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CONTINUITY OF MONOTONE FUNCTIONS

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Abstract. It is shown that a monotone function acting between euclidean spaces $\mathbf{R}^n$ and $\mathbf{R}^m$ is continuous almost everywhere with respect to the Lebesgue measure on $\mathbf{R}^n$.

As well known the set of all points of discontinuity of a real monotone function is at most countable. The paper deals with the set of all discontinuity points of a monotone function acting between euclidean spaces. We shall be concerned in order theoretic monotonicity, so let us agree that $\leq$ denotes the componentwise ordering of $\mathbf{R}^k$ ($x \leq y$ means $x_i \leq y_i$ for $i = 1, 2, \cdots, k$). If $a, b \in \mathbf{R}^k$, $a \leq b$, the set $[a, b] = \{x \in \mathbf{R}^k : a \leq x \leq b\}$ will be called a *k-cell.*

Let A be a nonempty subset of $\mathbf{R}^n$. Then A is said to be *solid*, if $a, b \in A$ and $a \leq x \leq b$ implies $x \in A$. The smallest solid set containing A is called the *solid cover of* A and equals

$$S(A) = \{x \in \mathbf{R}^n : a \leq x \leq b \quad \text{for some} \quad a, b \in A\}.$$

A function $f : A \longrightarrow \mathbf{R}$ is said to be nondecreasing (respectively nonincreasing) if

$$x, y \in A,\ x \leq y \implies f(x) \leq f(y) \quad (\text{ respectively} \quad f(y) \leq f(x)).$$

A function $g = (g_1, \cdots, g_m) : A \longrightarrow \mathbf{R}^m$ is called *monotone* if each of its components $g_i : A \longrightarrow \mathbf{R}$ is either nondecreasing or nonincreasing.

The set D of all points of discontinuity of a monotone function $f : A \longrightarrow \mathbf{R}^m$ is not necessarily countable if $n > 1$. By way of example take the characteristic function $h_C : \mathbf{R}^n \longrightarrow \mathbf{R}$ of the cone $C = \{x \in \mathbf{R}^n : x \geq 0\}$. However, D remains small also for $n > 1$.

We need first a property of solid subsets of $\mathbf{R}^n$.

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Proposition. *The boundary of a solid subset of* $\mathbf{R}^n$ *is of Lebesgue measure zero.*

Proof. Let A be a solid subset of $\mathbf{R}^n$. Denote by K the interior of the cone $C = \{x \in \mathbf{R}^n : x \geq 0\}$, set

$$A^- = \{x \in \operatorname{cl}(A) : A \cap (x - K) = \emptyset\},$$
$$A^+ = \{x \in \operatorname{cl}(A) : A \cap (x + K) = \emptyset\},$$

and note that A^-, A^+ are closed subsets of the boundary $\operatorname{bd}(A)$ of A. We claim that $\operatorname{bd}(A) = A^- \cup A^+$. By way of contradiction suppose $x \in \operatorname{bd}(A) \setminus (A^- \cup A^+)$. Then there exists elements $y, z \in A$ such that $y \in x - K$, $z \in x + K$, hence x is an interior point of the n-cell $[y, z]$ which is contained in A. This contradicts $x \in \operatorname{bd}(A)$, therefore $\operatorname{bd}(A) \subset A^- \cup A^+$ and the claim follows.

Thus, we have to prove that A^- and A^+ are of Lebesgue measure zero. To this end suppose that A^- is nonempty, note that

$$y \notin (x - K) \cup (x + K) \quad \text{for all} \quad x, y \in A^-, \tag{1}$$

and denote by P the orthogonal projection of $\mathbf{R}^n$ onto the subspace $E = \{x \in \mathbf{R}^n : x_1 + x_2 + \cdots + x_n = 0\}$. Since by (1) P is injective on A^-, there exists a function $h : P(A^-) \longrightarrow \mathbf{R}$ such that

$$A^- = \{u + h(u)e : \quad u \in P(A^-),\ e = (1, 1, \cdots, 1)\}.$$

An easy computation shows that (1) implies

$$|h(u) - h(v)| \leq \|u - v\|_\infty, \quad u, v \in P(A^-),$$

hence h is continuous. It follows that A^- and similarly A^+ is of Lebesgue measure zero, as desired. □

Corollary. *Every bounded solid subset of* $\mathbf{R}^n$ *is Jordan measurable.*

Proof. It is well known (see for example [1]) that a subset A of $\mathbf{R}^n$ is Jordan measurable if and only if A is bounded and $\operatorname{bd}(A)$ is of Lebesgue measure zero. □

We are now in a position to prove our main result.

Theorem. *Let A be a nonempty subset of $\mathbf{R}^n$ and let $f : A \longrightarrow \mathbf{R}^m$ be a monotone function. Then the set of all points of discontinuity of f is of Lebesgue measure zero.*

Proof. The components f_i of $f = (f_1, f_2, \cdots, f_m)$ are real-valued monotone functions, f is continuous at $x \in A$ if and only if all f_i are continuous at x, hence it suffices to prove the theorem for $m = 1$. Furthermore, f can be extended on the solid cover $S(A)$ of A by

$$\widehat{f}(z) = \sup\{f(x) : \ x \in A, : x \leq z\}, \quad z \in S(A),$$

so we may suppose that A is solid. Finally, since $\mathrm{int}(A)$ is a countable union of n-cells, we may assume by Proposition and by a homothetic argument that $A = [0, e]$, $e = (1, \cdots, 1)$, and that $f : [0, e] \longrightarrow \mathbf{R}$ is nondecreasing.

For every $x \in U = \mathrm{int}([0, e])$ set

$$g(x) = \inf\{f(x + te) - f(x - te), \quad 0 < t \in \mathbf{R}\}.$$

Observe that for sufficiently small $s > 0$ f maps the neighborhood $[x-se, x+se] \subset [0, e]$ of x into the real interval $[f(x - se), f(x + se)]$ containing $f(x)$. Therefore f is continuous at x if and only if $g(x) = 0$. Put

$$D_k = \{x \in U : \quad g(x) \geq \frac{1}{k}\}, \quad k = 1, 2, \cdots,$$

and note that the set D of all discontinuity points of f satisfies $D \cap U = \bigcup_{k \in \mathbf{N}} D_k$. Thus, we have to prove that each D_k is of Lebesgue measure zero.

We claim that $D_k = \mathrm{cl}(D_k) \cap U$. Take any $x \in U \setminus D_k$, and pick $s > 0$ such that

$$[x - se, x + se] \subset [0, e], \quad f(x + se) - f(x - se) < \frac{1}{k}.$$

Note that every $y \in [x - (s/2)e, x + (s/2)e]$ satisfies

$$[y - \frac{s}{2}e, y + \frac{s}{2}e] \subset [x - se, x + se],$$

hence $g(y) \leq f(y + (s/2)e) - f(y - (s/2)e) < 1/k$, and so $y \notin D_k$. Therefore,

$$[x - \frac{s}{2}e, x + \frac{s}{2}e] \cap D_k = \emptyset,$$

and the claim follows.

Assume now that D_k is nonempty and let $\epsilon > 0$. For each fixed $x \in [0, e]$ consider the real function $h : t \longmapsto f(x + te)$. Since h is nondecreasing and jumps for at least $1/k$ at every t satisfying $x + te \in D_k$, the set $D_k \cap (x + \mathbf{R}e)$ contains finitely many elements or it is empty.

Remove from the line $x + \mathbf{R}e$ finitely many disjoint relatively open intervals of common length less than ϵ and containing $\mathrm{cl}(D_k) \cap (x + \mathbf{R}e)$. Denote by $R(x)$ the remaining set, observe that $d = \mathrm{dist}(R(x), D_k) > 0$ and put

$$T(x) = \{y \in \mathbf{R}^n : \mathrm{dist}(y, x + \mathbf{R}e) < d\}.$$

From the open covering $\{T(x) : \ x \in [0, e]\}$ of $[0, e]$ extract a finite subcovering $\{T_i = T(x_i) : \ i = 1, \cdots, p\}$. Accept $T_0 = \emptyset$ and set

$$U_i = T_i \setminus \bigcup_{j<i} T_j, \quad E = \{z \in \mathbf{R}^n : z_1 + \cdots + z_n = 0\}.$$

By construction

$$\mu_n(U_i \cap D_k) \leq \epsilon \mu_{n-1}(U_i \cap E)$$

holds for all i (μ_m denotes the Lebesgue measure in $\mathbf{R}^m$). It follows from

$$\begin{aligned} \mu_n(D_k) &= \sum_{i=1}^{p} \mu_n(U_i \cap D_k) \leq \\ &\leq \epsilon \sum_{i=1}^{p} \mu_{n-1}(U_i \cap E) = \mu_{n-1}\left(\bigcup_{i=1}^{p} U_i \cap E\right) \end{aligned}$$

that $\mu_n(D_k) = 0$, so the proof is complete. □

Applying the Lebesgue's characterization of Riemann integrable functions (see [1] or [2]) we get the folowing result.

Corollary. *Let A be a nonempty Jordan measurable subset of $\mathbf{R}^n$ and let $f : A \longrightarrow \mathbf{R}^m$ be a bounded monotone function. Then f is Riemann integrable.*

References

[1] Marsden, J. E., *Elementary Classical Analysis*, Freeman and Company, San Francisco, 1974.
[2] Spivak, M., *Calculus on Manifolds*, Benjamin, New York, 1965.

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