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A PARALLELOGRAM CONFIGURATION CONDITION IN NETS

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Abstract. After describing a (general and special) coordinatization of k-nets there are found algebraic equivalents for the validity of certain quadrangle configuration conditions in k-nets with small degree k.

The present article deals with some quadrangle configuration conditions in nets. The degree k of the net will be supposed to be so small as possible. Necessary fundamental notions about k-nets are explained in §1. In §2 there is defined a special type of k-net configuration condition. Some simple interrelation is investigated in §3. In §4 the coordinatization of a k-net is introduced in the sense of [2]. Corresponding admissible relations are specialized so that they yield the coordinatization algebras and further (in case of affine planes) the Hughes ternary rings. In §5 the algebraic description of configuration conditions is found using a convenient coordinatization principle. As the quadrangle configuration conditions result from the most simple configuration with only four vertices and six edges, the study of such simplest configurations and of further configuration conditions containing the complete quadrilateral can be considered as an important starting point in the theory of net configuration conditions.

§1 Definition of a k-net

A *k-net* ($k \geq 3$) $\mathcal{N}$ is defined as a quadruplet $(\mathcal{P}, \mathcal{L}, \|, \mathrm{I}\ )$, where $\mathcal{P}$ and $\mathcal{L}$ are non-empty sets, I is a binary relation from $\mathcal{P}$ to $\mathcal{L}$ ($\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$) and $\|$ is a decomposition of $\mathcal{L}$ onto non-empty mutually disjoint subsets $\mathcal{L}_1, \dots, \mathcal{L}_k$. Further it is assumed that

1. for every $P \in \mathcal{P}$ and for every $i \in \{1, \dots, k\}$ there exists exactly one $h_i \in \mathcal{L}_i$ so that $P \ \mathrm{I} \ h_i$,

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2. if $i, j \in \{1, \dots, k\}$ $i \neq j$ and $h_i \in \mathcal{L}_i$, $h_j \in \mathcal{L}_j$ then
   $\#\{X \in \mathcal{P} \mid X \ \mathrm{I}\ h_i \ \&\ X \ \mathrm{I}\ h_j\} = 1$,
3. for every $h \in \mathcal{L}$ it follows $\#\{X \in \mathcal{P} \mid X \ \mathrm{I}\ h\} > 1$.

From the definition if follows $\#\mathcal{L}_1 = \cdots = \#\mathcal{L}_k$ and this cardinal number is called the *order* of $\mathcal{N}$. The number k is called the degree of $\mathcal{N}$.

Points A, B are said to be *joinable* if it exists a line h so that $A \ \mathrm{I}\ h \ \&\ B \ \mathrm{I}\ h$; in the contrary case A, B are called *non-joinable*.

If points A, B are joinable, we shall write $A\bullet\!\!-\!\!\bullet B$; the negation will be denoted by $A\bullet\!\!\not-\!\!\bullet B$. Especially, if A, B lie on the same line of $\mathcal{L}_i$, we shall write $A \overset{i}{\bullet\!\!-\!\!\bullet} B$.

If h is a line, then we put $\bar{h} = \{X \in \mathcal{P} \mid X \ \mathrm{I}\ h\}$. If P is a point, then we put $\bar{P} = \{x \in \mathcal{L} \mid P \ \mathrm{I}\ x\}$.

It can be easily shown that $\#\bar{h}$ does not depend on the choice of h and is equal to the order of the net.

If A, B are different joinable points, then the uniquely determined line C such that $A \ \mathrm{I}\ C$, $B \ \mathrm{I}\ C$ is called the *join line* of A, B and is denoted by $A \sqcup B$.

If A is a point and $i \in \{1, \dots, k\}$, then the line a such that $a \in \mathcal{L}_i \ \&\ A \ \mathrm{I}\ a$ is uniquely determined and is denoted by $A \sqcup i$.

If a, b are lines from distinct subsets $\mathcal{L}_i$, $\mathcal{L}_j$, then there is exactly one point C such that $C \ \mathrm{I}\ a$, $C \ \mathrm{I}\ b$. It is called the *intersection point* and is denoted by $a \sqcap b$.

§2 A configurational condition

Definition 2.1. 1. An ordered quadruplet of mutually distinct points (A, B, C, D) of a k-net is called a *quadrangle* if $A\bullet\!\!-\!\!\bullet B$, $B\bullet\!\!-\!\!\bullet C$, $C\bullet\!\!-\!\!\bullet D$, $D\bullet\!\!-\!\!\bullet A$ and if no three of points A, B, C, D lie on the same line.

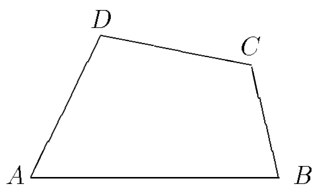

Fig. 1

2. A quadrangle (A, B, C, D) is said 1) *to have the first* or *the second diagonal* if it holds $A\bullet\!\!-\!\!\bullet B$, $B\bullet\!\!-\!\!\bullet C$, $C\bullet\!\!-\!\!\bullet D$, $D\bullet\!\!-\!\!\bullet A$, $A\bullet\!\!-\!\!\bullet C$ or $A\bullet\!\!-\!\!\bullet B$, $B\bullet\!\!-\!\!\bullet C$, $C\bullet\!\!-\!\!\bullet D$, $D\bullet\!\!-\!\!\bullet A$, $B\bullet\!\!-\!\!\bullet D$, respectively 2) *to be without the first* or *the second diagonal* if $A\bullet\!\!-\!\!\bullet C$, respectively $B\bullet\!\!-\!\!\bullet D$ does not hold, and 3) *to have both diagonals* if $A\bullet\!\!-\!\!\bullet B$, $B\bullet\!\!-\!\!\bullet C$, $C\bullet\!\!-\!\!\bullet D$, $D\bullet\!\!-\!\!\bullet A$, $A\bullet\!\!-\!\!\bullet C$, $B\bullet\!\!-\!\!\bullet D$ holds.

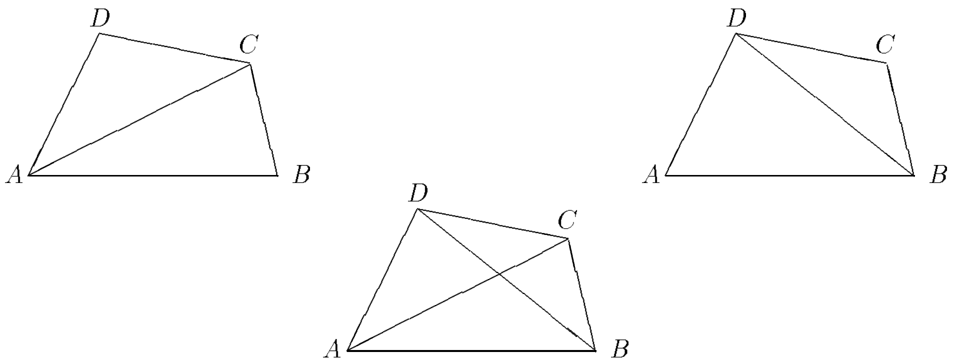

Fig. 2

A quadrangle (A, B, C, D) will be designated only by $ABCD$. The lines $A \sqcup B$, $B \sqcup C$, $C \sqcup D$, $D \sqcup A$ are its *sides* and the lines $A \sqcup C$, $B \sqcup D$, if they exist, are its *diagonals*.

3. A quadrangle $ABCD$ is called a *parallelogram* if $A \sqcup D \| B \sqcup C$, $A \sqcup B \| C \sqcup D$.

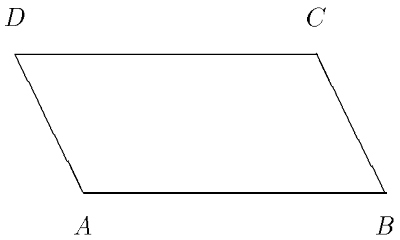

Fig. 3

4. We say that in a 4-net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \|, \mathrm{I})$ the condition LP_{1234} holds (the *parallelogram "letter" condition*) if every parallelogram $ABCD$, $A \overset{1}{\bullet\!\!-\!\!\bullet} D$, $A \overset{2}{\bullet\!\!-\!\!\bullet} B$ with the first diagonal $A \sqcup C$, $A \overset{3}{\bullet\!\!-\!\!\bullet} C$, has also the second diagonal $B \sqcup D$ such that $B \overset{4}{\bullet\!\!-\!\!\bullet} D$.

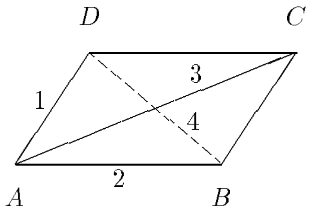

Fig. 4

Similarly the condition LP_{ijkh} (for a permutation (i, j, k, h) of $\{1, 2, 3, 4\}$) can be defined.

§3 An elementary property of given configuration condition

Lemma 3.1. *In a 4-net $\mathcal{N}$ let there hold the condition LP_{1234}. Then in $\mathcal{N}$ there holds the condition LP_{ijkh} for $(i,j) \in \{(1,2);(2,1)\}$, $(k,h) \in \{(3,4);(4,3)\}$.*

Proof. Let $\mathcal{N}$ be a net satisfying all assumptions of the lemma.

a) Choose arbitrary points A, B, C, D in $\mathcal{N}$ so that $A \overset{2}{\bullet\!-\!\bullet} D$, $A \overset{1}{\bullet\!-\!\bullet} B$, $A \overset{3}{\bullet\!-\!\bullet} C$, $B \overset{2}{\bullet\!-\!\bullet} C$, $D \overset{1}{\bullet\!-\!\bullet} C$. These points satisfy the assumption of LP_{2134}.

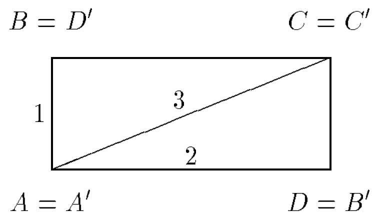

Fig. 5

Put $A' = A$, $B' = D$, $C' = C$, $D' = B$. Then the points A', B', C', D' satisfy the assumption of LP_{1234} so that there exists the line $D' \sqcup B'$, $D' \overset{4}{\bullet\!-\!\bullet} B'$. Therefore there exists the line $D \sqcup B$ with $D \overset{4}{\bullet\!-\!\bullet} B$ that is the conclusion of LP_{2134}.

b) Let A, B, C, D be arbitrary points of $\mathcal{N}$ such that $A \overset{1}{\bullet\!-\!\bullet} D$, $A \overset{2}{\bullet\!-\!\bullet} B$, $A \overset{4}{\bullet\!-\!\bullet} C$, $B \overset{1}{\bullet\!-\!\bullet} C$, $C \overset{2}{\bullet\!-\!\bullet} D$.

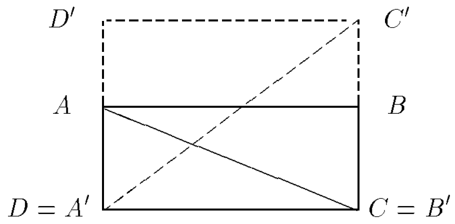

Fig. 6

Then the points A, B, C, D satisfy the assumption of LP_{1243}. Put $A' = D$, $B' = C$m $C' = (A' \sqcup 3) \sqcap (B' \sqcup 1)$, $D = (C' \sqcup 2) \sqcap (A' \sqcup 1)$. As the points A', B', C', D' satisfy the assumption of LP_{1234}, there is a line $B' \sqcup D'$, $B' \overset{4}{\bullet\!-\!\bullet} D'$. Therefore $D' = A$, $C' = B$. So there exists the line $D \sqcup B$ with $D \overset{3}{\bullet\!-\!\bullet} B$ which is the conclusion of LP_{1243}. The remaining condition LP_{2143} can be deduced similarly.□

§4 Coordinatization of a k-net

Definition 4.1. Let $S_1, \dots, S_k$ be mutually bijective at least two-element sets. We say that $\Omega \subseteq S_1 \times \dots \times S_k$ is an *admissible relation* if the following condition is satisfied

$$\forall \alpha, \beta \in \{1, \dots, k\}, \alpha \neq \beta \; \forall x \in S_\alpha, \forall y \in S_\beta \exists!(x_1, \dots, x_k) \in \Omega \quad x = x_\alpha, y = y_\beta.$$

Remark 4.2. Let an admissible relation $\Omega \subset S_1 \times \cdots \times S_k$ be given. Then for all $\alpha, \beta, \gamma \in \{1, \ldots, k\}$, $\alpha \neq \beta \neq \gamma \neq \alpha$, we shall define a mapping $(\alpha\beta\gamma) : (a_\alpha, a_\beta) \mapsto a_\gamma$ so that $(\alpha\beta\gamma)(a_\alpha, a_\beta) = a_\gamma \Leftrightarrow \exists(x_1, \ldots, x_k) \in \Omega \quad x_\alpha = a_\alpha,\ x_\beta = a_\beta,\ x_\gamma = a_\gamma$. This mapping is the binary operation of a 3-basic quasigroup. In this way we obtain $\dfrac{k!}{(k-3)!}$ 3-basic quasigroup at most.

Definition 4.3. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathrm{I}, \mathcal{L}_1 \ldots \mathcal{L}_k)$ be a k-net. The mapping $\sigma : \mathcal{P} \to \mathcal{L}_1 \times \cdots \times \mathcal{L}_k$ given by

$$P \mapsto (a_1, \ldots, a_k) \quad a_1 \in \mathcal{L}_1, \ldots, a_k \in \mathcal{L}_k,$$

will be called the *coordinatization* of $\mathcal{N}$ and the relation $\mathcal{R}_\mathcal{N} = \{\sigma(P) \mid P \in \mathcal{P}$ will be called the *coordinatizing relation* of $\mathcal{N}$. It is determined by the given k-net uniquely.

Lemma 4.4. *Let $(\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_k, \mathrm{I})$ be a k-net. Then $\mathcal{R}_\mathcal{N}$ is an admissible relation.*

Proof. Let there be satisfied the assumptions of Lemma 4.4. Take arbitrary lines $a_i, a_j; a_i \in \mathcal{L}_i,\ a_j \in \mathcal{L}_j;\ i, j \in \{1, \ldots, k\},\ i \neq j$. Then through the point $a_i \sqcap a_j$ there goes for every $h \in \{1, \ldots, k\}$, $h \neq i, j$, a unique line a_h. Thus $(a_1, \ldots, a_k) \in \mathcal{P}$, $a_1 \in \mathcal{L}_1, \ldots, a_k \in \mathcal{L}_k$ are elements of an admissible relation. □

Lemma 4.5. *Let $\mathcal{R} \subset S_1 \times \cdots \times S_k$ be an admissible relation. Then there exists a k-net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_k, \mathrm{I})$ such that $\mathcal{R}$ is isotopic*[1] *with $\mathcal{R}_\mathcal{N}$.*

Proof. Let there be given sets $S_1, \ldots, S_k$, $\#S_i \geq 2$, $i \in \{1, \ldots, k\}$ and let $\mathcal{R} \subset S_1 \times \cdots \times S_k$ be an admissible relation. Further let there be given the sets $S'_1, \ldots S'_k$ which are mutually disjoint and such that there exist bijections

$$\alpha_i \colon S_i \to S'_i \quad i \in \{1, \ldots, k\}.$$

Obviously $S'_1, \ldots, S'_k$ is an admissible relation.

Define

$$
\begin{aligned}
\mathcal{P} &:= S'_1 \times \cdots \times S'_k \\
\mathcal{L}_i &:= S'_i, i \in \{1, \ldots, k\} \qquad (1)\\
\mathcal{L} &:= \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k
\end{aligned}
$$

If $P \in h$, $P \in \mathcal{P}$, $h \in \mathcal{L}$ then we shall write $P \ \mathrm{I}\ h$. We will show that $(\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \ldots, \mathcal{L}_k, \mathrm{I})$ is a k-net.

(i) Let $A \in \mathcal{P}$, $i \in \{1, \ldots, k\}$. Then, by (1), there is just one $h_i \in \mathcal{L}_i$ such that $A \ \mathrm{I}\ h_i$.

(ii) Take arbitrary $h'_i \in \mathcal{L}_i$, $h'_j \in \mathcal{L}_j$, $i, j \in \{1, \ldots, k\}$, $i \neq j$. On the line h'_i there lies the point $(a'_1, \ldots, a'_{i-1}, h'_i, a'_{i+1}, \ldots, a'_{j-1}, h'_j, a'_{j+1}, \ldots, a'_k)$ and on the

[1]Two relations $\mathcal{A} \subseteq A_1 \times \cdots \times A_k$, $\mathcal{B} \subseteq B_1, \ldots, B_k$ are said to be *isotopic*, if there exist bijections $\gamma_1 : A_1 \to B_1, \ldots, \gamma_k : A_k \to B_k$ such that for every $(x_1, \ldots, x_k) \in A_1 \times \cdots \times A_k$ it holds $(x_1, \ldots, x_k) \in \mathcal{A} \leftrightarrows (\gamma_1(x_1), \ldots, \gamma_k(x_k)) \in \mathcal{B}$

line h'_j the point $(b'_1, \dots, b'_{i-1}, h'_i, b'_{i+1}, \dots, b'_{j-1}, h'_j, b'_{j+1}, \dots, b'_k)$. As $(a'_1, \dots, a'_k)$, $(b'_1, \dots, b'_k)$ are elements of an admissible k-relation $S'_1 \times \cdots \times S'_k$ it is $(a'_1, \dots, a'_k) = (b'_1, \dots, b'_k)$. Thus

$$\#\{X \in \mathcal{P} \mid X \ \mathrm{I}\ \ h'_i \ \& \ X \ \mathrm{I}\ \ h'_j\} = 1.$$

(iii) Now investigate an arbitrary line h and an arbitrary point $A' \ \mathrm{I}\ h$. Let $h \in \mathcal{L}_m$ for some $m \in \{1, \dots, k\}$. Further let $A' = (a'_1, \dots, a'_k)$, $a'_i \in S'_i$, $i \in \{1, \dots, k\}$. As $\#S'_i \geq 2$ there exist $B' = (b'_1, \dots, b'_k) \in h$, $b'_i \in S'_i$, $i \in \{1, \dots, k\}$, such that $a'_i \neq b'_i$ for all $i \in \{1, \dots, k\}$, $i \neq m$. Thus $A' \neq B'$ and $\#\{X \in \mathcal{P} \mid X \ \mathrm{I}\ h\} \geq 2$. □

By an *admissible algebra* we shall understand a set $S, \#S \geq 2$ together with a prominent element $0 \in S$, with a collection $(\varphi_i)_{i \in \{3,\dots,k\}}$ of permutation of S fixing the element 0 and with a collection of quasigroup operations $(+_i)_{i \in \{3,\dots,k\}}$ such that

1) $\varphi_3 = \mathrm{id}_S$,
2) $\forall c_1, c_2 \in S, \ i_1, i_2 \in \{3, \dots, k\}, \quad i_1 \neq i_2$
$\exists!(x, y) \in S \times S \quad \varphi_{i_1}(x) +_{i_1} y = c_1, \ \varphi_{i_2}(x) +_{i_2} y = c_2.$

Construction 4.6. Let there be given a k-net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, ||, \ \mathrm{I}\ )$. Choose an arbitrary line $a \in \mathcal{L}_1$ and an arbitrary line $b \in \mathcal{L}_2$. Further put $0 = a \sqcap b$ and define

1) $S := \mathcal{L}_1$,
2) $\lambda_2 : \mathcal{L}_2 \to S, \ h_2 \mapsto h'_2 = \{[(h_2 \sqcup a) \sqcup 3] \sqcap b\} \sqcup 1$,

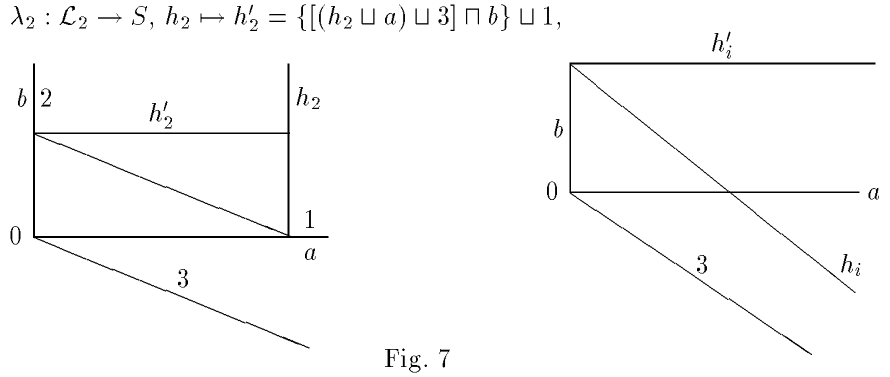

Fig. 7

$\lambda_i : \mathcal{L}_i \to S, \ h_i \mapsto h'_i = (h_i \sqcap b) \sqcup 1$ for all $i \in \{3, \dots, k\}$.

3) $\varphi_i : S \to S, \ x \mapsto \varphi_i(x) = \{[(x \sqcup 3) \sqcup a] \sqcup i\} \sqcap b$ for all $i \in \{3, \dots, k\}$ where $\varphi_3 = \mathrm{id}_S$.

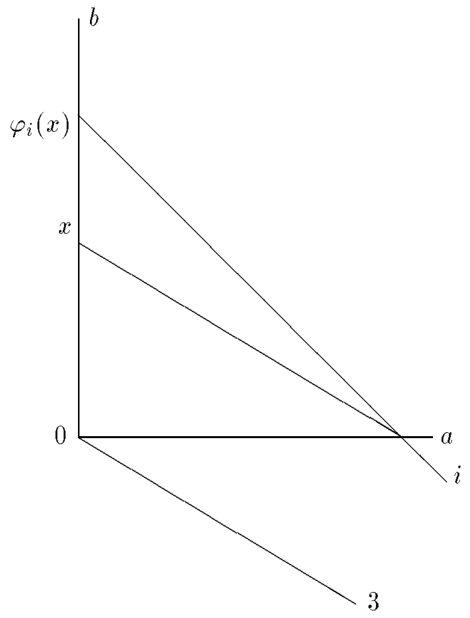

Fig. 8

4) $+_i : S \times S \to S$, $(x, y) \mapsto \varphi_i(x) +_i y := (21i)(x, y)$ for all $i \in \{3, \ldots, k\}$

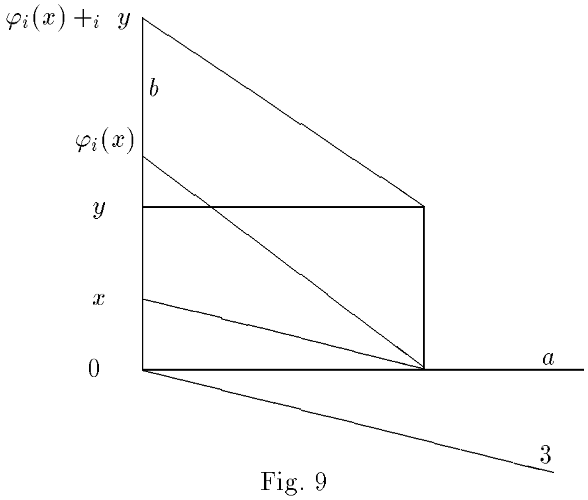

Fig. 9

It can be easily shown that $+_3$ is a loop operation. We shall call $(S, 0, (\varphi_i)_{i\in\{3,\ldots,k\}}, (+_i)_{i\in\{3,\ldots,k\}})$ the *coordinatizing algebra* corresponding to $\mathcal{N}$.

Remark 4.7. Let us have a k-net $\mathcal{N}$

$$(S, 0, (\varphi_i)_{i\in\{3,\ldots,k\}}, (+_i)_{i\in\{3,\ldots,k\}}).$$

Define a mapping $\mathcal{L}_1 \times \dots \mathcal{L}_k \to \overbrace{S \times \dots \times S}^{k}$, $(h_1, \dots, h_k) \mapsto (x_1, \dots, x_k)$ such that

$$\mathcal{L}_1 = S, \lambda_2 : \mathcal{L}_2 \to S,\ h_2 \mapsto h_2' = \{([h_2 \sqcap a] \sqcap 3) \sqcap b\} \sqcup 1, \quad a, b \in S,$$
$$\lambda_i : \mathcal{L}_i \to S, h_i \mapsto h_i' = \{h_i \sqcap b\} \sqcup 1,$$

then the quasigroup operations derived from $\mathcal{R}_{\mathcal{N}}$ go over onto quasigroup operations from point 4) of Construction 4.6.

Lemma 4.8. *Every admissible algebra is a coordinatizing algebra of a suitable k-net.*

Proof. Let $(S, 0, (\varphi_i)_{i\in\{3,\dots,k\}}, (t_i)_{i\in\{3,\dots,k\}})$ be an admissible algebra. Define $\mathcal{P} := S \times S$, $\mathcal{L}_i := \{\{(a, y) \mid y \in S\} \mid a \in S\}$, $\mathcal{L}_2 := \{\{(x, b) \mid x \in S\} \mid b \in S\}$, $\mathcal{L}_i := \{\{(x, y) \mid y = \varphi_i(x) +_i y = c_i\} \mid c_i \in S\}$ for all $i \in \{3, \dots, k\}$, $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \cup \dots \cup \mathcal{L}_k$. The incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ will be the relation "to be an element of the set". It can be easily shown that $(\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \dots, L_k, \mathrm{I})$ is a k-net having the desired property. $\square$

§5 Algebraic properties of k-nets satisfying given configurational conditions

Theorem 5.1. *Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathrm{I})$ be a net and $(S, 0, (\varphi_3, \varphi_4), (+_3, +_4))$ its coordinatizing algebra with respect to the origin 0. Then in $\mathcal{N}$ there holds condition LP_{1234} if and only if*

a) $+_3 = +_4 (= +)$ where $(S, +)$ is a commutative group,
b) $\varphi_4(x) = -x \ \mathrm{I}\ \forall x \in S$, where $-x$ is the opposite element of $x \in S$ in $(S, +)$.

Proof. Let $\mathcal{L}$ be a net satisfying all the assumptions of the Theorem 5.1. Further let there hold condition LP_{1234} in $\mathcal{N}$. By Lemma 3.2, in $\mathcal{N}$ there is satisfied also condition LP_{1243}. By help of operations $+_3, +_4$ we can explain condition LP_{1234} as follows:

$$(2) \qquad \forall r, u, s, v \in S \quad r +_3 v = s +_3 u \Rightarrow \varphi(r) +_4 u = \varphi(s) +_4 v$$

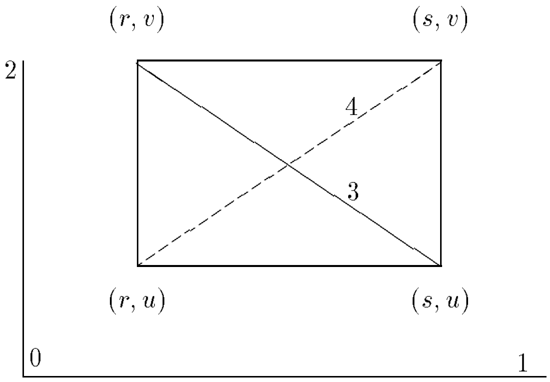

Fig. 10

and condition LP_{1243} similarly:

(3) $$\forall r, u, s, v \in S \quad \varphi(r) +_4 u = \varphi(s) +_4 v \Rightarrow r +_3 v = s +_3 u.$$

For $r = 0$, $s = v$, the conclusion $u = \varphi(v) +_4 v \; \forall u, v \in S$ follows from $v = v +_3 u$ by (2). If $u = 0$, then

(4) $$0 = \varphi(v) +_4 v$$

for every $v \in S$.

When putting $s = v$ in (2), from $r +_3 v = v +_3 u$ we get $\varphi(r) +_4 u = \varphi(v) +_4 v$ for all $u, r, v \in S$.

By (4) we obtain $\varphi(r) +_4 u = 0$. Again by (4) we get $r = u$.

So $u +_3 v = v +_3 u$ for all $u, v \in S$. Then the loop $(S, +_3)$ must be commutative.

For $r = 0$, $\varphi(s) = u$, the conclusion $v = \varphi^{-1}(u) +_u$ follows from $u = u +_4 v$ for all $u, v \in S$ by (3).

Herefrom $v = 0$ so that

(5) $$0 = \varphi^{-1}(u) +_3 u$$

for every $u \in S$.

If we put $r = 0$, $v = 0$, then we obtain, by (3), the conclusion $0 = s +_3 u$ for all $u, s \in S$ from $u = \varphi(s)$. So

(6) $$0 = s +_3 \varphi(s)$$

for all $s \in S$.

Choose an arbitrary element $a \in S$. Then by (5) it holds $0 = \varphi^{-1}(a) +_3 a$ and by (6) similarly $0 = a +_3 \varphi(a)$.

As the loop $(S, +_3)$ is commutative, we get

(7) $$\varphi(a) = \varphi^{-1}(a)$$

for every $a \in S$.

For $u = 0$ we get the equation $\varphi(r) = \varphi(s) +_4 v$ for all $r, s, v \in S$ by (2) from $r +_3 v = s$. Thus

(8) $$\varphi(r) = \varphi(r +_3 v) +_4 v$$

for all $r, v \in S$.

When putting $s = 0$, we get, by (2) from $r +_3 v = u$, the equation $\varphi(r) +_4 u = v$ for all $r, u, v \in S$. So

(9) $$\varphi(r) +_4 (r +_3 v) = v$$

for all $r, v \in S$.

If we substitute into (9) by (8), we get

$$[\varphi(r +_3 v) +_4 v] +_4 (r +_3 v) = v \tag{10}$$

for all $r, v \in S$. As $(S, +_3)$ is a loop, there exists $z \in S$ so that $z = r +_3 v$ and (10) goes over onto

$$(\varphi(z) +_4 v) +_4 z = v \tag{11}$$

for all $z, v \in S$.

Let the element $d \in S$ be such that $z = \varphi(d)$. Then from (11) we obtain by (7) the equation

$$(d +_4 v) +_4 \varphi(d) = v \tag{12}$$

for all $d, v \in S$.

Further let $c = d +_4 v$. Then from (12) we obtain

$$c +_4 \varphi(d) = v\ . \tag{13}$$

Now choose arbitrary elements a, b, $x \in S$ such that $b = a +_4 x$. When setting $a = d$, $b = c$, we get, by (13),

$$x = v = c +_4 \varphi(d) = b +_4 \varphi(a).$$

So from the equation

$$a +_4 x = b$$

it follows

$$x = b +_4 \varphi(a) \tag{14}$$

for all $a, b \in S$. Then by (9) we conclude, by (14),

$$r +_3 v = v +_4 \varphi(r)$$

for all $r, v \in S$. By (7) it holds $\varphi^2 =$id so that

$$r +_3 v = v +_4 r$$

for all $r, v \in S$.
As $(S, +_3)$ is a commutative loop,

$$v +_3 r = v +_4 r$$

holds for all $r, v \in S$. Thus $+_3 = +_4$. Put $+_3 = +$.

Taking arbitrary elements $a, b, c \in S$, then, by (10), we obtain

$$a + c = b + [\varphi(b) + (a + c)]. \tag{15}$$

As in $\mathcal{N}$ there holds condition LP_{1234}, from (15) it follows

$$\varphi(a) + [\varphi(b) + (a + c)] = \varphi(b) + c.$$

When adding the element a on both sides, we obtain

$$a + [\varphi(a) + [\varphi(b) + (a + c)]] = a + (\varphi(b) + c)\,.$$

When using (10), we get

$$\varphi(b) + (a + c) = a + (\varphi(b) + c) \tag{16}$$

for all $a, b, c \in S$.

Further put $a = x$, $c = y$, $b = \varphi(z)$. Then

$$(x + y) + z = (a + c) + \varphi(b)\,.$$

By (16) it is

$$(a + c) + \varphi(b) = a + \varphi(b) + c = a + (c + \varphi(b)) = x + (y + z)\,.$$

Thus $(x + y) + z = x + (y + z)$ for all x, y, $z \in S$.

The loop $(S, +)$ is thus associative and consequently it is a commutative group. Conditions a) and b) are thus satisfied.

On the contrary, let us now suppose that in $\mathcal{N}$ there hold conditions a), b). Choose arbitrary elements r, u, s, $v \in S$ such that

$$r + v = s + u\,. \tag{17}$$

Because $(S, +)$ is a commutative group, we can rewrite (17) as

$$r + v - r = s + u - r\,,$$

so that $v = s + u - r$. When adding the element $-s$ on both sides, we have

$$-s + v = u - r,$$

and from

$$r + v = s + u$$

it follows

$$-r + u = -s + v.$$

On the other side, it holds $\varphi(x) = -x$ for every $x \in S$. Thus from

$$r + v = s + u \quad \text{we get} \quad \varphi(r) + u = \varphi(s) + v$$

for all r, s, u, $v \in S$. Condition LP_{1234} is thus satisfied. □

Theorem 5.2. *Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{L}_6, \mathrm{I}\ )$ be a net and $(S, 0, \varphi_3, \varphi_4, \varphi_5, \varphi_6, +_3, +_4, +_5, +_6)$ its coordinatizing algebra with respect to the origin 0. Then in $\mathcal{N}$ there holds condition LP_{1256} if and only if*

a) $+_5 = +_6 (=: \oplus)$ and $(S, \oplus)$ is a commutative group

b) $+_6(x) = \ominus\varphi_5(x)\ \forall x \in S$. Here the symbol $\ominus$ designates the subtraction in $(S, \oplus)$.

Proof. Let $\mathcal{N}$ be a net with coordinatizing algebra as in assumptions of Theorem 5.2. Further assume that in $\mathcal{N}$ there holds condition LP_{1256}. By construction 4.6 it follows

$$a +_i b = (1\,2\,i)\varphi_i^{-1}(a), b)\ \ \forall a, b \in S \quad \forall i \in \{3, 4, 5, 6\}.$$

By Theorem 5.1 $+_5 = +_6$ holds. Denote $+_5 = \oplus$. The $(S, \oplus)$ is a commutative group and condition a) is satisfied. Further from condition LP_{1256} it follows $\varphi_5(x) \oplus \varphi_6(x) = 0$ for all $x \in S$.

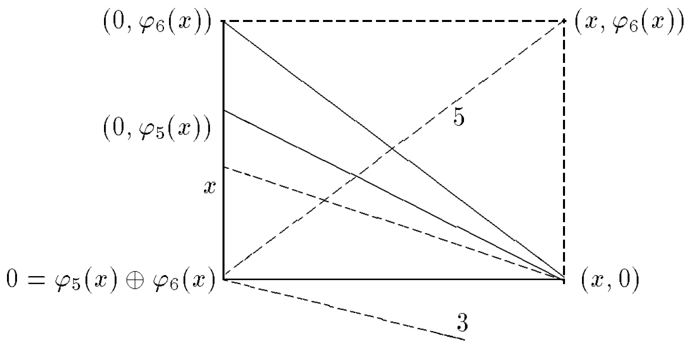

Fig. 11

Therefore condition b) is satisfied.

Assume on the contrary that in $\mathcal{N}$ there hold conditions a) and b). Choose arbitrary elements $x_1, x_2, y_1, y_2 \in S$ such that

$$\ominus\varphi_5(x_1) \oplus y_1 = \varphi_5(x_2) + y_2. \tag{18}$$

Since there are satisfied conditions a) and b), we can rewrite (18) as

$$\varphi_5(x_1) \oplus y_2 = \ominus\varphi_5(x_2) \oplus y_1.$$

When using the equation $\varphi_6(x) = \ominus\varphi_5(x)$, we obtain

$$\varphi_6(x_1) \oplus y_2 = \varphi_6(x_2) \oplus y_1.$$

Thus we have

$$\varphi_5(x_1) \oplus y_1 = \varphi_5(x_2) \oplus y_2 \Rightarrow \varphi_6(x_1) \oplus y_2 = \varphi_6(x_2) \oplus y_1$$

which is condition LP_{1256}. $\square$

Corollary 5.3. *Let $\mathcal{N}$ be a net with even degree $k \geq 4$ and $(S, 0, (\varphi_i)_{i \in \{3,\dots,k\}}, (+_i)_{i \in \{3,\dots,k\}})$ its coordinatizing algebra with respect to the origin 0.*
Then in $\mathcal{N}$ there holds condition $LP_{1,2(2h+1),(2h+2)}$ $\forall h \in \{1, 2, \dots, \frac{k}{2} - 1\}$ if and only if

a) $+_{2h+2} = +_{2h+1} (=: \oplus_h)$ and $(S, \oplus_h)$ is a commutative group.

b) $\varphi_{2h+2}(x) = \ominus_h \varphi_{2h+1}(x)$ $\quad \forall x \in S$ where the symbol $\ominus_h$ designates the subtraction in $(S, \oplus_h)$, $h \in \{1, \dots, \frac{k}{2} - 1\}$.

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