcal{X}(M)$, $x\in M$, $w\in T^r_xM$ and $\tau\in\mathbf{R}$ we can define $\gamma^\tau_{M,X,w}:T^r_xM\to V_wT^rM$ to be the composition of linear mappings

$$T^r_xM\overset{\psi_0}{=}V_0(T^rM)=V_0(T^rM)\times\{0\}\xrightarrow{Q_M(X)}V_0(T^rM)\times\{0\}\overset{\psi_0}{=}T^r_xM\overset{\psi_w}{=}V_w(T^rM),$$

where $\{0\}\subset T_\tau\mathbf{R}$ and $0\in T^r_xM$. Then for any $\tau\in\mathbf{R}$ we have a natural transformation $\mathcal{G}^\tau_M:\mathcal{X}(M)\to\mathcal{X}(T^rM)$, $M\in\mathcal{M}_n$, transforming vector fields on n-manifolds into vector fields on $T^r|\mathcal{M}_n$ such that

$$\mathcal{G}^\tau_M(X)(w)=\gamma^\tau_{M,X,w}(w)$$

for any $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, $x \in M$ and $w \in T^r_x M$. Then by the same arguments as for $\mathcal{E}^\tau$, $\mathcal{G}^\tau = 0$. Therefore

$$Q_M(X)(V_{(0,\tau)}(E^r M)) = \{0\} \tag{8.4}$$

for any $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, $x \in M$ and $\tau \in \mathbf{R}$, where $0 \in T^r_x M$ and $\{0\} \subset V_{(0,\tau)}(E^r M)$.

It follows from (8.2) that for every $\tau \in \mathbf{R}$ we have a natural transformation $Q^\tau_M(X)$, $M \in \mathcal{M}_n$, $X \in \mathcal{X}(M)$, transforming vector fields into affinors on $T^r|\mathcal{M}_n$ such that

$$Q^\tau_M(X)(v) = Q_M(X)(v) \in T_w(T^r M) = T_w(T^r M) \times \{0\} \subset T_{(w,\tau)}(E^r M)$$

for any $M \in \mathcal{M}_n, X \in \mathcal{X}(M), w \in T^r M$ and $v \in T_w(T^r M) = T_w(T^r M) \times \{0\} \subset T_{(w,\tau)}(E^r M)$. From (8.1) and (8.4) we deduce that Q^τ satisfies the assumptions of Proposition 7.1 with Q^τ playing the role of Q for every $\tau \in \mathbf{R}$. Then there exist $\lambda_1, ..., \lambda_{r-1} : \mathbf{R} \to \mathbf{R}$ (not necessarily smooth) such that

$$Q^\tau_M(X) = \lambda_1(\tau)\overset{(1)}{Q}_M(X) + ... + \lambda_{r-1}(\tau)\overset{(r-1)}{Q}_M(X)$$

for any $M \in \mathcal{M}_n$ and $X \in \mathcal{X}(M)$. It follows from (7.5) that

$$\overset{(s)}{Q}_{\mathbf{R}^n}(\partial_1)(T^r \partial_2(0)) \in V_0(T^r \mathbf{R}^n), \text{ where } 0 \in T^r_0 \mathbf{R}^n, s = 1, ..., r-1,$$

are linearly independent. Therefore $\lambda_1, ..., \lambda_{r-1}$ are of class C^∞. Of course, $Q = \lambda_1 \overset{(1)}{Q}{}^+ + ... + \lambda_{r-1} \overset{(r-1)}{Q}{}^+$, because of (8.3) and the definition of $\overset{(s)}{Q}{}^+$. $\square$

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Włodzimierz M. Mikulski
Institute of Mathematics
Jagellonian University
Reymonta 4
Krakøw, POLAND