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FURTHER HIGHER MONOTONICITY PROPERTIES OF STURM–LIOUVILLE FUNCTION

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Abstract. Suppose that the function $q(t)$ in the differential equation (1) $y'' + q(t)y = 0$ is decreasing on (b, ∞) where $b \geq 0$. We give conditions on q which ensure that (1) has a pair of solutions $y_1(t)$, $y_2(t)$ such that the n-th derivative ($n \geq 1$) of the function $p(t) = y_1^2(t) + y_2^2(t)$ has the sign $(-1)^{n+1}$ for sufficiently large t and that the higher differences of a sequence related to the zeros of solutions of (1) are ultimately regular in sign.

1. Introduction

Several authors (starting with [9]; see [11] for references) have considered higher monotonicity properties of solutions of the differential equation

$$y'' + q(t)y = 0 \tag{1}$$

when assumptions are made about the higher monotonicity behaviour of q. Hartman [6] showed that if $q(\infty) > 0$ and

$$(-1)^n q^{(n+1)}(t) \geq 0, \quad 0 \leq t < \infty, \quad n = 0, 1, \dots \tag{2}$$

then (1) has a pair of solutions $y_1(t)$, $y_2(t)$ such that $p(t) = y_1^2(t) + y_2^2(t)$ satisfies

$$(-1)^n p^{(n)}(t) \geq 0, \quad 0 \leq t < \infty, \quad n = 0, 1, \dots . \tag{3}$$

L. Lorch and P. Szego [9] showed, among other things, that if (3) holds, then

$$(-1)^n \Delta^{n+1} t_k \geq 0, \quad n = 0, 1, \dots, \quad k = 1, 2, \dots , \tag{4}$$

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[1]Miloš Háčik died suddenly on August 21, 1986 shortly after the first draft of this article was prepared. We dedicate it to his memory. Z.D., M.E.M.

where $\{t_1, t_2, \ldots\}$ is the sequence of zeros of any non–trivial solution of (1) on $(0, \infty)$. A typical example for both of these results is the transformed form

$$y'' + \left[1 + \left(\frac{1}{4} - \nu^2\right) t^{-2}\right] y = 0 \tag{5}$$

of the Bessel equation in the case $|\nu| \geq \frac{1}{2}$.

It is natural also to consider this equation for $|\nu| < \frac{1}{2}$. In this case we have

$$(-1)^n q^{(n)}(t) \geq 0, \quad 0 < t < \infty, \quad n = 0, 1, \ldots \tag{6}$$

and for a standard pair $y_1(t) = \sqrt{t} J_\nu(t)$, $y_2(t) = \sqrt{t} Y_\nu(t)$ of solutions of (5) it is known [9,p. 62] that $p(t) = y_1^2(t) + y_2^2(t)$ satisfies

$$(-1)^n p^{(n+1)}(t) > 0, \quad 0 < t < \infty, \quad n = 0, 1, \ldots . \tag{7}$$

However, in general, (6) does not imply (7).

The question of what additional assumptions are needed to make (6) imply (7) was discussed by P. Hartman in [7]. Z. Došlá [3] showed, in the case $q(\infty) > 0$, that if (6) holds and if certain assumptions are made about the orders of magnitude of the successive derivatives of $q(t)$, then there is a sequence $\{T_n\}$ such that

$$(-1)^n p^{(n+1)}(t) \geq 0, \quad T_n < t < \infty, \quad n = 0, 1, \ldots . \tag{8}$$

This is a weaker result than (7). However, P. Hartman [7] showed that in the special case $q(t) = 1 + \beta/t^\gamma, \beta > 0, \gamma > 0$, (7) holds for $0 < \gamma < 1$ and $\gamma = 2$.

In this paper we use a result of Hartman [6, Theorem 22.1_n] to show (Theorem 2.1) that if (6) holds and if

$$(-1)^n [q(t)^{-1} D_t]^{(n+1)} (q(t))^{-1} \geq 0, \quad 0 < t < \infty, \quad n = 0, 1, \ldots , \tag{9}$$

then (7) holds. In §6, we give examples of situations where (6) and (9) hold. We also give (Theorem 2.2) a sort of converse result where higher monotonicity assumptions on p are used to give similar properties of q. In §3 we give results analogous to (4) under assumption (6) and additional assumptions on q. However, these results, like those of [3] refer to *ultimate* monotonicity. Typically, they show the existence of a sequence $\{l_n\}$ such that

$$(-1)^n \Delta^{n+2} t_k \geq 0, \quad n = 0, 1, \ldots, k = l_n, l_n + 1, \ldots . \tag{10}$$

In §4, we state some auxiliary results and in §5 we present proofs of the Theorems. §6 is devoted to applications.

Finally, we remark that most of our results are stated and proved for "multiply monotonic" functions and sequences, i.e. inequalities like (6), (8) and (10) are supposed to hold for $n = 0, 1, \ldots, M$ for some finite M. Results concerning "completely monotonic" functions and sequences follow by letting M tend to infinity.

2. The functions $q(t)$ and $p(t)$

Our principal result here is the following.

Theorem 2.1. *Let (1) be oscillatory at ∞. Let*

$$(-1)^n D_t^n(q(t)) \geq 0, \quad 0 < t < \infty, \quad n = 0, 1, \dots, N+1 \tag{11}$$

and

$$(-1)^n D_\theta^{n+1}[(q(t)^{-1}] \geq 0, \quad 0 < t < \infty, \quad n = 0, 1, \dots, N+1\,, \tag{12}$$

where $\theta'(t) = q(t)$, i.e. $D_\theta = [q(t)]^{-1} D_t$. Then (1) has a pair of solutions $y_1(t), y_2(t)$ on $(0, \infty)$ such that $p(t) = y_1^2(t) + y_2^2(t)$ satisfies

$$(-1)^n D_t^{n+1} p(t) \geq 0, \quad 0 < t < \infty, \quad n = 0, 1, \dots, N\,. \tag{13}$$

$y_1(t)$ and $y_2(t)$ are unique to the extent that $p(t)$ is unique up to multiplication by a constant.

Furthermore, if $N \geq 2$, the function $p^2 q$ is nonincreasing on $(0, \infty)$ and it has a positive limit as $t \to \infty$.

Theorem 2.1 was about deducing properties of p from those of q. Theorem 2.2 goes in the opposite direction. It is valid on any interval I.

Theorem 2.2. *Let the pair of solutions $y_1(t), y_2(t)$ of (1) in I be such that $p(t) = y_1^2(t) + y_2^2(t)$ satisfies*

$$(-1)^n D_t^{n+1}[p(t)]^{1/2} \geq 0, t \in I, n = 0, 1, \dots, N\,. \tag{14}$$

Then

$$(-1)^n D_t^n q(t) \geq 0, t \in I, n = 0, 1, \dots, N\,. \tag{15}$$

3. Ultimate monotonicity

We start with a result whose conclusion is weaker than that of Theorem 2.1 in the sense that, for each n, (13) holds only on a subinterval (μ_n, ∞) of $(0, \infty)$. Its hypotheses are, in part, weaker than those of Theorem 2.1 but we need to impose an order condition on the derivatives of $q(t)$ and $q^{-1}(t)$. We use the usual notation $f(t) = O(t^{-\alpha}),\ t \to \infty$, to mean that

$$\limsup_{t\to\infty} |f(t)| t^\alpha < \infty\,.$$

Theorem 3.1. *Let (1) be oscillatory at $t = \infty$ and let $N \geq 4$ be a fixed integer. Let*

$$(-1)^n q^{(n)}(t) \geq 0, n = 0, 1, \ldots, N+2, \tag{16}$$

$$\lim_{t\to\infty} \frac{tq''(t)}{q'(t)} = \rho, \quad -\infty < \rho < 0. \tag{17}$$

If $q(\infty) = 0$, suppose, in addition, that

$$(q^{-1})' = O(t^{-1}), \quad t \to \infty. \tag{18}$$

Then (1) has a pair of solutions $y_1(t), y_2(t)$ such that $p(t) = y_1^2(t) + y_2^2(t)$ satisfies

$$(-1)^n p^{(n+1)}(t) \geq 0, \quad \mu_n < t < \infty, \quad n = 0, 1, \ldots, N, \tag{19}$$

where $\{\mu_n\}$ is a nondecreasing sequence and $\mu_n = \mu_{n+1}$ only if $\mu_n = 0$. Moreover, $p^2 q$ is ultimately nonincreasing and has a positive limit as $t \to \infty$.

Now we turn to deducing properties of zeros of a nontrivial solution of (1) from properties of $q(t)$.

We let $t_1, t_2, \ldots$ be the sequence of consecutive positive zeros of a solution $y(t)$ of (1). Our result on this sequence runs as follows.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 hold and let*

$$\lim_{t\to\infty} |q'(t)|t^2 = k, \quad 0 < k \leq \infty. \tag{20}$$

Then there is a nondecreasing sequence $\{l_n\}$ of non–negative integers with $l_n = l_{n+1}$ only if $l_n = 0$, such that

$$(-1)^n \Delta^{n+2} t_k \geq 0, \quad n = 0, 1, \ldots, N+1, \quad k = l_n, l_n + 1, \ldots. \tag{21}$$

Remark 1. The case of Theorem 3.1 in which $q(\infty) > 0$ was stated and proved in [3,Theorem 3.1] under stronger assumptions. The case $q(\infty) > 0$ of Theorem 3.2 is a supplement to [3,Theorem 3.2] for functions $q(t)$ tending "slowly" to a nonzero constant.

In the following section, we show that the function q considered in Theorems 3.1, 3.2 is slowly varying; this will be used to establish further asymptotic properties of q. Lemma 4.3 makes it possible to differentiate an O-term and to determine the sign of a general differential operator. This, together with [6,Theorem 22.1$_n$] and [11,Theorem 2.1], will be employed for the functions q^{-1} and p in the proofs of Theorems 3.1 and 3.2, respectively.

4. Auxiliary results

In what follows, we need

Definition 1. *A function f is n-times monotonic on an interval I, and we write $f \in M_n(I)$ if*

$$(-1)^i f^{(i)}(t) \geq 0, \quad t \in I, \quad i = 0, 1, \dots, n\,.$$

Clearly, sums and products of n-times monotonic functions are n-times monotonic.

Lemma 4.1. *Let $\limsup_{t\to\infty} |f_2(t)/f_1(t)| < 1$. Then there exists T such that $sgn[f_1(t) + f_2(t)] = sgn\ f_1(t)$ for $t \geq T$.*

Proof. . It is clear that there exists a T and an $\epsilon > 0$ such that $|f_2(t)/f_1(t)| < 1-\epsilon$ and hence $[f_1(t) + f_2(t)]/f_1(t) > \epsilon$ for $t \geq T$. □

In [4] J. Karamata's notion of regularly varying functions is presented as follows:

A function $f(t)$ is said to be *ρ–regularly varying at infinity* or simply ρ–varying at infinity, if it is real–valued, positive and measurable on $[b, \infty)$ for some $b > 0$ and if for each $x > 0$,

$$\lim_{t\to\infty} f(tx)/f(t) = x^\rho \tag{22}$$

for some ρ in the interval $-\infty < \rho < \infty$. ρ is called the *index* of regular variation. For example, for all real ρ, the functions t^ρ, $t^\rho \log(1+t)$, $[t \log(\log(e+t))]^\rho$ are ρ-varying at infinity.

A function $f(t)$ which is regularly varying with index $\rho = 0$ is called *slowly varying* (at infinity). Every function f for which $f(t)$ tends to a positive constant as $t \to \infty$ is slowly varying. So is $\log t$.

If $\lim_{t\to\infty} |f(t)| t^{-\alpha} = A, \quad 0 < A < \infty$, then f is α-varying.

Proposition 1. [12,p.7]. *Any function f which is defined, positive and has continuous first derivative on $[B, \infty)$ for some positive B, and satisfies*

$$\lim_{t\to\infty} tf'(t)/f(t) = \rho, \quad -\infty < \rho < \infty \tag{23}$$

is ρ-varying at infinity.

Proposition 2. *Suppose that f is ρ-varying at infinity and that f' exists and is monotone on $[b, \infty)$. Then (23) holds and for $\rho \neq 0$, the function $(sgn\rho) f'(t)$ is $(\rho - 1)$-varying at infinity.*

Proof. The main ideas for the proof are contained in [4], especially in the proofs of Theorems 2.7.1 (b) and Theorem 1.2.1 (b), but since the proposition does not appear to follow directly from results in [4] we give an indication of its proof here. Suppose that the hypotheses of Proposition 2 hold. We have

$$\frac{f(yt) - f(xt)}{f(t)} = \frac{tf'(t)}{f(t)} \int_x^y \frac{f'(st)}{f'(t)} ds\,.$$

Case (i): f' positive and decreasing or negative and increasing.

Suppose first that $1 < x < y$ and that $t > b$. Then the integrand is ≤ 1 throughout the range of integration, so

$$\frac{f(yt) - f(xt)}{f(t)} \leq \frac{tf'(t)}{f(t)}(y - x) \tag{24}$$

and we get, on taking lim inf as $t \to \infty$,

$$y^\rho - x^\rho \leq \liminf_{t\to\infty} \frac{tf'(t)}{f(t)}(y - x)\,.$$

Letting $y \to x$ and then $x \to 1$, we get

$$\liminf_{t\to\infty} \frac{tf'(t)}{f(t)} \geq \rho\,. \tag{25}$$

On the other hand, with $x < y < 1$ and $t > b$, the integrand is ≥ 1 throughout the range of integration, we have

$$\frac{f(yt) - f(xt)}{f(t)} \geq \frac{tf'(t)}{f(t)}(y - x) \tag{26}$$

and we get, on taking lim sup as $t \to \infty$,

$$y^\rho - x^\rho \geq \limsup_{t\to\infty} \frac{tf'(t)}{f(t)}(y - x).$$

Letting $y \to x$ and then $x \to 1$, we get

$$\limsup_{t\to\infty} \frac{tf'(t)}{f(t)} \leq \rho. \tag{27}$$

The two inequalities (25) and (27) together give (23).

Case (ii): f' positive and increasing or negative and decreasing. The inequalities (24) and (26) for the integral will now be reversed so we get, in this case, for $1 < x < y$,

$$y^\rho - x^\rho \geq \limsup_{t\to\infty} \frac{tf'(t)}{f(t)}(y - x)$$

and for $x < y < 1$

$$y^\rho - x^\rho \leq \liminf_{t\to\infty} \frac{tf'(t)}{f(t)}(y - x).$$

This leads to (23) as before.

Now let $\rho > 0$ so that f' is ultimately positive. Let $\epsilon > 0$ be given. By (23), there is a T such that for $t > T$,

$$\frac{\rho - \epsilon}{t} < \frac{f'}{f} < \frac{\rho + \epsilon}{t}. \tag{28}$$

Thus for $x > 0$ and $t > T/x$, we have

$$\frac{\rho-\epsilon}{tx}f(tx)\frac{t}{\rho+\epsilon}\frac{1}{f(t)} < \frac{f'(tx)}{f'(t)} < \frac{\rho+\epsilon}{tx}f(tx)\frac{t}{\rho-\epsilon}\frac{1}{f(t)}. \tag{29}$$

The right–hand side of (29) approaches $x^{\rho-1}(\rho+\epsilon)/(\rho-\epsilon)$ and the left–hand side approaches $x^{\rho-1}(\rho-\epsilon)/(\rho+\epsilon)$ as $t\to\infty$. But the coefficients $(\rho-\epsilon)/(\rho+\epsilon)$ and $(\rho+\epsilon)/(\rho-\epsilon)$ are arbitrarily close to 1. Thus the middle expression in (29) approaches $x^{\rho-1}$ as $t\to\infty$ and so f' is $(\rho-1)$–varying at infinity. This completes the proof in the case $\rho>0$. The case $\rho<0$ can be dealt with similarly. □

Remark 2. The function $q(t)$ in Theorems 3.1 and 3.2 is slowly varying. This is evident if $q(\infty)>0$. If $q(\infty)=0$, then (18) shows that $\limsup_{t\to\infty} q^{-2}(t)|q'(t)|t<\infty$, which implies that $\lim_{t\to\infty} q^{-1}(t)|q'(t)|t=0$, i.e. q is slowly varying by Proposition 1.

The following Lemma is a restatement, with a slight correction, of a result of [3,(16)].

Lemma 4.2. *Let f and g be n-times differentiable functions on an interval I. Then*

$$[fD_t]^n(g) = f^n g^{(n)} + \sum \phi(n,t) g^{(\beta)} f^{\gamma}$$

where $\phi(n,t)$ is a homogeneous form in $f', f'', \dots, f^{(n-1)}$ whose typical term is

$$const.(f')^{\alpha_1}(f'')^{\alpha_2}\dots(f^{(n-1)})^{\alpha_{n-1}}$$

with

$$1\le\beta,\gamma\le n-1,\ \ \alpha_1+2\alpha_2+\cdots+(n-1)\alpha_{n-1}+\beta=n \tag{30}$$

and

$$0\le\alpha_i\le n-i,\ \ \text{for}\ \ i=1,2,\dots,n-1.$$

Lemma 4.3. *Let $f^{(k)}$ be monotone for $k=1,2,\dots,n,\ \ 0<f(\infty)\le\infty$ and*

$$\lim_{t\to\infty} tf''(t)/f'(t)=\rho<0.$$

If $f(\infty)=\infty$, suppose, in addition, that $f'(t)=O(t^{-1}),\ t\to\infty$. If $f(\infty)<\infty$, suppose, in addition, that $\lim_{t\to\infty} tf'(t)=0$. Then

(i) $\lim_{t\to\infty}\frac{t^{k-1}f^{(k)}(t)}{f'(t)}=\rho(\rho-1)\dots(\rho-k+2)\ne 0,\ \ k=2,\dots,n.$
(ii) $f^{(k)}(t)=O(t^{-k}),\ \ t\to\infty,\ \ k=2,\dots,n,$ *if* $f(\infty)=\infty$ *and* $\lim_{t\to\infty} t^k f^{(k)}(t)=0,\ \ k=2,\dots,n,$ *if* $f(\infty)<\infty$.
(iii) *If the $f^{(k)}$ alternate in sign there exists a number $T_k=T(k)$ such that*

$$sgn(fD_t)^{(k)}[f(t)] = sgnf^{(k)}(t),\ \ t\ge T_k,\ \ k=1,2,\dots,n,$$

$T_k\le T_{k+1}$ *and* $T_k=T_{k+1}$ *only if* $T_k=0,\ k=1,2,\dots,n.$

Proof. In order to prove (i), we can use Propositions 1 and 2, leading to

$$\lim_{t\to\infty}\frac{t^{k-1}f^{(k)}(t)}{f'(t)}=\lim_{t\to\infty}\frac{tf^{(k)}(t)}{f^{(k-1)}(t)}\frac{tf^{(k-1)}(t)}{f^{(k-2)}(t)}\cdots\frac{tf''(t)}{f'(t)}$$

$$=\rho(\rho-1)\dots(\rho-k+2)\neq 0,\quad k=2,\dots,n.$$

In the cases $f(\infty)=\infty$ and $f(\infty)<\infty$, we have

$$\limsup_{t\to\infty} t^k|f^{(k)}(t)|=\lim_{t\to\infty}|\frac{t^{k-1}f^{(k)}(t)}{f'(t)}|\limsup_{t\to\infty}|f'(t)|t<\infty$$

and

$$\lim_{t\to\infty} t^k f^{(k)}(t)=\lim_{t\to\infty}\frac{t^{k-1}f^{(k)}(t)}{f'(t)}\lim_{t\to\infty} tf'(t)=0$$

respectively, so we have (ii). For (iii), we use Lemma 4.2 to get

$$\operatorname{sgn}(fD_t)^k(f)=\operatorname{sgn}[f^kf^{(k)}+\sum\phi(k,t)f^{(\beta)}f^{\gamma}]$$

and (iii) will follow from Lemma 4.1 once we show

$$\limsup_{t\to\infty}|\frac{\phi(k,t)f^{(\beta)}}{f^{(k)}}|\lim_{t\to\infty}[f(t)]^{\gamma-k}=0. \tag{31}$$

1. Let $f(\infty)=\infty$. Then $\lim_{t\to\infty}[f(t)]^{\gamma-k}=0$ because $\gamma-k<0$. We prove that $\limsup_{t\to\infty}|\phi(k,t)f^{(\beta)}/f^{(k)}|<\infty$. A typical term in the limes superior is a constant multiple of

$$\limsup_{t\to\infty}[f']^{\alpha_1}[f'']^{\alpha_2}\dots[f^{(k-1)}]^{\alpha_{k-1}}f^{(\beta)}[f^{(k)}]^{-1}. \tag{32}$$

In view of (i),(ii) and (30) we get

$$\lim_{t\to\infty}|\frac{f'}{t^{k-1}f^{(k)}}|\limsup_{t\to\infty} t^{k-1}[f']^{\alpha_1-1}[f'']^{\alpha_2}\dots[f^{(k-1)}]^{\alpha_{k-1}}f^{(\beta)}<\infty$$

in case $\alpha_1\geq 1$; in case $\alpha_1=0$ there exists $\alpha_s\geq 1, s\in\{2,\dots,n-1\}$ such that (32) approaches

$$\lim_{t\to\infty}[\frac{f^{(s)}t^{s-1}}{f'}]^{\alpha_s}\lim_{t\to\infty}|\frac{f'}{t^{k-1}f^{(k)}}|\limsup_{t\to\infty} t^{k-1-\alpha_s(s-1)}[f']^{\alpha_s-1}[f'']^{\alpha_2}\dots$$
$$\dots[f^{(s-1)}]^{\alpha_{s-1}}[f^{(s+1)}]^{\alpha_{s+1}}\dots[f^{(k-1)}]^{\alpha_{k-1}}f^{(\beta)}<\infty$$

because $\alpha_s-1+2\alpha_2+\dots+s\alpha_s+\dots+\beta=k-1-\alpha_s(s-1)$ and (ii) is used.
2. Let $0<f(\infty)<\infty$. In a similar way we get $\lim_{t\to\infty}|\phi(k,t)f^{(\beta)}/f^{(k)}|=0$ and (31) holds.

The conclusion (iii) now follows from Lemma 4.1.

It remains to prove $T_k \leq T_{k+1}$ and $T_k = T_{k+1}$ only if $T_k = 0,\ k = 1, 2, \ldots$. First, using the fact that if $f \in M_n(0, \infty)$ and if f is not constant then $(-1)^k f^{(k)}(t) > 0,\ k = 0, \ldots, n-1,\ t \in (0, \infty)$ (see [13,Lemma 0.3]), we have $f^{(k)} \neq 0$, for $t \in (0, \infty),\ k = 2, \ldots, n-1$.

Let $T_k = \min\{T | \operatorname{sgn}(fD_t)^k(f(t)) = \operatorname{sgn} f^{(k)}(t), \text{for } t > T\},\ k = 1, \ldots, n$. Suppose, contrary to what we wish to show, that there exists a $k \in \{1, \ldots, n-1\}$ such that $T_{k+1} < T_k$ or $T_{k+1} = T_k > 0$. If $k \in \{1, 2. \ldots, n-2\}$, suppose without loss of generality, that $f^{(k+1)}(t) < 0$ and $f^{(k)}(t) > 0$ for $t \in (0, \infty),\ k = 1, 2, \ldots, n-2$. Putting $F_k = (fD_t)^k(f),\ k = 1, \ldots, n-1$, we have $F_{k+1} = fD_t(F_k)$. It follows that $D_t F_k < 0$, for $t > T_{k+1}$, $F_k > 0$, for $t > T_k$. Thus, because of the continuity of F and the definition of T_k, we get $F_k(T_k) = 0$ and $F_k(t)$ is decreasing for $t > T_k \geq T_{k+1}$, i.e. $F_k(t) < 0$ for $t > T_k$, which is a contradiction.

Similarly, if $k = n-1$ then $F_{n-1}(T_{n-1}) = 0,\ F_{n-1}(t) < 0$ for $t > T_{n-1}$, $D_t F_{n-1}(t) \geq 0$ for $t > T_n \geq T_{n-1}$, which is a contradiction. □

Lemma 4.4. *Let the hypotheses of Theorem 3.1 hold. Then*

(i) $\lim_{t\to\infty} \frac{t[q^{-1}(t)]''}{[q^{-1}(t)]'} = \rho < 0$;

(ii) *$sgn[q^{-1}(t)]^{(k)} = -sgn q^{(k)}(t)$, for sufficiently large t.*

Proof. Because of Remark 2, we have

$$\lim_{t\to\infty}[tq'(t)/q(t] = 0, \quad \lim_{t\to\infty}[tq''(t)/q'(t)] = \rho, \quad -\infty < \rho < 0.$$

By a routine computation,

$$\lim_{t\to\infty} \frac{t[q^{-1}(t)]''}{[q^{-1}(t)]'} = -2 \lim_{t\to\infty} \frac{tq'(t)}{q(t)} + \lim_{t\to\infty} \frac{tq''(t)}{q'(t)} = \rho < 0,$$

so we get (i). To prove (ii), we show by mathematical induction that

$$(q^{-1})^{(k)} = -q^{-2}q^{(k)} + \sum_{\gamma=3}^{k+1} \phi(\gamma, k, t) q^{-\gamma}$$

where $\phi(\gamma, k, t)$ is a homogeneous form in $q', \ldots, q^{(k-1)}$ whose typical term is

$$\text{const.}(q')^{\alpha_1}(q'')^{\alpha_2} \ldots (q^{(k-1)})^{\alpha_{k-1}},$$

where $0 \leq \alpha_i \leq k,\ i = 1, \ldots, k-1,\ \sum_{i=1}^{k-1} i\alpha_i = k$, and $\sum_{i=1}^{k-1} \alpha_i = \gamma - 1$.

If we prove

$$L = \limsup_{t\to\infty} |\frac{\sum \phi(\gamma, k, t) q^{-\gamma}}{-q^{-2}q^{(k)}}| < 1,$$

then the conclusion (ii) will follow from Lemma 4.1. We shall use Lemma 4.3 (i),(ii) and the fact that $\lim_{t\to\infty}[tq'(t)/q(t)] = 0$. A typical term of L is a constant multiple of

$$q^{-\gamma+2}(q')^{\alpha_1}(q'')^{\alpha_2} \ldots (q^{(k-1)})^{\alpha_{k-1}}[q^{(k)}]^{-1}.$$

Now by the similar decomposition as in the proof of Lemma 4.3 (iii) we get that this is asymptotic to a constant multiple of $[tq'(t)/q(t)]^{\gamma-2}$ which approaches 0 as $t \to \infty$. The conclusion (ii) now follows from Lemma 4.1. □

Lemma 4.5. [10,pp. 1241-1242] *Let $g'(t)$ be $(n-1)$-times monotonic on I and let f be n-times monotonic on $g(I)$. Then $f[g(t)]$ is n-times monotonic on I. Let $g'(t)$ be $(n-1)$-times monotonic on $I$and let $D_x f(x)$ be n-times monotonic on $g(I)$. Then $D_t f[g(t)]$ is n-times monotonic on I.*

Remark 3. Lemma 4.5 shows that $(q^{-1})' \in M_n(b,\infty)$ implies that $(q^{-1/2})' \in M_n(b,\infty)$.

Lemma 4.6. [14,Lemma 1] *Let f be n-times monotonic on (b,∞). Then $\lim_{t\to\infty} t^k f^{(k)}(t) = 0$ for $k = 1,\dots,n-1$.*

5. Proofs of the theorems.

Proof of Theorem 2.1. The hypothesis (12) may be expressed as

$$(-1)^n \left[\frac{1}{q(t)} D_t\right]^n \left(D_t \frac{1}{q^2(t)}\right) \geq 0, \quad n = 0,1,\dots,N.$$

Thus it follows from a result of P. Hartman [6,Theorem 22.1_n] that under the present hypotheses, the equation (1) possesses a pair of solutions $y_1(t)$, $y_2(t)$ such that if $p(t) = y_1^2(t) + y_2^2(t)$, then $p'(t)$ is an N-times monotonic function of θ for $0 < t < \infty$. Moreover, $p(t)$ is determined up to a multiplicative constant. But $\theta'(t)$ is an N-times monotonic function of t. Thus by Lemma 4.5, $p'(t)$ is an N-times monotonic function of t, i.e. (13) holds.

If $N \geq 2$, we see from the Appell equation [1]

$$p''' + 4qp' + 2q'p = 0$$

that

$$(p^2 q)' = -pp'''/2 \leq 0, \tag{33}$$

so $p^2 q$ is nonincreasing. That its limit is positive is most easily seen from the Mammana identity [2]

$$p^2 q = (p')^2/4 - pp''/2 + w^2 \tag{34}$$

since p, p', p'' all $\to 0$ as $t \to \infty$ (Here w is the constant Wronskian of y_1 and y_2.)□

Proof of Theorem 2.2. If we write $r(t) = [p(t)]^{1/2}$, we have from (34) that

$$q(t) = -\frac{r''}{r} + \frac{w^2}{r^4}$$

See, for example, [2,p. 32]. But $-r''$ is obviously N-times monotonic, and from Lemma 4.5, so are $1/r$ and w^2/r^4. Hence, $q(t)$ is N-times monotonic. □

Proof of Theorem 3.1. According to Lemma 4.4 (ii), we have

$$\operatorname{sgn}(q^{-1})^{(k)} = -\operatorname{sgn} q^{(k)} = (-1)^{k+1}$$

for sufficiently large t. By Lemma 4.6, we get $\lim_{t\to\infty} tq'(t) = 0$. If $q(\infty) > 0$, then $\lim_{t\to\infty} t(q^{-1})' = -\lim_{t\to\infty} tq^{-2}q' = 0$, so $(q^{-1})' = O(t^{-1})$. This condition is part of our hypotheses in the case $q(\infty) = 0$. Thus, all the hypotheses of Lemma 4.3 (with f replaced by $1/q$) are satisfied and we have

$$\begin{aligned}\operatorname{sgn} D_\theta^{n+1}[q(t)]^{-1} &= \operatorname{sgn}(q^{-1}D_t)^{n+1}[q(t)]^{-1}\\ &= \operatorname{sgn}(q^{-1})^{(n+1)} = (-1)^n, \quad n = 0, 1, \ldots, N+1\end{aligned}$$

for $t \geq \mu_n$, for $t \geq \mu_n$ where $\mu_n \leq \mu_{n+1}$ and $\mu_n = \mu_{n+1}$ only if $\mu_n = 0$, $n = 0, \ldots, N+1$. Now, applying a modified form of Theorem 2.1, with hypotheses on (μ_{n+1}, ∞) rather than $(0, \infty)$, we get (19) and the last assertion of the Theorem.□

Proof of Theorem 3.2. According to [11,Theorem 2.1], we have

$$\operatorname{sgn}[\Delta^{n+1} t_k] = \operatorname{sgn}(pD_t)^n(p),$$

where $p(t)$ is given in Theorem 3.1. We shall show that the function $p(t)$ satisfies the hypotheses imposed on f in Lemma 4.3, in particular that

(a) $\limsup_{t\to\infty} tp'(t) < \infty$, if $p(\infty) = \infty$, and $\lim_{t\to\infty} tp'(t) = 0$, if $p(\infty) < \infty$,

(b) $\lim_{t\to\infty}[tp''(t)/p'(t)] = \rho < 0$. The condition $0 < p(\infty) \leq \infty$ is an automatic consequence of $0 \leq q(\infty) < \infty$, since $p^2 q$ approaches a finite positive constant as t approaches infinity. Then we will get $\operatorname{sgn}(pD_t)^n(p) = \operatorname{sgn} p^{(n)} = (-1)^{n+1}$ on $\mu_n < t < \infty$ and $\operatorname{sgn}[\Delta^{n+1} t_k] = (-1)^{n+1}$, $k = l_n, l_n + 1, \ldots$, which is what we have to prove. Moreover, $l_n = l(n)$ denotes the smallest integer for which the zero t_{l_n} is $\geq \mu_n$. Since the sequence $\{\mu_n\}$ is nondecreasing and $\mu_n = \mu_{n+1}$ only if $\mu_n = 0$, the same property must hold for $\{l_n\}$.

We have

$$p(t) = [q(t)]^{-1/2}[c + w(t)], \quad w(t) \to 0, \quad c = \text{const.} > 0 \tag{35}$$

Thus $p^2 q = (c+w)^2$ implies $(p^2 q)' = 2(c+w)w'$. On the other hand, $(p^2 q)' = -pp'''/2 \leq 0$ (see (33)) and from this

$$w' = -\frac{1}{4}\frac{pp'''}{c+w} = -\frac{1}{4}q^{-1/2}p'''. \tag{36}$$

From (35), (36), we get

$$p' = -\frac{1}{2}q^{-3/2}q'(c+w) - \frac{1}{4}q^{-1}p''' \tag{37}$$

or

$$p' = -\frac{1}{2}q^{-3/2}q'(c+w)[1 + \frac{1}{2}\frac{q^{1/2}p'''}{q'(c+w)}]. \tag{38}$$

If $p(\infty) < \infty$, and hence $q(\infty) > 0$, then $\lim_{t\to\infty} tp'(t) = 0$, from (37), because by Lemma 4.6, applied to q, p respectively, we have $\lim_{t\to\infty} tq'(t) = 0$, $\lim_{t\to\infty} t^2 p'''(t) = 0$. Let $p(\infty) = \infty$, and hence $q(\infty) = 0$. We then have

$$\lim_{t\to\infty} \frac{q^{1/2}p'''}{(c+w)q'} = \lim_{t\to\infty} q^{1/2} \lim_{t\to\infty} t^2 p''' \frac{1}{c \lim_{t\to\infty} t^2 q'} = 0. \tag{39}$$

(The first two limits are zero and the one in the denominator is non-zero, by (20).) Thus the quantity in square brackets in (38) approaches 0 as t approaches infinity. Since $(q^{-1})' = O(t^{-1})$, i.e. $\limsup_{t\to\infty} tq^{-2}|q'| < \infty$, we see from (38) that $\limsup_{t\to\infty} tp'(t) < \infty$.

To prove (b), we use (37) which we can write (since (39) holds both for $q(\infty) > 0$ and $q(\infty) = 0$),

$$p' = -\frac{1}{2}(c+w)q^{-3/2}q'[1 + O(1)].$$

Differentiating (37), we get, using (36),

$$p'' = \frac{3}{4}q^{-5/2}q'^2(c+w) - \frac{1}{2}q^{-3/2}q''(c+w) + \frac{3}{8}q^{-2}q'p''' - \frac{1}{4}q^{-1}p^{(4)}.$$

Thus $\lim_{t\to\infty} tp''(t)/p'(t)$ is a sum of four terms. The first of these is

$$-\frac{3}{2}\lim_{t\to\infty} \frac{tq'(t)}{q} = 0,$$

using Remark 2. The second term is

$$\lim_{t\to\infty} \frac{tq''(t)}{q'(t)} = \rho,$$

by hypothesis (17). The third term is

$$-\frac{3}{4}\lim_{t\to\infty} \frac{t^2 p'''(t)}{tq^{1/2}(t)(c+w)} = 0\,.$$

This is obvious if $q(\infty) > 0$ and if $q(\infty) = 0$, then

$$\lim_{t\to\infty} tq^{1/2}(t) = \lim_{t\to\infty} [tq(t)]^{1/2} t^{1/2} = \infty\,,$$

since from (20) and the fact that $\lim_{t\to\infty} tq'(t)/q(t) = \lim_{t\to\infty} [t^2 q'(t)]/[tq(t)] = 0$, it follows that $\lim_{t\to\infty} tq(t) > 0$. The fourth and last term is

$$\frac{1}{2}\lim_{t\to\infty} \frac{tp^{(4)}(t)q^{1/2}(t)}{q'(t)(c+w)[1+O(1)]} = \frac{1}{2}\lim_{t\to\infty} q^{1/2}(t) \lim_{t\to\infty} \frac{t^3 p^{(4)}(t)}{t^2 q'(t)(c+w)} = 0\,.$$

Putting these four terms together, we have

$$\lim_{t\to\infty} tp''(t)/p'(t) = \rho, \quad \rho < 0,$$

and the proof is complete. □

6. Applications

Theorem 2.1 may be applied to the generalized Airy equation

$$y'' + t^{\alpha} y = 0 \tag{40}$$

in case $-1/2 \le \alpha \le 0$. This shows in the usual notation for Bessel functions that for $1/2 \le \nu \le 2/3$, the first derivative of the function

$$p(t) = t[J_\nu^2(2\nu t^{1/(2\nu)}) + Y_\nu^2(2\nu t^{1/(2\nu)})] \tag{41}$$

is completely monotonic on $(0, \infty)$. We may contrast this with the facts that:

(i) for $1/3 \le \nu < 1/2$ the function (41) is completely monotonic on $(0, \infty)$ [11, Theorem 5.1];
(ii) for $|\nu| \ge 1/2$ the function $t[J_\nu^2(t) + Y_\nu^2(t)]$ is completely monotonic on $(0, \infty)$.

We note that Theorem 3.1 is not applicable to (40). However (35) and Remark 3 suggest the following

Conjecture:. *(7) holds for (40) in the case $-2 < \alpha < 0$ at least for sufficiently large t.*

In the case $q(\infty) = 0$, the results of Theorems 3.1 and 3.2 are applicable to the equation

$$y'' + (\log t)^{-1} y = 0$$

and more generally, to the equation

$$y'' + [l_k(t)]^{-\rho} y = 0$$

on a suitable t-interval, where $0 < \rho \le 1$ and $l_k(t)$ is the iterated logarithm, i.e. $l_0(t) = t, \quad l_k(t) = \log(l_{k-1}(t), \quad k = 1, 2, \ldots$. (On the contrary, a function not satisfying ((18)) is $q(t) = t^{\alpha}, \alpha < 0$). In the case $q(\infty) > 0$, Theorem 3.1 is applicable to the equations

$$y'' + [1 + t^{-\gamma}] y = 0, \gamma > 0\,, \tag{42}$$

see also [3,Corollary 1]

$$y'' + [1 + (\log t)^{-1}] y = 0\,, \tag{43}$$

and

$$y'' + [1 + (l_k(t))^{-\rho}] y = 0\,. \tag{44}$$

The conclusion of Theorem 3.2 is applicable to the equations (43) and (44). For equation (42), the corresponding conclusion was proved in [3, Corollary 1].

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