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RESULTS ON THE COMMUTATIVE NEUTRIX CONVOLUTION PRODUCT OF DISTRIBUTIONS

Brian Fisher, Emin Özçağ

Abstract. Let f, g be distributions in $\mathcal{D}'$ and let $f_n(x) = f(x)\tau_n(x)$, $g_n(x) = g(x)\tau_n(x)$, where $\tau_n(x)$ is a certain function which converges to the identity function as n tends to infinitity. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit exists. The neutrix convolution product $\ln x_- \boxed{*} x_+^r$ is evaluated for $r = 0, 1, 2, \dots$, from which other neutrix convolution products are deduced.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The following definition for the convolution product of certain distributions f and g in $\mathcal{D}'$, was given by Gel'fand and Shilov [6].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.
Then the convolution product $f * g$ is defined by

$$\langle (f * g)(x), \phi\rangle = \langle f(y), \langle g(x), \phi(x + y)\rangle\rangle$$

for arbitrary ϕ in $\mathcal{D}$.

It follows that if the convolution product $f * g$ exists by Definition 1, then

$$f * g = g * f, \tag{1}$$

$$(f * g)' = f * g' = f' * g. \tag{2}$$

In the next definition, the function τ is an infinitely differentiable, even function with the properties:

$$(i)\ 0 \le \tau(x) \le 1, \quad (ii)\ \tau(x) = 1 \quad \text{for } |x| \le \frac{1}{2}, \quad (iii)\ \tau(x) = 0 \text{ for } |x| \ge 1,$$

see Jones [7].

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Definition 2. Let f and g be distributions and let

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \dots$, where τ is defined as in Definition 2 of [5]. Let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \dots$. Then the commutative neutrix convolution product $f \boxed{*} g$ is defined as the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\underset{n\to\infty}{\mathrm{N-lim}}\, \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n, \quad (\lambda > 0\,;\, r = 1, 2, \dots)$$

and all functions $\epsilon(n)$ for which $\lim_{n\to\infty} \epsilon(n) = 0$.

The convolution product $f_n * g_n$ in this definition is again in the sense of Definition 1, the support of f_n being contained in the interval $[-n - n^{-n}, n + n^{-n}]$. It was proved in [3] that if a convolution product exists by Definition 1, then the commutative neutrix convolution product exists and defines the same distribution.

The following theorems were proved in [3] and [4] respectively.

Theorem 1. *The neutrix convolution product* $x_-^\lambda \boxed{*} x_+^\mu$ *exists and*

$$x_-^\lambda \boxed{*} x_+^\mu = B(-\lambda - \mu - 1, \mu + 1)x_-^{\lambda+\mu+1} + B(-\lambda - \mu - 1, \lambda + 1)x_+^{\lambda+\mu+1},$$

for $\lambda, \mu, \lambda + \mu \neq 0, \pm 1, \pm 2, \dots$, *where* B *denotes the Beta function.*

Theorem 2. *The neutrix convolution product* $x_-^\lambda \boxed{*} x_+^{r-\lambda}$ *exists and*

$$\begin{aligned} x_-^\lambda \boxed{*} x_+^{r-\lambda} &= B(-r - 1, r + 1 - \lambda)x_-^{r+1} + B(-r - 1, \lambda + 1)x_+^{r+1} \\ &\quad + \frac{(-1)^r (\lambda)_{r+1}}{(r + 1)!} x^{r+1} \ln |x|, \end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ *and* $r = -1, 0, 1, 2, \dots$.

In this theorem, B again denotes the Beta function but is defined as in [2] by

$$B(\lambda, \mu) = \underset{n\to\infty}{\mathrm{N-lim}} \int_{1/n}^{1-1/n} t^{\lambda-1}(1 - t)^{\mu-1}\, dt.$$

Before proving the next theorem, we need the following result:

$$\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(r-i+1)^2}=\frac{(-1)^r\Phi(r+1)}{r+1} \tag{3}$$

for $r=0,1,2,\dots$, where

$$\Phi(r)=\begin{cases}0, & r=0,\\ \sum_{i=1}^{r}1/i, & r\geq 1.\end{cases}$$

Proof. Putting

$$s(x)=\sum_{i=0}^{r}\binom{r}{i}\frac{(-x)^{r-i+1}}{(r-i+1)^2},$$

we have

$$xs'(x)=\sum_{i=0}^{r}\binom{r}{i}\frac{(-x)^{r-i+1}}{r-i+1}$$

and so

$$[xs'(x)]'=-\sum_{i=0}^{r}\binom{r}{i}(-x)^{r-i}=-(1-x)^r.$$

Thus

$$s'(x)=\frac{(1-x)^{r+1}-1}{(r+1)x}.$$

It now follows that

$$\begin{aligned}s(1)&=(-1)^{r+1}\sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^i}{(r-i+1)^2}\\ &=\frac{1}{r+1}\int_0^1[(1-x)^{r+1}-1]\,d\ln x=\int_0^1\ln x(1-x)^r\,dx\\ &=\frac{\partial}{\partial\lambda}B(\lambda,r+1)\bigg|_{\lambda=1}\\ &=\frac{(r+1)!\Gamma'(1)-\Gamma'(r+2)}{(r+1)(r+1)!}=-\frac{\Phi(r+1)}{r+1}\end{aligned}$$

and equation (3) follows.

We now prove the following theorem.

Theorem 3. *The neutrix convolution product* $\ln x_-\boxed{*}\,x_+^r$ *exists and*

$$\ln x_-\boxed{*}\,x_+^r=\frac{(-1)^{r+1}}{r+1}x_-^{r+1}\ln x_-+\frac{(-1)^r\Phi(r+1)}{r+1}x_-^{r+1}+\frac{\Phi(r)}{r+1}x_+^{r+1} \tag{4}$$

for $r = 0, 1, 2, \dots$.

Proof. We put

$$(\ln x_-)_n = \ln x_- \tau_n(x), \quad (x_+^r)_n = x_+^r \tau_n(x),$$

so that $(\ln x_-)_n$ and $(x_+^r)_n$ are summable functions. Then

$$\begin{aligned}
\langle (\ln x_-)_n * (x_+^r)_n, \phi(x)\rangle &= \langle (\ln y_-)_n, \langle (x_+^r)_n, \phi(x+y)\rangle\rangle \\
&= \int_{-n-n^{-n}}^{0} \ln(-y)\tau_n(y) \int_a^b (x-y)_+^r \tau_n(x-y)\phi(x)\,dx\,dy \\
&= \int_a^b \phi(x) \int_{-n}^{0} \ln(-y)(x-y)_+^r \tau_n(x-y)\,dy\,dx \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)_+^r \tau_n(x-y)\,dy\,dx
\end{aligned} \tag{5}$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a, b]$.

When $x < 0$ and $-n \le y \le 0$, $\tau_n(x-y) = 1$ on the support of ϕ. Thus with $x < 0$ and $-n \le y \le 0$, we have, on using the substitution $y = xu^{-1}$

$$\begin{aligned}
\int_{-n}^{0} \ln(-y)(x-y)_+^r \tau_n(x-y)\,dy &= \int_{-n}^{x} \ln(-y)(x-y)^r\,dy \\
&= (-x)^{r+1}\ln(-x) \int_{-x/n}^{1} u^{-r-2}(1-u)^r\,du \\
&\quad - (-x)^{r+1} \int_{-x/n}^{1} u^{-r-2}\ln u(1-u)^r\,du \\
&= I_{1n} - I_{2n}.
\end{aligned}$$

We have

$$\begin{aligned}
\int_{-x/n}^{1} u^{-r-2}(1-u)^r\,du &= \sum_{i=0}^{r}(-1)^i \binom{r}{i} \int_{-x/n}^{1} u^{i-r-2}\,du \\
&= \sum_{i=0}^{r}(-1)^i \binom{r}{i} \left[\frac{1}{i-r-1} - \frac{(-x/n)^{i-r-1}}{i-r-1}\right]
\end{aligned}$$

and it follows that

$$\begin{aligned}
\mathrm{N}\!-\!\lim_{n\to\infty} I_{1n} &= \sum_{i=0}^{r} \frac{(-1)^{i+1} r!}{i!(r-i)!(r-i+1)}(-x)^{r+1}\ln(-x) \\
&= -\frac{1}{r+1}\sum_{i=0}^{r}(-1)^i \binom{r+1}{i} x_-^{r+1}\ln x_- \\
&= \frac{(-1)^{r+1}}{r+1} x_-^{r+1}\ln x_-.
\end{aligned} \tag{6}$$

Further,

$$\int_{-x/n}^{1} u^{-r-2}\ln u(1-u)^r\,du = \sum_{i=0}^{r}(-1)^i\binom{r}{i}\int_{-x/n}^{1} u^{i-r-2}\ln u\,du$$
$$= \sum_{i=0}^{r}(-1)^i\binom{r}{i}\left[\frac{u^{i-r-1}\ln u}{i-r-1}-\frac{u^{i-r-1}}{(i-r-1)^2}\right]_{-x/n}^{1}$$

and it follows that

$$\text{(7)}\qquad \operatorname*{N-lim}_{n\to\infty} I_{2n} = \sum_{i=0}^{r}\binom{r}{i}\frac{(-1)^{i+1}}{(r-i+1)^2}(-x)^{r+1} = \frac{(-1)^{r+1}\Phi(r+1)}{r+1}x_-^{r+1},$$

by virtue of equation (3). It follows from equations (6) and (7) that when $x<0$,

$$\text{(8)}\qquad \operatorname*{N-lim}_{n\to\infty}\int_{-n}^{0}\ln(-y)(x-y)_+^r = \frac{(-1)^{r+1}}{r+1}x_-^{r+1}\ln x_- + \frac{(-1)^r\Phi(r+1)}{r+1}x_-^{r+1}.$$

When $x>0$ and $-n\le y\le 0$, we have

$$\text{(9)}\qquad \begin{aligned}&\int_{-n}^{0}\ln(-y)(x-y)_+^r\tau_n(x-y)\,dy\\ &= \int_{x-n}^{0}\ln(-y)(x-y)^r\,dy + \int_{x-n-n^{-n}}^{x-n}\ln(-y)(x-y)^r\tau_n(x-y)\,dy.\end{aligned}$$

Using the substitution $y=x(1-u^{-1})$, we have

$$\int_{x-n}^{0}\ln(-y)(x-y)^r\,dy = x^{r+1}\ln x\int_{x/n}^{1}u^{-r-2}\,du + x^{r+1}\int_{x/n}^{1}u^{-r-2}\ln(1-u)\,du$$
$$- x^{r+1}\int_{x/n}^{1}u^{-r-2}\ln u\,du$$
$$= I_{3n}+I_{4n}-I_{5n}.$$

We have

$$\int_{x/n}^{1}u^{-r-2}\,du = -\frac{1}{r+1}+\frac{n^{r+1}}{(r+1)x^{r+1}}$$

and it follows that

$$\text{(10)}\qquad \operatorname*{N-lim}_{n\to\infty} I_{3n} = -\frac{x^{r+1}\ln x}{r+1}.$$

Next we have

$$\int_{x/n}^{1}u^{-r-2}\ln(1-u)\,du = -\sum_{i=1}^{\infty}\int_{x/n}^{1}\frac{u^{i-r-2}}{i}\,du$$
$$= \sum_{i=1}^{r}\frac{1-(n/x)^{r-i+1}}{i(r-i+1)} + \sum_{i=r+2}^{\infty}\frac{1-(n/x)^{r-i+1}}{i(r-i+1)}$$
$$+\frac{\ln x-\ln n}{r+1}$$

and it follows that

$$\text{(11)} \qquad \begin{aligned} \operatorname*{N-lim}_{n\to\infty} I_{4n} &= \sum_{i=1}^{r} \frac{x^{r+1}}{i(r-i+1)} + \sum_{i=r+2}^{\infty} \frac{x^{r+1}}{i(r-i+1)} + \frac{x^{r+1}\ln x}{r+1} \\ &= \frac{\Phi(r)}{r+1}x^{r+1} - \frac{x^{r+1}}{(r+1)^2} + \frac{x^{r+1}\ln x}{r+1}. \end{aligned}$$

Finally, we have

$$\int_{x/n}^{1} u^{-r-2}\ln u\,du = \frac{n^{r+1}[\ln x - \ln n]}{(r+1)x^{r+1}} - \frac{1}{(r+1)^2} + \frac{n^{r+1}}{(r+1)^2 x^{r+1}}$$

and it follows that

$$\text{(12)} \qquad \operatorname*{N-lim}_{n\to\infty} I_{5n} = -\frac{x^{r+1}}{(r+1)^2}.$$

Further, with $n > x+1$

$$\begin{aligned} \left| \int_{x-n-n^{-n}}^{x-n} \ln(-y)(x-y)^r \tau_n(x-y)\,dy \right| &\le \int_{n}^{n+n^{-n}} \ln(y-x)y^r\,dy \\ &= o(n^{r-n}\ln n) \end{aligned}$$

and so

$$\text{(13)} \qquad \lim_{n\to\infty} \int_{x-n-n^{-n}}^{x-n} \ln(-y)(x-y)^r \tau_n(x-y)\,dy = 0.$$

It now follows from equations (9), (10), (11), (12) and (13) that

$$\text{(14)} \qquad \operatorname*{N-lim}_{n\to\infty} \int_{-n}^{0} \ln(-y)(x-y)_+^r \tau_n(x-y)\,dy = \frac{\Phi(r)}{r+1}x^{r+1}.$$

Next, with $-\frac{1}{2}n < a \le x \le b < \frac{1}{2}n$, we have

$$\begin{aligned} \left| \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)_+^r \tau_n(x-y)\,dy \right| &\le \int_{-n-n^{-n}}^{-n} \ln(-y)(x-y)^r\,dy \\ &= o(n^{r-n}\ln n) \end{aligned}$$

and so

$$\text{(15)} \qquad \lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)_+^r \tau_n(x-y)\,dy = 0.$$

It now follows from equations (5), (8), (14) and (15) that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} \langle (\ln x_-)_n * (x_+^r)_n, \phi(x)\rangle = (r+1)^{-1}\langle (-1)^{r+1}x_-^{r+1}\ln x_- \\ + (-1)^r\Phi(r+1)x_-^{r+1} + \Phi(r)x_+^{r+1}, \phi(x)\rangle \end{aligned}$$

and equation (4) follows. This completes the proof of the theorem. □

Corollary. *The neutrix convolution products* $\ln x_+ \boxed{*}\, x_-^r$, $\ln x_- \boxed{*}\, x^r$, $\ln x_+ \boxed{*}\, x^r$, $\ln |x| \boxed{*}\, x_-^r$, $\ln |x| \boxed{*}\, x_+^r$ *and* $\ln |x| \boxed{*}\, x^r$ *exist and*

$$\ln x_+ \boxed{*}\, x_-^r = \frac{(-1)^{r+1}}{r+1} x_+^{r+1} \ln x_+ + \frac{(-1)^r \Phi(r+1)}{r+1} x_+^{r+1} + \frac{\Phi(r)}{r+1} x_-^{r+1}, \tag{16}$$

$$\ln x_- \boxed{*}\, x^r = \frac{\Phi(r)}{r+1} x_+^{r+1}, \tag{17}$$

$$\ln x_+ \boxed{*}\, x^r = \frac{(-1)^r \Phi(r)}{r+1} x_-^{r+1}, \tag{18}$$

$$\ln |x| \boxed{*}\, x_-^r = \frac{(-1)^{r+1}}{r+1} x^{r+1} \ln |x| + \frac{(-1)^r \Phi(r+1)}{r+1} x^{r+1} + \frac{\Phi(r)}{r+1} x_-^{r+1}, \tag{19}$$

$$\ln |x| \boxed{*}\, x_+^r = \frac{1}{r+1} x^{r+1} \ln |x| - \frac{\Phi(r+1)}{r+1} x^{r+1} + \frac{\Phi(r)}{r+1} x_+^{r+1}, \tag{20}$$

$$\ln |x| \boxed{*}\, x^r = \frac{\Phi(r)}{r+1} [x_+^{r+1} + (-1)^r x_-^{r+1}] \tag{21}$$

for $r = 0, 1, 2, \dots$.

Proof. Equation (16) follows immediately on replacing x by $-x$ in equation (4).

The convolution products $\ln x_+ * x_+^r$ and $\ln x_- * x_-^r$ exist by Gel'fand and Shilov's definition and it is easily proved that

$$\ln x_+ * x_+^r = \frac{1}{r+1} x_+^{r+1} \ln x_+ - \frac{\Phi(r+1)}{r+1} x_+^{r+1}, \tag{22}$$

$$\ln x_- * x_-^r = \frac{1}{r+1} x_-^{r+1} \ln x_- - \frac{\Phi(r+1)}{r+1} x_-^{r+1} \tag{23}$$

for $r = 0, 1, 2, \dots$. Since

$$\ln |x| = \ln x_+ + \ln x_-, \quad x^r = x_+^r + (-1)^r x_-^r$$

and since the neutrix convolution products are clearly distributive with respect to addition, equations (17), (18), (19), (20) and (21) follow easily.

In the next theorem and its corollary, the distributions x_+^{-s} and x_-^{-s} are defined by

$$x_+^{-s} = \frac{(-1)^{s-1} (\ln x_+)^{(s)}}{(s-1)!}, \quad x_-^{-s} = -\frac{(\ln x_-)^{(s)}}{(s-1)!}$$

for $s = 1, 2, \dots$. □

Theorem 5. *The neutrix convolution product* $x_-^{-s} \boxed{*}\, x_+^r$ *exists and*

$$\begin{aligned} x_-^{-s} \boxed{*}\, x_+^r = \binom{r}{s-1} \Big\{ (-1)^{r-s} x_-^{r-s+1} \ln x_- - (-1)^{r-s} \Big[\Phi(r-s+1) \\ - \Phi(s-1) \Big] x_-^{r-s+1} - \Phi(r) x_+^{r-s+1} \Big\} \end{aligned} \tag{24}$$

for $s = 1, 2, \ldots, r+1$ *and* $r = 0, 1, 2, \ldots$ *and*

$$x_-^{-s} \boxed{*} x_+^r = \frac{r!(s-r-2)!}{(s-1)!} x_-^{r-s+1} \tag{25}$$

for $s = r+2, r+3, \ldots$ *and* $r = 0, 1, 2, \ldots$.

Proof. The convolution product $(\ln x_-)_n * (x_+^r)_n$ exists by Gel'fand and Shilov's definition and so equations (2) hold. Thus if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, where we may suppose that $a < 0 < b$,

$$\begin{aligned}\langle[(\ln x_-)_n * (x_+^r)_n]', \phi(x)\rangle &= -\langle(\ln x_-)_n * (x_+^r)_n, \phi'(x)\rangle \\ &= -\langle(x_-^{-1})_n * (x_+^r)_n, \phi(x)\rangle + \langle[\ln x_- \tau_n'(x)] * (x_+^r)_n, \phi(x)\rangle\end{aligned}$$

and so

$$\langle(x_-^{-1})_n * (x_+^r)_n, \phi(x)\rangle = \langle(\ln x_-)_n * (x_+^r)_n, \phi'(x)\rangle + \langle[\ln x_- \tau_n'(x)] * (x_+^r)_n, \phi(x)\rangle.$$

The support of $\ln x_- \tau_n'(x)$ is contained in the interval $[-n-n^{-n}, -n]$ and so with $n > -a > n^{-n}$, it follows as above that

$$\begin{aligned}\langle[\ln x_- \tau_n'(x)] * (x_+^r)_n, \phi(x)\rangle &= \\ &= \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx \\ &= \int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r\,dy\,dx \\ &\quad - \int_{-n^{-n}}^0 \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r\,dy\,dx \\ &\quad + \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx,\end{aligned} \tag{26}$$

where on the domain of integration, $\ln(-y)$ and $(x-y)^r$ are locally summable functions.

Putting $M = \sup\{|\tau'(x)|\}.\sup\{|\phi(x)|\}$, we have

$$\begin{aligned}&\left|\int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx\right| \le \\ &\qquad \le Mn^n \int_{-n^{-n}}^{n^{-n}} \int_{-n-n^{-n}}^{-n} \ln(-y)(x-y)^r\,dy\,dx \\ &\qquad = o(n^{r-n}\ln n)\end{aligned}$$

and it follows that

$$\lim_{n\to\infty} \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx = 0.$$

Similarly,

$$\lim_{n\to\infty}\int_{-n^{-n}}^{0}\phi(x)\int_{-n-n^{-n}}^{-n}\ln(y)\tau_n'(y)(x-y)^r\,dy\,dx = 0.$$

Integrating by parts, it follows that

$$\begin{aligned}(27)\qquad &\int_{-n-n^{-n}}^{-n}\ln(-y)\tau_n'(y)(x-y)^r\,dy\\ &= \ln n(x+n)^r - \int_{-n-n^{-n}}^{-n}\left[y^{-1}(x-y)^r - r\ln(-y)(x-y)^{r-1}\right]\tau_n(y)\,dy,\end{aligned}$$

where

$$\operatorname*{N-lim}_{n\to\infty}\ln n(x+n)^r = 0.$$

Further, it follows as in the proof of equation (15) that

$$\lim_{n\to\infty}\int_{-n-n^{-n}}^{-n}[(-y)^{-1}(x-y)^r - r\ln(-y)(x-y)^{r-1}]\tau_n(y)\,dy = 0.$$

Thus

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty}\langle(x_-^{-1})_n*(x_+^r)_n,\phi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty}\langle(\ln x_-)_n*(x_+^r)_n,\phi'(x)\rangle\\ &= \langle\ln x_-\boxed{*}\,x_+^r,\phi'(x)\rangle.\end{aligned}$$

This proves that the neutrix convolution product $x_-^{-1}\boxed{*}\,x_+^r$ exists and

$$\begin{aligned}x_-^{-1}\boxed{*}\,x_+^r &= -(\ln x_-\boxed{*}\,x_+^r)'\\ &= (-1)^{r-1}[x_-^r\ln x_- - \Phi(r)x_-^r] - \Phi(r)x_+^r\end{aligned}$$

on using equation (4). Equation (24) follows when $s=1$ and $r=0,1,2,\ldots$.

Now assume that equation (24) holds for some positive integer $s<r+1$ and $r=0,1,2,\ldots$. Let ϕ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a,b]$. Then it follows as above that

$$\langle[(x_-^{-s})_n*(x_+^r)_n]',\phi(x)\rangle = s\langle(x_-^{-s-1})_n*(x_+^r)_n,\phi(x)\rangle + \langle[x_-^{-s}\tau_n'(x)]*(x_+^r)_n,\phi(x)\rangle$$

and so

$$s\langle(x_-^{-s-1})_n*(x_+^r)_n,\phi(x)\rangle = -\langle(x_-^{-s})_n*(x_+^r)_n,\phi'(x)\rangle - \langle[x_-^{-s}\tau_n'(x)*(x_+^r)_n,\phi(x)\rangle.$$

The support of $x_-^{-s}\tau_n'(x)$ is contained in the interval $[-n-n^{-n}, -n]$ and so with $n > -a > n^{-n}$,

$$
\begin{aligned}
&\langle [x_-^{-s}\tau_n'(x)] * (x_+^r)_n, \phi(x)\rangle \\
&\qquad = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx \\
&\qquad = \int_a^0 \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r\,dy\,dx \\
&\qquad\quad - \int_{-n^{-n}}^0 \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r\,dy\,dx \\
&\qquad\quad + \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx,
\end{aligned}
\tag{29}
$$

where on the domain of integration, $(-y)^{-s}$ and $(x-y)^r$ are locally summable functions.

As above,

$$
\lim_{n\to\infty} \int_{-n^{-n}}^{n^{-n}} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r \tau_n(x-y)\,dy\,dx = 0 \tag{30}
$$

$$
= \lim_{n\to\infty} \int_{-n^{-n}}^{0} \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r\,dy\,dx \tag{31}
$$

Further, integrating by parts,

$$
\begin{aligned}
&\int_{-n-n^{-n}}^{-n} (-y)^{-s}\tau_n'(y)(x-y)^r\,dy = n^{-s}(x+n)^r \\
&\quad - \int_{-n-n^{-n}}^{-n} [s(-y)^{-s-1}(x-y)^r - r(-y)^{-s}(x-y)^{r-1}]\tau_n(y)\,dy.
\end{aligned}
\tag{32}
$$

Now

$$
n^{-s}(x+n)^r = \sum_{i=0}^{r} \binom{r}{i} x^i n^{r-s-i}
$$

and so

$$
\mathrm{N}\!-\!\lim_{n\to\infty} n^{-s}(x+n)^r = \binom{r}{s} x^{r-s}. \tag{33}
$$

As above

$$
\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} [s(-y)^{-s-1}(x-y)^r - r(-y)^{-s}(x-y)^{r-1}]\tau_n(y)\,dy = 0 \tag{34}
$$

and it follows from equations (29) to (34) that

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\langle[x_-^{-s}\tau_n'(x)]*(x_+^r)_n,\phi(x)\rangle &= \binom{r}{s}\int_a^0 x^{r-s}\phi(x)\,dx \\
&= \binom{r}{s}(-1)^{r-s}\int_a^b x_-^{r-s}\phi(x)\,dx \\
&= \binom{r}{s}(-1)^{r-s}\langle x_-^{r-s},\phi(x)\rangle.
\end{aligned}$$

It now follows from equation (28) that

$$\begin{aligned}
&\operatorname*{N-lim}_{n\to\infty} s\langle (x_-^{-s-1})_n*(x_+^r)_n,\phi(x)\rangle \\
&= -\langle x_-^{-s}\boxed{*}\,x_+^r,\phi'(x)\rangle - \binom{r}{s}(-1)^{r-s}\langle x_-^{r-s},\phi(x)\rangle.
\end{aligned}$$

From our assumption, this proves that $x_-^{-s-1}\boxed{*}\,x_+^r$ exists and

$$\begin{aligned}
x_-^{-s-1}\boxed{*}\,x_+^r &= \binom{r}{s}(-1)^{r-s-1}\Big\{x_-^{r-s}\ln x_- + (r-s+1)^{-1}x_-^{r-s} \\
&\quad - \big[\Phi(r-s+1)-\Phi(s-1)\big]x_-^{r-s} \\
&\quad + (-1)^{r-s}\Phi(r)x_+^{r-s} + s^{-1}x_-^{r-s}\Big\} \\
&= \binom{r}{s}\Big\{(-1)^{r-s-1}x_-^{r-s}\ln x_- \\
&\quad - (-1)^{r-s-1}[\Phi(r-s)-\Phi(s)]x_-^{r-s} - \Phi(r)x_+^{r-s}\Big\}.
\end{aligned}$$

Equation (24) follows by induction for $s = 1, 2, \ldots, r+1$ and $r = 0, 1, 2, \ldots$.

When $s \ge r+1$, equations (28) to (33) still hold but this time

$$\operatorname*{N-lim}_{n\to\infty} n^{-s}(x+n)^r = 0$$

and so

$$(x_-^{-s}\boxed{*}\,x_+^r)' = s x_-^{-s-1}\boxed{*}\,x_+^r$$

for $s = r+1, r+2, \ldots$ and $r = 0, 1, 2, \ldots$. In particular,

$$x_-^{-r-1}\boxed{*}\,x_+^r = -\ln x_- - \Phi(r)$$

and so on differentiating we get

$$(r+1)x_-^{-r-2}\boxed{*}\,x_+^r = x_-^{-1},$$

proving equation (25) for the case $s = r+2$. Equation (25) now follows easily by induction for $s = r+2, r+3, \ldots$ and $r = 0, 1, 2, \ldots$, completing the proof of the theorem. $\square$

Corollary. *The neutrix convolution products* $x_+^{-s} \boxed{*}\, x_-^r$, $x_-^{-s} \boxed{*}\, x^r$, $x_+^{-s} \boxed{*}\, x^r$, $x^{-s} \boxed{*}\, x_-^r$, $x^{-s} \boxed{*}\, x_+^r$ *and* $x^{-s} \boxed{*}\, x^r$ *exist and*

$$\begin{aligned} x_+^{-s} \boxed{*}\, x_-^r = \binom{r}{s-1}\Big\{&(-1)^{r-s} x_+^{r-s+1} \ln x_+ \\ &- (-1)^{r-s}\Big[\Phi(r-s+1) - \Phi(s-1)\Big] x_+^{r-s+1} \\ &-\Phi(r) x_-^{r-s+1}\Big\}, \end{aligned} \tag{35}$$

$$x_-^{-s} \boxed{*}\, x^r = \binom{r}{s-1}\Big[(-1)^{r-s}\Phi(s-1) x_-^{r-s+1} - \Phi(r) x_+^{r-s+1}\Big], \tag{36}$$

$$x_+^{-s} \boxed{*}\, x^r = \binom{r}{s-1}\Big[(-1)^{s}\Phi(s-1)] x_+^{r-s+1} - (-1)^r\Phi(r) x_-^{r-s+1}\Big], \tag{37}$$

$$\begin{aligned} x^{-s} \boxed{*}\, x_-^r = \binom{r}{s-1}\Big[&(-1)^{r-s} x^{r-s+1} \ln |x| \\ &+ (-1)^{r-s}\Phi(s-1) x_+^{r-s+1} \\ &- (-1)^{r-s}\Phi(r-s+1) x^{r-s+1} - \Phi(r) x_-^{r-s+1}\Big], \end{aligned} \tag{38}$$

$$\begin{aligned} x^{-s} \boxed{*}\, x_+^r = \binom{r}{s-1}\Big[&(-1)^{s-1} x^{r-s+1} \ln |x| \\ &+ (-1)^{r}\Phi(s-1) x_-^{r-s+1} \\ &+(-1)^{s}\Phi(r-s+1) x^{r-s+1} - (-1)^s\Phi(r) x_+^{r-s+1}\Big], \end{aligned} \tag{39}$$

$$\begin{aligned} x^{-s} \boxed{*}\, x^r = \binom{r}{s-1}&[\Phi(s-1) - \Phi(r)][(-1)^s x_+^{r-s+1} \\ &+ (-1)^r x_-^{r-s+1}] \end{aligned} \tag{40}$$

for $s = 1, 2, \ldots, r+1$ *and* $r = 0, 1, 2, \ldots$, *and*

$$x_+^{-s} \boxed{*}\, x_-^r = \frac{r!(s-r-2)!}{(s-1)!} x_+^{r-s+1}, \tag{41}$$

$$x_-^{-s} \boxed{*}\, x^r = 0, \tag{42}$$

$$x_+^{-s} \boxed{*}\, x^r = 0, \tag{43}$$

$$x^{-s} \boxed{*}\, x_-^r = \frac{r!(s-r-2)!}{(s-1)!} x^{r-s+1}, \tag{44}$$

$$x^{-s} \boxed{*}\, x_+^r = \frac{(-1)^{r+1} r!(s-r-2)!}{(s-1)!} x^{r-s+1}, \tag{45}$$

$$x^{-s} \boxed{*}\, x^r = 0 \tag{46}$$

for $s = r+2, r+3, \ldots$ *and* $r = 0, 1, 2, \ldots$.

Proof. Equations (35) and (41) follow immediately on replacing x by $-x$ in equations (24) and (25) respectively.

Differentiating equations (22) and (23) s times using equations (2) gives

$$x_+^{-s} * x_+^r = (-1)^{s-1}\binom{r}{s-1}[x_+^{r-s+1}\ln x_+ - \Phi(r-s+1)x_+^{r-s+1}],$$
$$x_-^{-s} * x_-^r = (-1)^{s-1}\binom{r}{s-1}[x_-^{r-s+1}\ln x_- - \Phi(r-s+1)x_-^{r-s+1}]$$

for $s = 1, 2, \dots, r+1$ and $r = 0, 1, 2, \dots$, and

$$x_+^{-s} * x_+^r = \frac{(-1)^{r+1}r!(s-r-2)!}{(s-1)!}x_+^{r-s+1},$$
$$x_-^{-s} * x_-^r = \frac{(-1)^{r+1}r!(s-r-2)!}{(s-1)!}x_-^{r-s+1}$$

for $s = r+1, r+2, \dots$ and $r = 0, 1, 2, \dots$. Equations (36), (37), (42) and (43) follow immediately. Equations (38), (39), (40), (44), (45) and (46) also follow on noting that

$$x^{-s} = x_+^{-s} + (-1)^s x_-^{-s}$$

for $s = 1, 2, \dots$. $\square$

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Brian Fisher, Emin Özçağ
Department of Mathematics and Computer Science
University of Leicester
Leicester, LE1 7RH, GREAT BRITAIN