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A CHARACTERIZATION OF KRULL RINGS WITH ZERO DIVISORS

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Abstract. It is proved that a Marot ring is a Krull ring if and only if its monoid of regular elements is a Krull monoid.

It was first noticed by L. Skula [7] that a domain R is a Krull domain if and only if the multiplicative monoid $R \setminus \{0\}$ is a Krull monoid (or, equivalently, admits a divisor theory). For independent proofs and historical remarks see [1] and [3].

In this note we extend the above-mentioned result to Krull rings with zero divisors as treated in [4]. All rings in this note are commutative and possess a unit element. If R is a ring, we denote by $R^\bullet$ the monoid of regular elements of R, by $R^\times$ the group of invertible elements of R and by $T(R)$ a total quotient ring of R; clearly, $T(R)^\bullet = T(R)^\times$. For a prime ideal P of R, we set $R_{(P)} = (R^\bullet \setminus P)^{-1}R \subset T(R)$. Throughout, we shall assume that R is a Marot ring, and we shall use the Marot property in the following form.

Lemma. *A ring R is a Marot ring if and only if the following condition is satisfied:*

$$(\mathbf{M}) \quad \begin{cases} \text{For any two } R\text{-submodules } M_1, M_2 \text{ of } T(R), \\ M_1 \cap T(R)^\bullet = M_2 \cap T(R)^\bullet \neq \emptyset \text{ implies } M_1 = M_2. \end{cases}$$

Proof. By [4], Theorem 7.1, R is a Marot ring if and only if every regular R-submodule of $T(R)$ is generated by its regular elements. Therefore every Marot ring satisfies $(\mathbf{M})$.

Now let R be a ring satisfying $(\mathbf{M})$ and let $M \subset T(R)$ be a regular R-submodule. Let $M_0 \subset M$ be the R-submodule generated by $M \cap T(R)^\bullet$; it satisfies $M_0 \cap T(R)^\bullet = M \cap T(R)^\bullet \neq \emptyset$, and therefore $M_0 = M$. □

For the valuation theory of monoids and the theory of Krull monoids we refer to [3]. The main result of this note is the following Theorem.

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Theorem. *Let R be a Marot ring. Then R is a Krull ring if and only if $R^\bullet$ is a Krull monoid.*

Proof of the Theorem (Part 1). Let R be a Krull ring. Then there exists a set Ω of rank one valuations $v : T(R) \to \mathbb{Z} \cup \{\infty\}$ such that $R = \{x \in T(R) \mid v(x) \geq 0 \text{ for all } v \in \Omega\}$ and, for any $x \in T(R)^\bullet$, $v(x) = 0$ for all but finitely many $v \in \Omega$. If $v \in \Omega$, then $v(x) \in \mathbb{Z}$ for all $x \in T(R)^\bullet$, and $v^\bullet = v|T(R)^\bullet : T(R)^\bullet \to \mathbb{Z}$ is a valuation of $R^\bullet$. The set $\{v^\bullet \mid v \in \Omega\}$ is a defining set of valuations of $R^\bullet$, and therefore $R^\bullet$ is a Krull monoid. □

The proof of the non-trivial part of the Theorem rests on the following Proposition.

Proposition. *Let R be a Marot ring, $v : T(R)^\bullet \to \mathbb{Z}$ an essential valuation of $R^\bullet$, and let $P \triangleleft R$ be the ideal generated by $\{x \in R^\bullet \mid v(x) > 0\}$.*

i) *If $n \in \mathbb{N}$, $x_1, \ldots, x_n \in R^\bullet$, $\alpha_1, \ldots, \alpha_n \in R$ and $x = \alpha_1 x_1 + \cdots + \alpha_n x_n \in R^\bullet$, then $v(x) \geq \min\{v(x_1), \ldots, v(x_n)\}$.*

ii) *P is a prime ideal of R, then*

$$R_{(P)} \cap T(R)^\bullet = \{x \in T(R)^\bullet \mid v(x) \geq 0\},$$
$$PR_{(P)} \cap T(R)^\bullet = \{x \in T(R)^\bullet \mid v(x) > 0\}$$

and

$$R_{(P)}^\times = \{x \in T(R)^\bullet \mid v(x) = 0\}.$$

iii) *$R_{(P)}$ is a discrete rank one valuation ring.*

Proof. i) We may suppose that $n \geq 2$ and $v(x_1) = \min\{v(x_1), \ldots, v(x_n)\}$. For $2 \leq \nu \leq n$, we have $x_1^{-1}x_\nu \in T(R)^\bullet$, $v(x_1^{-1}x_\nu) \geq 0$, and since v is essential for $R^\bullet$, there exists an element $z_\nu \in R^\bullet$ such that $v(z_\nu) = 0$ and $z_\nu x_1^{-1} x_\nu \in R^\bullet$. Putting $z = z_2 \cdot \ldots \cdot z_n \in R^\bullet$, we obtain $v(z) = 0$, $zx_1^{-1}\alpha_\nu x_\nu \in R$ for $2 \leq \nu \leq n$, and hence

$$zx_1^{-1}x = \alpha_1 z + \sum_{\nu=2}^{n} zx_1^{-1}\alpha_\nu x_\nu \in R^\bullet;$$

consequently, $0 \leq v(zx_1^{-1}x) = -v(x_1) + v(x)$, and the assertion follows.

ii) By **i)**, we obtain

$$P \cap R^\bullet = \{x \in R^\bullet \mid v(x) > 0\}.$$

For any $x, y \in R^\bullet$, $xy \in P$ implies $0 < v(xy) = v(x) + v(y)$, and since $v(x) \geq 0$, $v(y) \geq 0$, we conclude $v(x) > 0$ or $v(y) > 0$, i.e. $x \in P$ or $y \in P$. Hence P is a prime ideal by [4], Theorem 7.10.

By construction, every $x \in R_{(P)} \cap T(R)^\bullet$ satisfies $v(x) \geq 0$; $x \in PR_{(P)} \cap T(R)^\bullet$ implies $v(x) > 0$, and $x \in R_{(P)}^\times$ implies $v(x) = 0$. For the converse, let $x \in T(R)^\bullet$ be an element satisfying $v(x) \geq 0$. Since v is essential for $R^\bullet$, there exists some $z \in R^\bullet$ such that $xz \in R^\bullet$ and $v(z) = 0$. This implies $z \notin P$, and consequently $x = \frac{xz}{z} \in R_{(P)}$. If $v(x) > 0$, then $v(xz) = v(x) > 0$, whence $xz \in P$ and $x \in PR_{(P)}$. If $v(x) = 0$, then x and x^{-1} both lie in $R_{(P)}$, whence $x \in R_{(P)}^\times$.

iii) By [6], Proposition 22, we must prove that $PR_{(P)}$ is the only regular prime ideal of $R_{(P)}$, and that it is an invertible ideal.

Let $t \in T(R)^\bullet$ be an element satisfying $v(t) = 1$. By **ii)**, $t \in PR_{(P)}$, and we claim that $PR_{(P)} = R_{(P)}t$. Clearly, it is sufficient to prove that $PR_{(P)} \cap T(R)^\bullet \subset R_{(P)}t$. If $x \in PR_{(P)} \cap T(R)^\bullet$, then $v(x) > 0$ and hence $v(xt^{-1}) = v(x) - 1 \geq 0$, which implies $xt^{-1} \in R_{(P)}$ and $x \in R_{(P)}t$. Being a regular principal ideal, $PR_{(P)}$ is invertible.

If $Q \triangleleft R_{(P)}$ is a regular prime ideal and $x \in Q \cap T(R)^\bullet$, then $v(x) > 0$ by **ii)**. This implies $v(t^{v(x)}x^{-1}) = 0$, hence $t^{v(x)}x^{-1} = e \in R_{(P)}^\times$ and $t^{v(x)} = xe \in Q$, whence $t \in Q$ and $PR_{(P)} \subset Q$. Since $(R_{(P)} \setminus PR_{(P)}) \cap T(R)^\bullet = R_{(P)}^\times$, the ideal $PR_{(P)}$ is a maximal regular ideal, and therefore $PR_{(P)} = Q$. □

Proof of the Theorem (Part 2). Let $R^\bullet$ be a Krull monoid and Ω the set of essential valuations of $R^\bullet$. For $v \in \Omega$, let $P_v \triangleleft R$ be the ideal generated by $\{x \in R^\bullet \mid v(x) > 0\}$. By the Proposition, P_v is a prime ideal and $R_{(P_v)}$ is a discrete rank one valuation ring. Therefore it is sufficient to prove that

$$R = \bigcap_{v \in \Omega} R_{(P_v)},$$

and every $x \in T(R)^\bullet$ lies in $R_{(P_v)}^\times$ for all but finitely many $v \in \Omega$.

By [3], Satz 1, Ω is a defining set of valuations for $R^\bullet$, which means that $R^\bullet = \{x \in T(R)^\bullet \mid v(x) \geq 0 \text{ for all } v \in \Omega\}$ and, for all $x \in T(R)^\bullet$, $v(x) = 0$ for all but finitely many $v \in \Omega$. By the Proposition, this implies

$$R^\bullet = \bigcap_{v \in \Omega} R_{(P_v)} \cap T(R)^\bullet,$$

and hence

$$R = \bigcap_{v \in \Omega} R_{(P_v)}$$

by the Lemma; furthermore, if $x \in T(R)^\bullet$, then $x \in R_{(P_v)}^\times$ for all but finitely many $v \in \Omega$. □

Remark. That the monoid of regular elements of a Krull ring is a Krull monoid, was already observed in [2]. Yet another characterization of Krull rings with zero divisors was given in [5].

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