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SPECIAL TANGENT VALUED FORMS AND THE FRÖLICHER–NIJENHUIS BRACKET

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Abstract. We define the tangent valued $\mathcal{C}$-forms for a large class of differential geometric categories. We deduce that the Frölicher-Nijenhuis bracket of two tangent valued $\mathcal{C}$-forms is a $\mathcal{C}$-form as well. Then we discuss several concrete cases and we outline the relations to the theory of special connections.

It has been clarified recently, see e.g. [4], [9], [10], that the Frölicher-Nijenhuis bracket is an important tool for the theory of general connections on arbitrary fibred manifolds, as well as for some other problems in differential geometry. However, it seems that only two kinds of special tangent valued forms were studied in detail up to now, namely the projectable forms on an arbitrary fibred manifold, [10], and the right-invariant forms on a principal fibre bundle, [1], [2]. In the present paper we develop a systematic approach to tangent valued k-forms corresponding to a category $\mathcal{C}$ over manifolds satisfying a simple additional condition and we discuss their relations to the theory of special connections. Sections 1 and 2 are devoted to the foundations of such a theory. Next we determine all vector bundle k-forms and all affine bundle k-forms. In Section 4 we treat one of the simple algebraic models for higher order differential geometry, the category $2\mathcal{GLB}$ of 2-graded linear bundles, [7], [8]. The complete description of all $2\mathcal{GLB}$-forms in Proposition 7, which represents a natural modification of the vector bundle case, suggests that the theory of $\mathcal{C}$-forms for several categories of structured bundles, [3], can be rich. On the other hand, in the last section we deduce that the categories of symplectomorphisms and volume-preserving diffeomorphisms admit only trivial tangent valued forms.

All manifolds and maps are assumed to be infinitely differentiable.

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1. Categories over manifolds and related vector fields. Let $\mathcal{M}f$ denote the category of all manifolds and all smooth maps. A *category over manifolds* is a category $\mathcal{C}$ endowed with a faithful functor $m : \mathcal{C} \to \mathcal{M}f$. Hence the $\mathcal{C}$-morphisms between two $\mathcal{C}$-objects A and B are identified with some smooth maps between the undelying manifolds mA and mB.

Roughly speaking, a $\mathcal{C}$-field on a $\mathcal{C}$-object A is a vector field $X : mA \to T(mA)$ on the underlying manifold such that all transformations forming the flow of X belong to $\mathcal{C}$. However, since the flow is formed by local diffeomorphisms in general, we must be somewhat more careful in the definition.

An open subobject B of a $\mathcal{C}$-object A is a $\mathcal{C}$-object over an open subset $mB \subset mA$ such that the inclusion $i_{mB} : mB \hookrightarrow mA$ is a $\mathcal{C}$-morphism and the following property holds: if for a smooth map $f : mC \to mB$ the composition $i_{mB} \circ f : mC \to mA$ is a $\mathcal{C}$-morphism $C \to A$, then f is a $\mathcal{C}$-morphism $C \to B$. By a locally defined $\mathcal{C}$-morphism of A_1 into A_2 we mean a smooth map $f : U_1 \to U_2$ between open subsets $U_1 \subset mA_1$ and $U_2 \subset mA_2$ with the property that there exist open subobjects B_1 of A_1 and B_2 of A_2, $U_1 \subset mB_1 \subset mA_1$, $U_2 \subset mB_2 \subset mA_2$, and a $\mathcal{C}$-morphism $g : B_1 \to B_2$ such that f is the restriction of g to U_1, U_2.

Definition 1. A vector field $X : mA \to T(mA)$ on a $\mathcal{C}$-object A is called a $\mathcal{C}$-field, if its flow is formed by locally defined $\mathcal{C}$-morphisms of A.

To prove that the $\mathcal{C}$-fields on a $\mathcal{C}$-object A form a subalgebra of the Lie algebra of all vector fields on mA, we need an additional assumption on the category $\mathcal{C}$. (But this property holds for all classical categories in differential geometry.)

Definition 2. A category $\mathcal{C}$ over manifolds is called infinitesimally closed, if every vector field tangent to a local one-parameter family of locally defined $\mathcal{C}$-isomorphisms is a $\mathcal{C}$-field.

Proposition 1. *Let $\mathcal{C}$ be an infinitesimally closed category. If X and Y are two $\mathcal{C}$-fields on a $\mathcal{C}$-object A, then kX for all $k \in \mathbb{R}$, $X + Y$ and $[X, Y]$ are $\mathcal{C}$-fields as well.*

Proof. It is well known that kX is constructed by reparametrizing the flow of X, $X+Y$ by composing the flows of X and Y and $[X,Y]$ by constructing the commutator of the flows of X and Y. Hence Proposition 1 follows from the assumption that $\mathcal{C}$ is infinitesimally closed.

2. Tangent valued $\mathcal{C}$-forms and the Frölicher-Nijenhuis bracket. We recall that a tangent valued k-form on a manifold M is a linear morphism $\omega : \Lambda^k TM \to TM$. For $k = 0$ this means a vector field on M.

Definition 3. Let $\mathcal{C}$ be an infinitesimally closed category. A tangent valued k-form $\omega : \Lambda^k T(mA) \to T(mA)$ on a $\mathcal{C}$-object A is called a $\mathcal{C}$-form, if $\omega(X_1, \dots, X_k)$ is a $\mathcal{C}$-field for every $\mathcal{C}$-fields $X_1, \dots, X_k$.

For $k = 0$, a tangent valued $\mathcal{C}$-form is a $\mathcal{C}$-field.

Frölicher and Nijenhuis defined the bracket $[\omega, \varphi]$ of a tangent valued k-form ω and of a tangent valued l-form φ, which is a tangent valued $(k+l)$-form. Their approach was based on the theory of graded derivations in the exterior algebra of M.

In this setting it is not so easy to show that the Frölicher-Nijenhuis bracket of two $\mathcal{C}$-forms is a $\mathcal{C}$-form as well. However, M. Modugno, [10], and, independently, P.W. Michor, [9], deduced the following expression for the Frölicher-Nijenhuis bracket in terms of the bracket of vector field

$$
\begin{aligned}
k!l![\omega,\varphi](X_1,\dots,X_{k+l}) &= \sum_\sigma \operatorname{sgn}\sigma[\omega(X_{\sigma 1},\dots,X_{\sigma k}),\varphi(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})]\\
&-l\sum_\sigma \operatorname{sgn}\sigma\varphi([\omega(X_{\sigma 1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots,X_{\sigma(k+l)})\\
&+(-1)^{kl}k\sum_\sigma \operatorname{sgn}\sigma\omega([\varphi(X_{\sigma 1},\dots,X_{\sigma l}),X_{\sigma(l+1)}],X_{\sigma(l+2)},\dots,X_{\sigma(k+l)})\\
&+(-1)^{k-1}\frac{kl}{2}\sum_\sigma \operatorname{sgn}\sigma\varphi(\omega([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+l)})\\
&+(-1)^{(k-1)l}\frac{kl}{2}\sum_\sigma \operatorname{sgn}\sigma\omega(\varphi([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots,X_{\sigma(l+1)}),X_{\sigma(l+2)},\dots,X_{\sigma(k+l)}),
\end{aligned}
$$

with summation with respect to all permutations σ of $k+l$ letters. Then Proposition 1 implies directly

Proposition 2. *The Frölicher-Nijenhuis bracket of two tangent valued $\mathcal{C}$-forms is a $\mathcal{C}$-form as well.*

Example 1. In the case of the category $\mathcal{FM}$ of all fibred manifolds, one sees directly that the vector fields whose flows are formed by local $\mathcal{FM}$-morphisms are just the projectable fields. (We recall that a vector field $X: Y \to TY$ on a fibred manifold $p: Y \to M$ is said to be projectable, if there exists a vector field $X_0: M \to TM$ such that $Tp \circ X = X_0 \circ p$.) Obviously, $\mathcal{FM}$ is infinitesimally closed. Let x^i be some local coordinates on M, y^p some fibre coordinates on Y and $z^a = (x^i, y^p)$. Consider a k-form $A: \Lambda^k TY \to TY$ with coordinate expression

$$
a^i_{a_1\dots a_k} dz^{a_1}\wedge\dots\wedge dz^{a_k}\otimes\frac{\partial}{\partial x^i} + a^p_{a_1\dots a_k} dz^{a_1}\wedge\dots\wedge dz^{a_k}\otimes\frac{\partial}{\partial y^p}.
$$

Taking into account the vector fields of the form $b^i\frac{\partial}{\partial x^i} + b^p_q y^q\frac{\partial}{\partial y^p}$ with constant $b's$, we find that $A(X_1,\dots,X_k)$ is a projectable vector field iff $a^i_{j_1\dots j_k} = a^i_{j_1\dots j_k}(x)$ are functions of x only and all other $a^i_{a_1\dots a_k}$ are zero. On the other hand, A is called projectable, if there exists a k-form $A_0: \Lambda^k TM \to TM$ such that $A_0 \circ p = \Lambda^k Tp \circ A$. Hence we have proved that the tangent valued $\mathcal{FM}$-forms coincide with the projectable tangent valued forms. Such forms were studied by Modugno in [10].

Example 2. Fix a Lie group G and consider the category $\mathcal{PB}(G)$ of principal G-bundles and their morphisms. Hence the local $\mathcal{PB}(G)$-morphisms are the local $\mathcal{FM}$-morphisms commuting with the right translations R_g. It is well known that two vector fields are f-related with respect to a smooth map f iff their flows are f-related. Hence the $\mathcal{PB}(G)$-fields on a principal fibre bundle are the TR_g-related ones for all $g \in G$, i.e. the classical right-invariant vector fields on P. This implies directly that $\mathcal{PB}(G)$ is infinitesimally closed and the tangent valued $\mathcal{PB}(G)$-forms coincide with the right-invariant tangent valued forms studied by the first author and D. Canarutto, [1], [2].

3. Vector and affine bundles. If $p : E \to M$ is a vector bundle, then $Tp : TE \to TM$ is also a vector bundle. Every $\mathcal{VB}$-field $X : E \to TE$ is projectable over a vector field $X_0 : M \to TM$. One sees directly that if X is tangent to a local one-parameter family of local $\mathcal{VB}$-morphisms, then X is a vector bundle morphism $E \to TE$ over $X_0 : M \to TM$. We present a complete proof of the fact that every such a field is a $\mathcal{VB}$-field (we shall modify it in the next section to a more complicated situation). Given a vector field X of the form

$$X^i(x)\frac{\partial}{\partial x^i} + X^p_q(x)y^q\frac{\partial}{\partial y^p}\,.$$

Its flow $\varphi^i(x,t)$, $\varphi^p(x,y,t)$ is determined by the differential equations

$$\frac{dx^i}{dt} = X^i(x), \quad \frac{dy^p}{dt} = X^p_q(x)y^q\,.$$

Write $\Phi^p(x,y,k,t) = \varphi^p(x,ky,t) - k\varphi^p(x,y,t)$. We have $\Phi^p(x,y,k,0) = 0$ by definition and

$$\frac{\partial\Phi^p}{\partial t} = X^p_q(\varphi^i(x,t))\Phi^q(x,y,k,t)\,.$$

Hence Φ^p satisfy a system of linear differential equations with zero initial condition, so that $\Phi^p = 0$. This means

$$\varphi^p(x,ky,t) = k\varphi^p(x,y,t)\,.$$

By the homogeneous function theorem, [6], φ^p is linear in y.

Let us start with the description of $\mathcal{VB}$-one-forms. Since the procedure from Example 1 holds even in the $\mathcal{VB}$-case, every $\mathcal{VB}$-one-form $A : TE \to TE$ is projectable, i.e. of the form

$$a^i_j(x)\,dx^j \otimes \frac{\partial}{\partial x^i} + (a^p_i(x,y)\,dx^i + a^p_q(x,y)\,dy^q) \otimes \frac{\partial}{\partial y^p}\,.$$

We require that $A(X)$ is a $\mathcal{VB}$-field for every $\mathcal{VB}$-field X. Take first $X^i = b^i =$const, $X^p_q = 0$. This yields $a^p_i = a^p_{iq}(x)y^q$. Next consider $X^i = 0$, $X^p_q = b^p_q$ =const. Since $a^p_q(x,y)b^q_r y^r$ must be linear in y, it is multiplied by k when replacing y by ky, $k \in \mathbb{R}$, i.e.

$$a^p_q(x,ky)b^q_r y^r = a^p_q(x,y)b^q_r y^r\,, \quad k \neq 0\,.$$

Letting $k \to 0$, we obtain $a^p_q(x,0)b^q_r y^r$ on the left-hand side, while the right hand side remains unchanged. Since b^q_r are arbitrary quantities, this implies $a^p_q(x,y) =$

$a^p_q(x)$. In other words, the $\mathcal{VB}$-one-forms $A : TE \to TE$ are those projectable forms which are linear morphisms of $TE \to TM$ into $TE \to TM$ over the base map $A_0 : TM \to TM$.

Consider now an arbitrary $\mathcal{VB}$-k-form $A : \Lambda^k TE \to TE$ over $A_0 : \Lambda^k TM \to TM$, which is of the form

$$
\begin{aligned}
&A_0 + (a^p_{i_1 \dots i_k}(x,y)\, dx^{i_1} \wedge \dots \wedge dx^{i_k} + a_{i_1 \dots i_{k-1} q}(x,y)\, dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} \wedge dy^q + \\
&\qquad \dots + a^p_{q_1 \dots q_k}(x,y)\, dy^{q_1} \wedge \dots \wedge dy^{q_k}) \otimes \frac{\partial}{\partial y^p}\,. \qquad (1)
\end{aligned}
$$

Since $A(X_1, \dots, X_k)$ must be a $\mathcal{VB}$-field for every $\mathcal{VB}$-fields $X_1, \dots, X_k$, we obtain first

$$
a^p_{i_1 \dots i_k} = a^p_{i_1 \dots i_k q}(x) y^q \qquad (2)
$$

and then, by the same change $y \to ky$ as above,

$$
a^p_{i_1 \dots i_{k-1} q} = a^p_{i_1 \dots i_{k-1} q}(x),\, a^p_{i_1 \dots i_{k-2} q_1 q_2} = 0, \dots, a^p_{q_1 \dots q_k} = 0\,. \qquad (3)
$$

We are going to interpret (1) - (3) geometrically. Taking into account the inclusion $i : \Lambda^k TE \to \otimes^k TE$, consider the map $\mathrm{id}_{TE} \otimes \bigotimes^{k-1} Tp : \bigotimes^k TE \to TE \otimes \bigotimes^{k-1} TM$. Since $Tp : TE \to TM$ is a vector bundle, $Tp \otimes \bigotimes^{k-1} \mathrm{id}_{TM} : TE \otimes \bigotimes^{k-1} TM \to TM \otimes \bigotimes^{k-1} TM$ is a vector bundle as well. Define

$$
L_k E = (\mathrm{id}_{TE} \otimes \bigotimes^{k-1} Tp)(\Lambda^k TE)
$$

which is a vector subbundle of $TE \otimes \bigotimes^{k-1} TM$ over $\Lambda^k TM$. Then (1) - (3) is equivalent to the following assertion.

Proposition 3. *A k-form $A : \Lambda^k TE \to TE$ is a $\mathcal{VB}$-form iff all the following three conditions hold*

(i) A is projectable into $A_0 : \Lambda^k TM \to TM$,
(ii) A factorizes through $L_k E$ into $A_L : L_k E \to TE$,
(iii) A_L is a linear morphism $L_k E \to TE$ over A_0.

Furthermore, consider an affine bundle $p : Y \to M$. Then $Tp : TY \to TM$ is also an affine bundle, [3]. First we deduce that a vector field $X : Y \to TY$ is an $\mathcal{AB}$-field iff it is an affine bundle morphism $Y \to TY$ over $X_0 : M \to TM$. The coordinate expression of such a field in affine fibre coordinates is

$$
X^i(x) \frac{\partial}{\partial x^i} + (X^p_q(x) y^q + X^p(x)) \frac{\partial}{\partial y^p}\,. \qquad (4)
$$

On one hand, it is clear that a vector field tangent to a local one-parameter family of local affine morphisms is of the form (4). On the other hand, the flow of (4) is given by

$$
\frac{dx^i}{dt} = X^i(x), \qquad \frac{dy^p}{dt} = X^p_q(x) y^q + X^p(x)\,. \qquad (5)
$$

Consider first the equation $\dfrac{dy^p}{dt} = X^p(x)$, which yields $y^p = \varphi^p(x,t)$. Then the "rest" of (5) corresponds to a $\mathcal{VB}$-field and we can use our previous result.

Let $A : \Lambda^k TY \to TY$ be an $\mathcal{AB}$-k-form. Since the procedure from Example 1 works, A is projectable over $A_0 : \Lambda^k TM \to TM$. For constant $X^i = b^i$, $X^p = b^p$, $X^p_q = 0$ we find that all $a^p_{a_1 \dots a_k}$ in the coordinate expression of A are affine functions. Using constant $X^i = b^i$, $X^p_q = b^p_q$, $X^p = 0$, we then obtain

$$\begin{aligned} a^p_{i_1 \dots i_{k-1} q} &= a^p_{i_1 \dots i_{k-1} q}(x) \\ a^p_{i_1 \dots i_{k-2} q_1 q_2} &= 0, \dots, a^p_{q_1 \dots q_k} = 0 \end{aligned} \tag{6}$$

Hence (6) and the previous relation

$$a^p_{i_1 \dots i_k} = a^p_{i_1 \dots i_k q}(x) y^q + \tilde{a}^p_{i_1 \dots i_k}(x) \tag{7}$$

characterize the $\mathcal{AB}$-forms.

The geometric interpretation of (6) and (7) is quite similar to the $\mathcal{VB}$-case. Since $Tp : TY \to TM$ is an affine bundle, $Tp \otimes \bigotimes^{k-1} \mathrm{id}_{TM} : TY \otimes \bigotimes^{k-1} TM \to TM \otimes \bigotimes^{k-1} TM$ is an affine bundle as well. If we define $L_k Y = (\mathrm{id}_{TY} \otimes \bigotimes^{k-1} Tp)(\Lambda^k TY)$, this is an affine subbundle of $TY \otimes \bigotimes^{k-1} TM$ over $\Lambda^k TM$. Then (6) and (7) is equivalent to the following assertion.

Proposition 4. *A k-form $A : \Lambda^k TY \to TY$ is an $\mathcal{AB}$-form iff all the following three conditions hold*

(i) A is projectable into $A_0 : \Lambda^k TM \to TM$,

(ii) A factorizes through $L_k Y$ into $A_L : L_k Y \to TY$,

(iii) A_L is a affine morphism $L_k Y \to TY$ over A_0.

4. Algebraic models for higher order differential geometry. The vector and affine bundles are the basic algebraic models for the first order differential geometry. The algebraic models for the higher order geometry have more complicated character, see e.g. [7], [8]. In this section we discuss the category $2\mathcal{GLB}$ of 2-graded linear bundles, [7]. The simplest example of a 2-graded linear bundle is the space $T^2_1 M = J^2_0(\mathbb{R}, M)$ of all second order one-dimensional velocities on a manifold M in the sense of Ehresmann, which is called the second order tangent bundle of M in higher order mechanics. In [7] it is proved that T^2_1 is a functor with values in the category $2\mathcal{GLB}$. (We remark that the 2-graded linear maps are equivalent to the morphisms of linear 2-towers, [7]. The latter concept is more geometrical, but the former one seems to be more suitable for our present aims.) In Proposition 7 below we characterize all tangent valued $2\mathcal{GLB}$-forms.

Let $V, W, \bar{V}, \bar{W}$ be vector spaces. A 2-graded linear map is a triple $f = (f_1, f_2, f_3)$ where $f_1 \in L(V, \bar{V})$ and $f_2 \in L(W, \bar{W})$ are linear maps and $f_3 \in L^2(V, \bar{W})$ is quadratic map of V into $\bar{W}$. Such a triple is interpreted as a map

$$f : V \times W \to \bar{V} \times \bar{W}, \quad f(v, w) = (f_1(v), f_2(w) + f_3(v)).$$

One verifies directly that the composition of 2-graded linear maps is 2-graded linear as well, so that these maps form a category $2\mathcal{GL}$. The objects in $2\mathcal{GL}$ are the products $V \times W$ of vector spaces, but we underline that the product vector structure on $V \times W$ is not preserved under $2\mathcal{GL}$-isomorphisms.

For every category $\mathcal{S}$ over manifolds one defines the category $\mathcal{SB}$ of $\mathcal{S}$-bundles, [3]. Since $2\mathcal{GL}$ is a category over manifolds in a canonical way, we obtain the category $2\mathcal{GLB}$ as a special case. By [3], the trivial $2\mathcal{GL}$-bundles are of the form $M \times V \times W$, where M is a manifold. A $2\mathcal{GLB}$-morphism into another trivial $2\mathcal{GL}$-bundle $\bar{M} \times \bar{V} \times \bar{W}$ is a quadruple (f_0, f_1, f_2, f_3), where $f_0 : M \to \bar{M}$, $f_1 : M \to L(V, \bar{V})$, $f_2 : M \to L(W, \bar{W})$, $f_3 : M \to L^2(V, \bar{W})$ are smooth maps, which is interpreted as a map $f : M \times V \times W \to \bar{M} \times \bar{V} \times \bar{W}$ of the form

$$f(x, v, w) = (f_0(x), f_1(x)(v), f_2(x)(w) + f_3(x)(v)) . \tag{8}$$

In general, (8) represents the local expression of an arbitrary $2\mathcal{GLB}$-morphism.

To make some geometric facts more transparent, let us introduce a category $2\mathcal{FM}$ of 2-fibred manifolds, whose objects are pairs of surjective submersions $p : Z \to Y$ and $q : Y \to M$ written as $Z \to Y \to M$, and whose morphisms preserve both fiberings. Obviously, every $2\mathcal{GL}$-bundle $Z \to Y \to M$ is 2-fibred manifold and the underlying fibering $Y \to M$ is a vector bundle.

Proposition 5. *For an arbitrary $2\mathcal{GL}$-bundle $Z \to Y \to M$ the tangent bundle $TZ \to TY \to TM$ is also a 2-graded linear bundle.*

Proof. In the case of a trivial $2\mathcal{GL}$-bundle $M \times V \times W$, $T(M \times V \times W) = TM \times TV \times TW$ is also a trivial $2\mathcal{GL}$-bundle and the tangent map to (8), whose second component is linear in v and dv over TM and third component is linear in w, dw and quadratic in v, dv over TM, is a trivial $2\mathcal{GLB}$-morphism as well. The rest of our claim follows form the general theory of structured bundles, [3]. □

We are going to show that the $2\mathcal{GLB}$-fields can be characterized analogously to the $\mathcal{VB}$- and $\mathcal{AB}$-cases. Consider a vector field $X : Z \to TZ$ an a $2\mathcal{GL}$-bundle $Z \to Y \to M$ tangent to a local one -parameter family of local $2\mathcal{GLB}$-morphisms. This implies, among others, that X is projectable into a vector field $X_1 : Y \to TY$ and the latter field is also projectable into a vector field $X_0 : M \to TM$. On the other hand, since $TZ \to TY \to TM$ is a $2\mathcal{GL}$-bundle, we have defined the concept of a $2\mathcal{GLB}$-section $Z \to TZ$ (i.e. a $2\mathcal{GLB}$ -morphism, which is a section of $TZ \to Z$ at the same time).

Proposition 6. *A vector field X on a $2\mathcal{GL}$-bundle $Z \to Y \to M$ is a $2\mathcal{GLB}$-field iff it is a $2\mathcal{GLB}$-section $Z \to TZ$.*

Proof. On one hand, if X is tangent to a local one-parameter family of local $2\mathcal{GLB}$-morphisms of Z, then one sees directly that $X : Z \to TZ$ is a $2\mathcal{GLB}$-section. Conversely, consider some local adapted coordinates x^i, v^p, w^a on $Z = M \times V \times W$. The coordinate form of a $2\mathcal{GLB}$-section is

$$dx^i = X^i(x), \quad dv^p = X^p_q(x) v^q, \quad dw^a = X^a_{pq}(x) v^p v^q + X^a_b(x) w^b . \tag{9}$$

In the vector bundle case we deduced that the flow of the first two equations of (9) is

$$\bar{x}^i = \varphi(x,t), \quad \bar{v}^p = \varphi^p_q(x,t)y^q\,. \tag{10}$$

Denote by $\varphi^a(x,v,w,t)$ the solution of the additional equation

$$\frac{dw^a}{dt} = X^a_{pq}(x)v^p v^q + X^a_b(x)w^b\,. \tag{11}$$

Write $\Phi^a = \varphi^a(x,kv,k^2w,t) - k^2\varphi^a(x,v,w,t)$, $k \in \mathbb{R}$, so that $\Phi^a = 0$ for $t = 0$ by definition. Then (11) with (10) imply

$$\frac{d\Phi^a}{\partial t} = X^a_b(\varphi^i(x,t))\Phi^b(x,v,w,k,t)\,.$$

Hence Φ^a satisfy a system of linear differential equations with zero initial condition, so that $\Phi^a = 0$. This yields

$$\varphi^a(x,kv,k^2w,t) = k^2\varphi^a(x,v,w,t)\,.$$

By the homogeneous function theorem, [6], φ^a is linear in w^a and quadratic in v^p. This means that the flow of X is formed by local $2\mathcal{GLB}$-morphisms. □

To describe the $2\mathcal{GLB}$-forms, we first construct a $2\mathcal{GL}$-bundle DZ related with $TZ \otimes TZ$. Consider the projections $Tp\otimes \mathrm{id}_{TM} : TZ \otimes TM \to TY \otimes TM$ and $\mathrm{id}_{TY} \otimes Tq : TY \otimes TY \to TY \otimes TM$. Then the Whitney sum over the pullback $p^*(TY \otimes TM)$ of $TY \otimes TM$ over Z

$$TZ \otimes TM \times_{p^*(TY\otimes TM)} TY \otimes TY =: DZ$$

is a vector bundle over Z.

Lemma. *$DZ \to TY \otimes TM \to \otimes^2 TM$ is a $2\mathcal{GL}$-bundle.*

Proof. In the trivial case $Z = \mathbb{R}^m \times V \times W$ we have

$$T(\mathbb{R}^m \times V) \otimes T\mathbb{R}^m = (\mathbb{R}^m \times \mathbb{R}^m \otimes \mathbb{R}^m) \times (V \times V) \otimes \mathbb{R}^m$$

and

$$DZ = T(\mathbb{R}^m \times V) \otimes T\mathbb{R}^m \times (W \times W \otimes \mathbb{R}^m \times V \otimes V)\,.$$

Having a $2\mathcal{GLB}$-morphism $f : Z = \mathbb{R}^m \times V \times W \to \bar{Z} = \bar{\mathbb{R}}^n \times \bar{V} \times \bar{W}$ of the form (8), one verifies directly that the induced map $Df : DZ \to D\bar{Z}$ is a $2\mathcal{GLB}$-morphism as well. The rest follows from the general theory of $\mathcal{S}$-bundles, [3]. □

Since

$$(Tq \otimes \mathrm{id}_{TM}) \circ (\mathrm{id}_{TZ} \otimes T(q\circ p)) = (\mathrm{id}_{TY} \otimes Tq) \circ (Tp \otimes Tp)$$

is the same map $TZ \otimes TZ \to TY \otimes TM$, the formula

$$Q_Z(B_1 \otimes B_2) = (B_1 \otimes T(q\circ p)(B_2), Tp(B_1) \otimes Tp(B_2))$$

induces a map $Q_Z : TZ \otimes TZ \to DZ$. Then $Q_Z \otimes \bigotimes^{k-2} T(q\circ p) : \bigotimes^k TZ \to DZ \otimes \bigotimes^{k-2} TM$ and we define

$$D_k Z = (Q_Z \otimes \bigotimes^{k-2} T(q\circ p))(\Lambda^k TZ)\,.$$

This is a $2\mathcal{GL}$-bundle $D_k Z \to L_k Y \to \Lambda^k TM$.

Proposition 7. *A one-form $TZ \to TZ$ is a $2\mathcal{GLB}$-form iff it is a $2\mathcal{GLB}$-morphism. A k-form $A : \Lambda^k TZ \to TZ$ with $k \geq 2$ is a $2\mathcal{GLB}$-form iff all the following three conditions hold*

(i) A is projectable into $A_0 : \Lambda^k TM \to TM$,
(ii) A factorizes through $D_k Z$ into $A_D : D_k Z \to TZ$,
(iii) A_D is a $2\mathcal{GLB}$-morphism $D_k Z \to TZ$ over A_0.

Proof. Let us start with the case $k = 2$. By functoriality, every $2\mathcal{GLB}$-two-form $A : \Lambda^2 TZ \to TZ$ is projectable into a $\mathcal{VB}$-two-form $A_1 : \Lambda^2 TY \to TY$. Let

$$(a^a_{ij} dx^i \wedge dx^j + a^a_{ip} dx^i \wedge dv^p + a^a_{ib} dx^i \wedge dw^b + a^a_{pq} dv^p \wedge dv^q + \\ + a^a_{pb} dv^p \wedge dw^b + a^a_{bc} dw^b \wedge dw^c) \otimes \frac{\partial}{\partial w^a}$$

be the coordinate expression of the "remaining" part of A. Hence (12) must be linear in w and quadratic in v for any two $2\mathcal{GLB}$-fields X and $\bar{X}$. Let us discuss the following possibilities (the non-indicated components are zero)

1) $X^a = c^a_c w^c, \bar{X}^b = \bar{c}^b_d w^d$	yield $a^a_{bc} = 0$,
2) $X^a = c^a_c w^c, \bar{X}^b = c^b_{pq} v^p v^q$	yield $a^a_{pb} = 0$,
3) $X^p = c^p_r v^r, \bar{X}^q = \bar{c}^q_s v^s$	yield $a^p_{qr} = a^p_{qr}(x)$ depend on x only,
4) $X^i = c^i, \bar{X}^a = c^a_b w^b$	yield $a^a_{ib} = a^a_{ib}(x)$,
5) $X^i = c^i, \bar{X}^p = c^p_q v^q$	yield $a^p_{iq} = a^p_{iqr}(x) v^r$,
6) $X^i = c^i, \bar{X}^j = \bar{c}^j$	yield $a^a_{ij} = a^a_{ijpq}(x) v^p v^q + a^a_{ijb}(x) w^b$.

This is just the coordinate form of our assertion. The cases $k = 1$ and $k \geq 3$ can be studied quite similarly. □

Our description of the $\mathcal{C}$-forms in Propositions 3, 4 and 7 has an interesting relation to the theory of connections of special types. For an arbitrary category $\mathcal{S}$ over manifolds, the following approach to the connections on an arbitrary $\mathcal{S}$-bundle $Y \to M$ is presented in [6]. A connection $\Gamma : Y \to J^1 Y$ is said to be an $\mathcal{SB}$-connection, if the Γ-lift of every vector field on M is an $\mathcal{SB}$-field. One sees directly this is equivalent to the requirement that the corresponding tangent valued one-form ω_Γ is an $\mathcal{SB}$-form, provided $\mathcal{SB}$ is infinitesimally closed. But the curvature of Γ can be defined as the Frölicher-Nijenhuis bracket $[\omega_\Gamma, \omega_\Gamma]$, so that it is an $\mathcal{SB}$-two-form. In the $\mathcal{VB}$- or $\mathcal{AB}$-case we rededuce the well known fact that the curvature of a $\mathcal{VB}$- or $\mathcal{AB}$-connection is linear or affine, respectively. But Proposition 7 gives a new characterization of some properties of the curvature of $2\mathcal{GLB}$-connections.

5. Symplectic and volume-preserving cases. In the last section we intend to show that there are some categories over manifolds, in which the tangent valued $\mathcal{C}$-forms are of trivial character. We first discuss the category $\mathcal{S}p$ of all symplectic manifolds and local symplectomorphisms. Clearly, the $\mathcal{S}p$-fields are the locally Hamiltonian vector fields characterized by

$$\mathcal{L}_X \omega = 0\,, \tag{13}$$

i.e. the Lie derivative of the symplectic form ω vanishes. This implies directly that $\mathcal{S}p$ is infinitesimally closed. Using the well known formula $\mathcal{L}_X = i_X d + d i_X$ and the fact that ω is closed, we can write (13) in the form

$$d(i_X\omega) = 0\,. \tag{14}$$

Having the canonical local expression of ω

$$dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n}$$

condition (14) reads

$$0 = d(X^1 dx^2 - X^2 dx^1 + \cdots + X^{2n-1}dx^{2n} - X^{2n}dx^{2n-1})\,. \tag{15}$$

Proposition 8. *The only $\mathcal{S}p$-one-forms on a connected symplectic manifold (M,ω) are the constant multiples of id_{TM}. The only $\mathcal{S}p$-k-form for $k>1$ is the zero form.*

Proof. A one form $A = a^i_j dx^j \otimes \dfrac{\partial}{\partial x^i}$ is a $\mathcal{S}p$-form iff

$$0 = d(a^1_i X^i dx^2 - a^2_i X^i dx^1 + \cdots + a^{2n-1}_i X^i dx^{2n} - a^{2n}_i X^i dx^{2n-1}) \tag{16}$$

for every $\mathcal{S}p$-field X^i. Consider first the field $b^i \dfrac{\partial}{\partial x^i}$ with constant components. By (15) all of them are $\mathcal{S}p$-fields. Then (16) implies

$$0 = da^1_i \wedge dx^2 - da^2_i \wedge dx^1 + \cdots + da^{2n-1}_i \wedge dx^{2n} - da^{2n}_i \wedge dx^{2n-1} \tag{17}$$

for all $i = 1, \dots 2n$. This simplifies (16) to the form

$$0 = a^1_i dX^i \wedge dx^2 - a^2_i dX^i \wedge dx^1 + \cdots + a^{2n-1}_i dX^i \wedge dx^{2n} - a^{2n}_i dX^i \wedge dx^{2n-1}\,. \tag{18}$$

Consider now the vector fields of linear coordinate form

$$X^i = b^i_j x^j\,, \qquad b^i_j = \text{const},$$

so that (15) reads

$$0 = b^1_i dx^i \wedge dx^2 - b^2_i dx^i \wedge dx^1 + \cdots + b^{2n-1}_i dx^i \wedge dx^{2n} - b^{2n}_i dx^i \wedge dx^{2n-1}\,.$$

This is equivalent to the conditions

$$b^{2l-1}_{2k-1} + b^{2k}_{2l} = 0 \qquad b^{2l-1}_{2k} - b^{2k-1}_{2l} = 0 \qquad b^{2l}_{2k-1} - b^{2k}_{2l-1} = 0 \tag{19}$$

for all $k, l = 1, \dots, n$. The coefficient by $dx^1 \wedge dx^2$ in (18) with individual b's implies

$$a^1_1 - a^2_2 = 0 \qquad a^1_\alpha = 0,\ a^2_\beta = 0, \quad \alpha = 2, \dots, 2n, \quad \beta = 1, 3, \dots, 2n.$$

Repeating such a procedure, we obtain

$$a^i_i - a^j_j = 0 \qquad \text{no summation,}$$

with all other a's vanishing. Then (17) gives

$$\frac{\partial a^1_1}{\partial x^1} = 0, \dots, \frac{\partial a^1_1}{\partial x^{2n}} = 0$$

so that a^i_i =const.

For an $\mathcal{S}p$-k-form A with $k > 1$ the same procedure yields $A = 0$. $\square$

A similar phenomenon appears in the case of the category $\mathcal{V}ol$ of manifolds with volume form and of the volume-preserving local diffeomorphisms. On such a manifold (M, φ), the $\mathcal{V}ol$-fields are the so-called divergence-free vector fields characterized by $\mathcal{L}_X \varphi = 0$. Even $\mathcal{V}ol$ is a infinitesimally closed category. In the canonical local coordinates, in which φ has the form

$$dx^1 \wedge \cdots \wedge dx^m$$

a divergence-free vector field is characterized by

$$\frac{\partial X^1}{\partial x^1} + \cdots + \frac{\partial X^m}{\partial x^m} = 0\,.$$

In the same way as in Proposition 8, one deduces for connected M

Proposition 9. *The only $\mathcal{V}ol$-one-forms on (M, φ) are the constant multiples of id_{TM}. The only $\mathcal{V}ol$-k–form for $k > 1$ is the zero form.*

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