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A GEOMETRIC APPROACH TO UNIVERSAL QUASIGROUP IDENTITIES

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Abstract. In the present paper we construct the accompanying identity $\hat{\mathcal{I}}$ of a given quasigroup identity $\mathcal{I}$. After that we deduce the main result: $\mathcal{I}$ is isotopically invariant (i.e., for every guasigroup $\mathbb{Q}$ it holds that if $\mathcal{I}$ is satisfied in $\mathbb{Q}$ then $\mathcal{I}$ is satisfied in every quasigroup isotopic to $\mathbb{Q}$) if and only if it is equivalent to $\hat{\mathcal{I}}$ (i.e., for every quasigroup $\mathbb{Q}$ it holds that in $\mathbb{Q}$ either $\mathcal{I}, \hat{\mathcal{I}}$ are both satisfied or both not).

§1 Coordinatizing loops of a given 3-web

Let $\mathcal{W} = (\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ be a given 3-web as a quintupel of the point set $\mathcal{L}$, line set $\mathcal{L}$ and line pencils $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ (in this order); here lines are taken as point sets. The order of $\mathcal{W}$ is the common cardinality of pencils.

Choose a set Q with the cardinality equal to the order of $\mathcal{W}$. Further choose bijections $\lambda_i : \mathcal{L}_i \to Q$, $i \in \{1,2,3\}$. Now there is just one ternary relation $\tau \subseteq Q \times Q \times Q$ such that $(x_1, x_2, x_3) \in \tau :\Leftrightarrow \lambda_1^{-1}(x_1), \lambda_2^{-1}(x_2), \lambda_3^{-1}(x_3)$ are concurrent lines. This ternary relation τ can be written out as a set of six quasigroup operations

$$\binom{k}{ij}(x_i, x_j) = x_k :\Leftrightarrow (x_1, x_2, x_3) \in \tau$$

where $\binom{1\ 2\ 3}{i\ j\ k}$ are all permutations of the set $\{1,2,3\}$. Binary operations $\cdot := \binom{3}{12}$, $\Big\backslash := \binom{2}{13}$, $\Big/ := \binom{1}{32}$ will be denoted as *main operations* of the 6-tupel (of mutually *parastrophic* operations $\binom{k}{ij}$). We shall speak about *coordinatizing objects* $Q, \lambda_1, \lambda_2, \lambda_3, \tau, \binom{k}{ij}, (Q, \cdot)$.

The coordinatizing quasigroup $(Q, \cdot)$ gets a loop, with neutral element e, if and only if there exists a point 0 (called the *origin*) such that the points $0 1 \sqcap \lambda_2(x)$, $0 2 \sqcap \lambda_1(x)$ lie in the line $\lambda_3(x)$ for all $x \in Q$. Here P_i denotes the line of $\mathcal{L}_i$ containing the point P, $i \in \{1,2,3\}$, and $\ell \sqcap \ell'$ denotes the intersection point of lines ℓ, ℓ' from different pencils. In this case the knowledge of one of bijections $\lambda_1, \lambda_2, \lambda_3$ suffices

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for the determination of both remaining ones. If we investigate simultaneously two triplets of coordinatizing bijections $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1', \lambda_2', \lambda_3')$ with the same starting set $Q = Q'$ then the corresponding ternary relations τ, τ' are bound by the *isotopy equivalence*

$$(x_1, x_2, x_3) \in \tau \Leftrightarrow (\lambda_1'\lambda_2^{-1}(x_1), \lambda_2'\lambda_1^{-1}(x_2), \lambda_3'\lambda_2^{-1}(x_3)) \in \tau'$$

for all $x_1, x_2, x_3 \in Q$.

We will describe the transformation of coordinates, i.e. the mutual relation between two coordinatizing loops of the same 3-web $\mathcal{W}$. Let us choose coordinatizing bijections $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_1', \lambda_2', \lambda_3')$ leading to coordinatizing loops. The corresponding origins will be denoted by 0, respectively $0'$, and we have the same starting set $Q = Q'$ and the common bijection $\lambda_3 = \lambda_3'$. We know that $02 \sqcap \lambda_1^{-1}(\xi)$, $01 \sqcap \lambda_2^{-1}(\xi) \in \lambda_3^{-1}(\xi)$ and analogously $02' \sqcap \lambda_1'^{-1}(\xi)$, $01' \sqcap \lambda_2'^{-1}(\xi) \in \lambda_3'^{-1}(\xi)$ for all $\xi \in Q$. Assume that $0' = \lambda_1^{-1}(\alpha) \sqcap \lambda_2^{-1}(\beta)$ for uniquely determined "coordinates" α, β of the point $0'$ with respect to the first coordinatization.

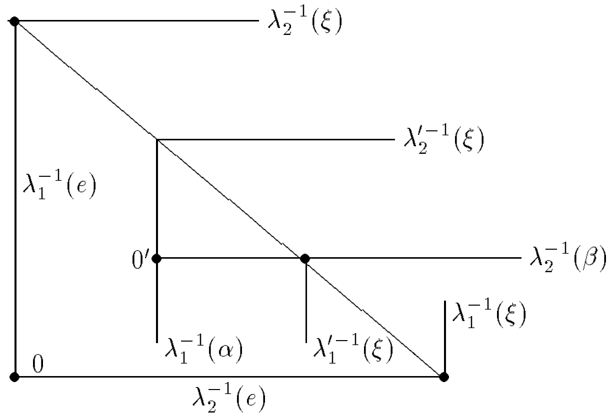


Fig. 1

Now we ask which relation occurs between coordinates ξ, η and ξ', η' of the same point $\lambda_1^{-1}(\xi) \sqcap \lambda_2^{-1}(\eta) = \lambda_1^{-1}(\xi') \sqcap \lambda_1'^{-1}(\eta')$.

For the sake of simplicity let us put $Q = \mathcal{L}_3$, $\lambda_3 = \lambda_3' = \mathrm{id}_Q$. Then $\xi \cdot \beta = \xi'$, $\alpha \cdot \eta = \eta'$, so that $\xi' = R_\beta(\xi)$, $\eta' = L_\alpha(\eta)$, $\eta = R_\beta^{-1}(\xi')$, $\eta = L_\alpha'(\eta')$. From the definition of coordinatizing operations $\cdot$ and $\cdot'$ it follows $\xi \cdot \eta = \xi' \cdot' \eta'$, i.e. $\xi \cdot \eta = R_\beta(\xi) \cdot' L_\alpha(\eta)$, respectively $\xi' \cdot' \eta' = R_\beta^{-1}(\xi') \cdot L_\alpha^{-1}(\eta')$ for all ξ, $\eta \in Q$, respectively for all ξ', $\eta' \in Q$. We see that the coordinatizing loop $(Q, \cdot')$ is the image of the coordinatizing loop $Q, \cdot'$ under the main isotopy $(R_\beta^{-1}, L_\alpha^{-1}, \mathrm{id}_Q)$. If the first loop has the neutral element e then the second one has the neutral element $e' = \alpha \cdot b$.

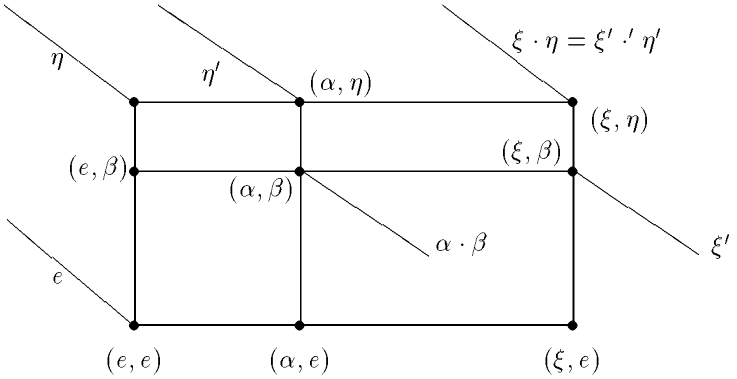


Fig. 2

If 0 is fixed and $0'$ runs over all the set $\mathcal{P}$ then the loop $(Q, \cdot')$ runs over the set of all the loops isotopic to $(Q, \cdot')$ up to isomorphisms. This important circumstance will be used later in §3.

§2 Transformations of line labels

Let $\mathbb{Q} = (Q, \cdot)$ be a non-trivial quasigroup, i.e., of order at least 2. Then the 3-web $\mathcal{W}_{\mathbb{Q}}$ *over* $\mathbb{Q}$ is defined as a 3-web $(\mathcal{P}, \mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ such that $\mathcal{P} = Q \times Q$, $\ell_\iota^{(1)} := \{(\iota, y)\big| y \in Q\}$, $\ell_\iota^{(2)} := \{(x, \iota)\big| x \in Q)$, $\ell_\iota^{(3)} := \{(x, y)\big| x \cdot y = \iota\}$ for all $\iota \in Q$ and $\mathcal{L}_i := \{\ell_i^{(\iota)}\big| \iota \in Q\}$ for all $i \in \{1, 2, 1\}$, $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. We proclaim the lines of $\mathcal{L}_1$ as *vertical*, the lines of $\mathcal{L}_2$ as *horizontal* and the lines of $\mathcal{L}_3$ as *skew*. The index $\iota \in Q$ will be called the *label* of the line $\ell_\iota^{(i)}$, $i \in \{1, 2, 3\}$. If the label ι belongs to the line $\ell_\iota^{(i)}$ then we say that it has the *position* i.

Now we fix an $i \in \{1, 2, 3\}$, choose some elements $\alpha, \beta \in Q$ and transform the label $\iota \in Q$ of the line $\ell_\iota^{(i)}$ as follows: For $i = 1$ let the transformation of ι into position 2 be $\alpha \backslash (i \cdot \beta)$ and into position 3 $\quad i \cdot \beta$, respectively. For $i = 2$ let the transformation of ι into position 1 be $(\alpha \cdot \iota) / \beta$ and into position 3 $\quad \alpha \cdot \iota$, respectively. Finally, for $i = 3$ let the transformation of ι into position 1 be ι / β and into position 2 $\quad \alpha \backslash \iota$, respectively. In this way the labeling of lines become a geometrical meaning: one can go over from one position to the other.

The coordinatizing quasigroup determined by the origin $\ell_\alpha^{(1)} \sqcap \ell_\beta^{(2)}$ is a loop, with neutral element α. In the sequel we shall apply the above transformations also onto variables of quasigroup identities.

Let there be given a quasigroup identity $\mathcal{I}$ with a finite set $\mathcal{X}$ of variables by main quasigroup operations $\cdot, \backslash, /$. Each variable $x \in \mathcal{X}$ enters in $\mathcal{I}$ in some subterms $\binom{k_1}{i_1 j_1}(x,\cdot)$, $\binom{k_2}{i_2 j_2}(\cdot,x)$, $x=\binom{k_3}{i_3 j_3}((\cdot,\cdot))$ (under the restriction onto the cases (i_1,j_1), (i_2,j_2), $(i_3,j_3)\in\{(1,2),\,(1,3),\,(3,2)\}$) and as such it has *positions* i_1, j_2 and k_3, respectively. Remark that the last case occurs if the left or right side of $\mathcal{I}$ consists only of x. For every $x\in\mathcal{X}$ we choose one of its positions, for example i, and call it the *starting position* of x. All remaining positions of x will be called *subordinate*. Let us take constants α,β and replace every variable $x\in\mathcal{X}$ in a subordinate position by its corresponding transformations. Further let us replace every consecutive term $\binom{k}{ij}(\cdot,\cdot)$ of the successive building of $\mathcal{I}$ (with position k) by its corresponding transformation. Finally we obtain an equality $\mathcal{I}_{\alpha,\beta}$ (a "specialized identity"), with variables from $\mathcal{X}$ and with two constants α,β. If we replace the constants α,β by new variables ξ,η we obtain an identity $\mathcal{I}_{\xi,\eta}$ called the *accompanying identity* of $\mathcal{I}$. The identity $\tilde{\mathcal{I}}=\mathcal{I}_{\xi,\eta}$ has the following *Position Property*: For every subterm t of $\tilde{\mathcal{I}}$, occurring in subterms $\binom{k_1}{i_1 j_1}(t,\cdot)$, $\binom{k_2}{i_2 j_2}(\cdot,t)$, $t=\binom{k_3}{i_3 j_3}(\cdot,\cdot)$ of $\tilde{\mathcal{I}}$, it holds $i_1=j_2=k_3$.

If we put $\xi=\eta$ in $\tilde{\mathcal{I}}$, we obtain the identity $\mathcal{I}_{\xi,\xi}$, the *secondary accompanying identity* of $\mathcal{I}$, with the set $\mathcal{X}\cup\{\xi\}$ of variables. Whilst the variables from $\mathcal{X}$ have always a unique position in $\tilde{\mathcal{I}}$, the variable ξ occurs either in position 1 or in position 2.

Before analyzing the connection between the given identity and its accompanying identities we give three illustrating examples.

1st example. Let $\mathcal{I}$ be the quasigroup identity $x\cdot x=x$ of idempotency. Let the starting position of x be equal to 2. We replace the variable x in position 2 and 3 (the position of x on the right side is taken to be 3 as is induced by the left side) by its transformations and obtain the equality $\mathcal{I}_{\alpha,\beta}$ and the identity $\mathcal{I}_{\xi,\eta}$

$$\Big((\alpha\cdot x)\Big/\beta\Big)\cdot x=\alpha\cdot x,\qquad \Big((\xi\cdot x)\Big/\eta\Big)\cdot x=\xi\cdot x.$$

$\mathcal{I}_{\xi,\eta}$ is equivalent with $\xi\cdot x\Big/\eta=\xi\cdot x\Big/x$ and this identity is equivalent with $\eta=x$. Thus the only quasigroups, in which the accompanying identity holds, are trivial quasigroups.

2nd example. Let $\mathcal{I}$ be the identity $(x\cdot x)\cdot x=x\cdot(x\cdot x)$ of monoassociativity. The starting position of x will be 2 again. Using convenient transformation of the variable x and consecutive subterms of $\mathcal{I}$ we obtain the equalities $\mathcal{I}_{\alpha,\beta}$ and $\mathcal{I}_{\xi,\eta}$

$$((((\alpha\cdot x)\Big/\beta)\cdot x)\Big/\beta)\cdot x=((\alpha\cdot x)\Big/\beta)\cdot(\alpha\Big\backslash(((\alpha\cdot x)\Big/\beta)\cdot x)),$$
$$((((\xi\cdot x)\Big/\eta)\cdot x)\Big/\eta)\cdot x=((\xi\cdot x)\Big/\eta)\cdot(\xi\Big\backslash(((\xi\cdot x)\Big/\beta)\cdot x)).$$

The accompanying identities $\mathcal{I}_{\xi,\eta}$, $\mathcal{I}_{\xi,\xi}$ are not equivalent.

3th example. Let $\mathcal{I}$ be the identity $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ of associativity. The starting position for the variables x,y,z are chosen to be 1, 2 and 3, respectively.

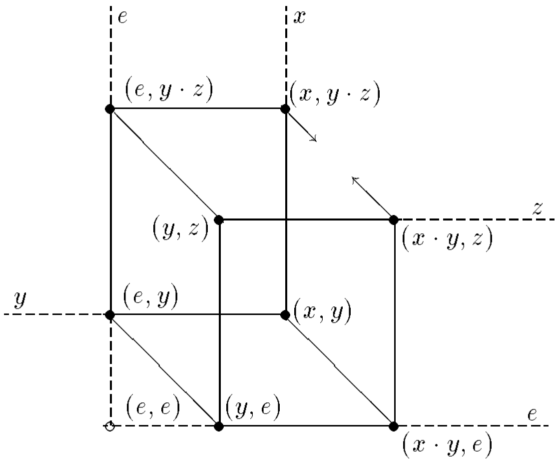
It suffices to transform the variable y on the right side into position 1 and the subterm on the left side into position 1 too. We obtain the equality $\mathcal{I}_{\alpha,\beta}$ and the identity $\mathcal{I}_{\xi,\eta}$

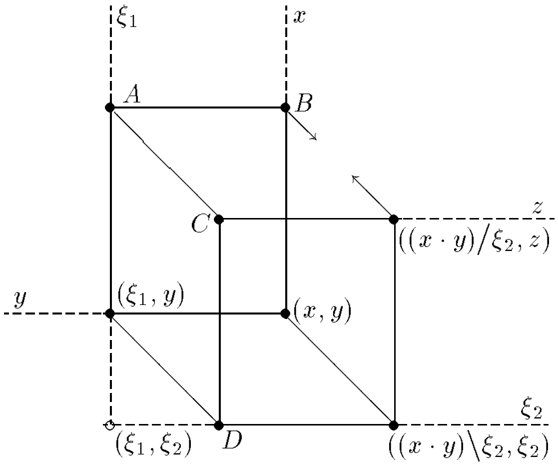
$$((\alpha \cdot (x \cdot y))\Big/\beta) \cdot z = x \cdot (\alpha\Big\backslash(((\alpha \cdot y)\Big/\beta) \cdot z)),$$
$$((\xi \cdot (x \cdot y))\Big/\eta) \cdot z = x \cdot (\xi\Big\backslash(((\xi \cdot y)\Big/\eta) \cdot z)).$$

The secondary accompanying identity $\mathcal{I}_{\xi,\xi}$ is

$$((\xi \cdot (x \cdot y))\Big/\xi) \cdot z = x \cdot (\xi\Big\backslash(((\xi \cdot y)\Big/\xi) \cdot z))$$

and is equivalent to $\mathcal{I}_{\xi,\eta}$ (a known property of Reidemeister closure condition in 3-webs).





where A means $\Big(\xi_1, \xi_1\Big\backslash(((\xi_1 \cdot y)/\xi_2) \cdot z)\Big)$, B means $\Big(x, \xi_1\Big\backslash(((\xi_1 \cdot y)/\xi_1) \cdot z)\Big)$, C means $\Big((\xi_1 \cdot z)\Big/\xi_2, z\Big)$ and D means $\Big((\xi_1 \cdot y)\Big/\xi_2, \xi_2\Big)$.

Theorem 1. *A quasigroup identity $\mathcal{I}$ holds in a loop isotopic to a given quasigroup $\mathbb{Q} = (Q, \cdot)$ if and only if there are elements $\alpha, \beta \in Q$ such that the equality $\mathcal{I}_{\alpha,\beta}$ holds in $\mathbb{Q}$.*

Corollary. *A quasigroup identity $\mathcal{I}$ holds in every loop isotopic to a given quasigroup $\mathbb{Q}$ if and only if the accompanying identity $\mathcal{I}_{\xi,\eta}$ holds in $\mathbb{Q}$.*

(Remark that the given quasigroup $\mathbb{Q}$ may be taken as a loop and we obtain the case investigated by V.D.Belousov in [5], Chapter IV, pp. 52-70 and in his further articles.)

Proof. Let a quasigroup identity $\mathcal{I}$ hold in a non-trivial* loop $\mathbb{L}$. We use the 3-web $\mathcal{W}_{\mathbb{L}}$ and interprete $\mathcal{I}$ as a relation between lines taking variables of $\mathcal{I}$ as variable lines of $\mathcal{W}_{\mathbb{L}}$ (we identify lines with their labels). In this way, $\mathcal{I}$ goes over onto a conditional identity (closure condition) which depends on coordinate axes of $\mathcal{W}_{\mathbb{L}}$. As every quasigroup $\mathbb{Q} = (Q, \cdot)$ isotopic to $\mathbb{L}$ arises (up to isomorphisms) as one of coordinatizing quasigroups of $\mathcal{W}_{\mathbb{L}}$ it follows that the coordinatizing loop $\mathbb{L}$ of $\mathcal{W}_{\mathbb{L}}$ is determined by a convenient choice of the origin and consequently of elements $\alpha, \beta \in Q$ used in transformation expressions leading to $\mathcal{I}_{\alpha\beta}$.

If $\mathcal{I}_{\alpha,\beta}$ holds in $\mathbb{Q} = (Q, \cdot)$ then the passage to $\mathcal{I}$ to a loop isotopic to $\mathbb{Q}$ can be made similarly as in the preceding procedure. $\square$

We say that a quasigroup identity $\mathcal{I}$ is *universal* (in other words: *invariant under quasigroup isotopies*) if every quasigroup $\mathbb{Q}$ satisfies the following hereditary condition with regard to isotopies: If $\mathcal{I}$ holds in $\mathbb{Q}$ then it holds in every quasigroup isotopic to $\mathbb{Q}$.

Theorem 2. *A quasigroup identity $\mathcal{I}$ is universal if and only if it is equivalent to its accompanying identity $\mathcal{I}_{\xi,\eta}$.*

Proof. Let $\mathcal{I}$ be equivalent to $\mathcal{I}_{\xi,\eta}$. If $\mathcal{I}_{\xi,\eta}$ holds in a quasigroup $\mathbb{Q}$ then the "geometric" form of $\mathcal{I}_{\xi,\eta}$ guarantees that $\mathcal{I}_{\xi,\eta}$ is valid in every quasigroup isotopic to $\mathbb{Q}$. But $\mathcal{I}_{\xi,\eta}$ is equivalent to $\mathcal{I}$ so that $\mathcal{I}$ must hold in every quasigroup isotopic to $\mathbb{Q}$.

Now let $\mathcal{I}$ be universal. Thus, if $\mathbb{Q}$ is an arbitrary quasigroup and $\mathcal{I}$ is valid in $\mathbb{Q}$, then $\mathcal{I}$ is valid in every quasigroup isotopic to $\mathbb{Q}$ and, especially, in every loop $\mathbb{L}$ isotopic to $\mathbb{Q}$. Thus assume $\mathcal{I}$ to be valid in a quasigroup $\mathbb{Q}$. Then $\mathcal{I}$ is valid in every loop isotopic to $\mathbb{Q}$ and, by Corollary to Theorem 1, $\mathcal{I}_{\xi,\eta}$ is valid in $\mathbb{Q}$. Conversely, if $\mathcal{I}_{\xi,\eta}$ is valid in $\mathbb{Q}$ then $\mathcal{I}$ is valid in every loop $\mathbb{L}$ isotopic to $\mathbb{Q}$ by Corollary to Theorem 1. From the assumed universality of $\mathcal{I}$ it follows that $\mathcal{I}$ must hold in all quasigroups isotopic to $\mathbb{L}$, where $\mathbb{L}$ runs over all loops isotopic to $\mathbb{Q}$. Thus $\mathcal{I}$ must hold in all quasigroups isotopic to $\mathbb{Q}$. $\square$

*If the given quasigroup $\mathbb{Q}$ is trivial then the assertion of Theorem 1 is obvious.

There exists a quasigroup identity $\mathcal{I}$ which does not satisfy the Position Property but is equivalent to its accompanying identity $\mathcal{I}_{\xi,\eta}$. It suffices to take for $\mathcal{I}$ the identity

$$((w \cdot (x \cdot y)) \Big/ u) \cdot z = x \cdot (u \Big\backslash (((u \cdot y) \Big/ u) \cdot z)) \tag{1}$$

with four variables u, x, y, z from our 2^{nd} example. Assigning to u the position 1 and going over to the accompanying identity of $\mathcal{I}$, we get

$$((u \cdot (x \cdot y)) \Big/ (\xi \Big\backslash (u \cdot \eta))) \cdot z = x \cdot (u \backslash (((u \cdot y) \Big/ (\xi \Big\backslash (u \cdot \eta))) \cdot z)). \tag{2}$$

Here we can substitute $\xi \Big\backslash (u \cdot \eta) = v$ and obtain an equivalent identity

$$(u \cdot (x \cdot y) \Big/ v) \cdot z = x \cdot ((u \Big\backslash ((u \cdot y) \Big/ v)) \cdot z) \tag{3}$$

having the Position Property. As (1) and (3) are equivalent (as already pointed out in 2^{nd} example) we have reached an example as claimed.

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