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REFLECTIONS WITH RESPECT TO SUBMANIFOLDS IN CONTACT GEOMETRY

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Abstract. We study to what extent some structure-preserving properties of the geodesic reflection with respect to a submanifold of an almost contact manifold influence the geometry of the submanifold and of the ambient space.

1. Introduction

Reflections with respect to points and curves and, more generally, with respect to submanifolds in Riemannian manifolds are generalizations of reflections with respect to linear subspaces of a Euclidean space. The reflections with respect to points and curves have been studied by different authors. It turns out that their properties strongly influence the curvature of the manifold and that one can characterize certain classes of manifolds (e.g., locally symmetric spaces and real space forms) by using properties of the reflections with respect to their points or their geodesics. For a survey of results of this type, we refer to [4], [14].

Later, one also started investigating similar problems concerning reflections with respect to submanifolds. As before, the study shows that the properties of these reflections influence both the curvature properties of the ambient manifold and the geometry of the submanifold. (We refer to [14] for a survey and for references to basic papers treating the subject.) In [3], the authors initiated the study of reflections with respect to submanifolds in the framework of contact geometry. In particular, they investigated the submanifolds of so-called Sasakian space forms admitting isometric reflections. It turns out that one can completely characterize these submanifolds, and that their structure is closely related to the Sasakian structure on M. (We refer to Lemma 3 for the exact statement of the result.) In this paper, we continue the study of this type of problems. In particular, we will study the submanifolds of a general almost contact metric manifold admitting so-called φ-preserving or ϕ-preserving reflections (we refer to Section 2 for the definitions). These are the analogs of holomorphic and symplectic reflections with

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respect to submanifolds in almost Hermitian manifolds. We will derive a list of necessary conditions for the manifolds in order to have such reflections. In the case where the ambient manifold is a Sasakian space form, we completely characterize the submanifolds admitting such reflections.

The paper is organized as follows. In Sections 2 and 3 we introduce some preliminary material concerning contact geometry and reflections with respect to submanifolds in Riemannian geometry. Then, in Sections 4 and 5, we treat our main results. Finally, we will (in Section 6) briefly treat the reflection for a special example of a submanifold in an almost contact metric space.

2. Contact geometry

A smooth $(2n+1)$-dimensional manifold M^{2n+1} is said to be an *almost contact metric manifold* if it admits a (non-zero) vector field ξ (called the *characteristic vector field*), a one-form η, a tensor field φ of type (1,1) and a (so-called *associated*) Riemannian metric g, satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{1}$$

for all X and Y tangent to M. These conditions imply that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and that

$$\eta(X) = g(X, \xi) \tag{2}$$

for any vector X tangent to M.

The *fundamental two-form* or *Sasaki form* of an almost contact metric manifold $(M, \xi, \eta, \varphi, g)$ is the two-form ϕ defined by

$$\phi(X,Y) = g(X, \varphi Y) \tag{3}$$

for all X and Y tangent to M. If ϕ satisfies

$$\phi = d\eta,$$

the manifold M is said to be a *contact metric manifold.* If the characteristic vector field ξ of a contact metric manifold is a Killing vector field, then M is said to be a *K-contact (metric) manifold.* It can be shown that an almost contact metric manifold M is a K-contact manifold if and only if

$$\nabla_X \xi = -\varphi X \tag{4}$$

for all X tangent to M, where ∇ denotes the Levi Civita connection associated to (M, g).

If the structure tensors of an almost contact metric manifold $(M, \xi, \eta, \varphi, g)$ satisfy

$$(\nabla_X \varphi) Y = g(X,Y)\xi - \eta(Y)X \tag{5}$$

for all X and Y tangent to M, the manifold M is called a *Sasakian manifold*. We remark that all Sasakian manifolds are examples of K-contact manifolds. Further, the Riemann curvature tensor

$$R_{XY}Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$$

of a Sasakian manifold $(M, \xi, \eta, \varphi, g)$ satisfies

$$R_{XY}\xi = \eta(X)Y - \eta(Y)X \tag{6}$$

for all X and Y tangent to M.

A plane section in the tangent space T_mM, $m \in M$ of a Sasakian manifold is said to be a *φ-section* if it is spanned by two unit vectors X and φX, orthogonal to ξ. The sectional curvature $R_{X\varphi XX\varphi X}$ of such a φ-section is called the associated *φ-sectional curvature*. If the φ-sectional curvature is constant, i.e. independent of the point $m \in M$ and of the chosen φ-section, then M is said to be a *Sasakian space form*, and it is frequently denoted by $M^{2n+1}(c)$ (where c is the constant φ-sectional curvature of M). The Riemann curvature tensor R of $M^{2n+1}(c)$ is given by

$$\begin{aligned} R_{XYZW} = {} & \frac{c+3}{4}\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\} \\ & + \frac{c-1}{4}\{\eta(Y)\eta(Z)g(X,W) - \eta(X)\eta(Z)g(Y,W) \\ & - g(X,Z)\eta(Y)\eta(W) + g(Y,Z)\eta(X)\eta(W) - g(Z,\varphi Y)g(\varphi X,W) \\ & + g(Z,\varphi X)g(\varphi Y,W) - 2g(X,\varphi Y)g(\varphi Z,W)\}. \end{aligned} \tag{7}$$

We refer to [1], [9], [15] for more information about almost contact metric manifolds and related topics, and for an extensive list of references to the literature.

Let $(M, \xi, \eta, \varphi, g)$ be an almost contact metric manifold. A submanifold P is said to be an *invariant* submanifold if the characteristic vector field ξ is tangent to P everywhere and if P is *invariant* with respect to φ, i.e. $\varphi(T_pP) \subset T_pP$ for all $p \in P$. If, for all $p \in P$, $\varphi(T_pP) \subset T_p^\perp P$, the submanifold P is said to be *anti-invariant*. One has

Lemma 1. *Let P be a submanifold of a K-contact metric manifold M^{2n+1} and suppose that ξ is everywhere normal to P. Then P is an anti-invariant submanifold of M and $\dim P \leq n$.*

This result is proved in [16] under the assumption that the ambient space M is Sasakian. One can immediately adapt the proof to the case of K-contact metric manifolds. In Section 6 we will give an example showing that this result cannot be generalized to the class of almost contact metric manifolds. We refer to [15], [16] for more information on structure-related submanifolds in Sasakian geometry.

A (local) diffeomorphism $f : M \to M$ of an almost contact metric manifold M is said to be *ϕ-preserving* if it preserves the Sasaki form ϕ, i.e.

$$f^\star\phi = \phi, \tag{8}$$

and f is said to be *φ-preserving* if it preserves the structure tensor φ, i.e. if

$$\varphi \circ f_\star = f_\star \circ \varphi. \tag{9}$$

For later use, we state the following result from [11] :

Lemma 2. *Let M be a contact metric manifold with structure tensors (ξ, η, φ, g) and let f be a φ-preserving local diffeomorphism. Then there exists a positive constant α such that*

$$f_\star \xi = \alpha \xi, \quad f^\star \eta = \alpha \eta, \quad f^\star g = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta. \tag{10}$$

3. Reflections with respect to submanifolds

Let (M, g) be an n-dimensional Riemannian manifold and suppose P is a connected, relatively compact, (topologically) embedded submanifold of dimension q. The mapping

$$\psi_P : p = \exp_m(ru) \mapsto \psi_P(p) = \exp_m(-ru)$$

for all $m \in P$, all $u \in T_m^\perp P$, $\|u\| = 1$ and all sufficiently small r, is an involutive local diffeomorphism of M, called the *(local) reflection* with respect to P. In what follows, we will study the relation between the properties of the reflection ψ_P and the geometry of P in the case where the ambient space (M, g) carries an almost contact metric structure.

In order to study this relation, we first introduce a general framework which will allow us to treat our problems analytically. We start by constructing special coordinate systems, the so-called *Fermi coordinate systems.* (We refer to [14] for a more detailed treatment and for references to the basic papers treating this subject.) First, let $m \in P$ and choose a local orthonormal frame field $\{E_1, \ldots, E_n\}$ defined along P in a neighborhood of m and such that $E_1, \ldots, E_q$ are tangent to P while $E_{q+1}, \ldots, E_n$ are normal to P. Next, let $(y^1, \ldots, y^q)$ be a coordinate system in a neighborhood of m in P for which

$$\frac{\partial}{\partial y^i}(m) = E_i(m), \qquad i = 1, \ldots, q.$$

Note that, in a sufficiently small neighborhood U of m in M, every point p of U can be expressed in a unique way as

$$p = \exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)$$

for some point $b \in P$. Hence, putting

$$x^i\Big(\exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)\Big) = y^i(b), \qquad i = 1, \ldots, q,$$

$$x^a\Big(\exp_b\Big(\sum_{\alpha=q+1}^{n} t_\alpha E_\alpha\Big)\Big) = t_a, \qquad a = q+1, \ldots, n,$$

we obtain a coordinate system $(x^1, \ldots, x^n)$ on U, called a *Fermi coordinate system* (relative to m, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$). With respect to such a Fermi coordinate system, the reflection ψ_P with respect to the submanifold P takes the following (local) form:

$$\psi_P : (x^1, \ldots, x^q, x^{q+1}, \ldots, x^n) \mapsto (x^1, \ldots, x^q, -x^{q+1}, \ldots, -x^n).$$

Further, there exists a strong relation between the basic vector fields $\frac{\partial}{\partial x^\alpha}$ of the Fermi coordinate system and some special Jacobi vector fields along geodesics in M (see for example [14] for more details). To describe this relation, let $p = \exp_m(ru)$ be a point in a small neighborhood of P in M (with $m \in P$, $u \in T_m^\perp P$, $\|u\| = 1$ and small r), and denote by $\gamma : s \mapsto \exp_m(su)$ the unit speed geodesic joining m and p. Further, let $(x^1, \ldots, x^n)$ be the Fermi coordinate system with respect to m, $(y^1, \ldots, y^q)$ and $(E_{q+1}, \ldots, E_n)$ as before, where we now choose the frame field $\{E_1, \ldots, E_n\}$ in such a way that $E_n(m) = u = \gamma'(0)$. Finally, we denote by Y_α the Jacobi vector fields along γ satisfying the initial conditions

$$\begin{aligned} Y_i(0) &= E_i(m), & Y_a(0) &= 0, \\ Y_i'(0) &= \nabla_u \frac{\partial}{\partial x^i}, & Y_a'(0) &= E_a(m), \end{aligned} \tag{11}$$

for all $i = 1, \ldots, q$ and $a = q+1, \ldots, n-1$. These Jacobi vector fields are then related to the basic vector fields $\frac{\partial}{\partial x^\alpha}$ of the Fermi coordinate system $(x^1, \ldots, x^n)$ by

$$\begin{aligned} Y_i(s) &= \frac{\partial}{\partial x^i}(\gamma(s)), & i &= 1, \ldots, q, \\ Y_a(s) &= s\frac{\partial}{\partial x^a}(\gamma(s)), & a &= q+1, \ldots, n-1. \end{aligned} \tag{12}$$

This relation plays an important role in the study of the geometry of a tubular neighborhood of P. We describe now some useful facts about it. First, we denote by $\{F_1, \ldots, F_n\}$ the frame field along γ, obtained by parallel translation (with respect to ∇) of $\{E_1(m), \ldots, E_n(m)\}$, and define an endomorphism-valued function D by

$$Y_\alpha(s) = D(s)F_\alpha, \qquad \alpha = 1, \ldots, n-1. \tag{13}$$

This function D satisfies the Jacobi equation

$$D'' + R \circ D = 0. \tag{14}$$

The initial conditions for D are given by

$$D(0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad D'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\perp(u) & I \end{pmatrix}, \tag{15}$$

where T and $\perp$ are defined, via the Levi Civita connection $\tilde{\nabla}$ of P, by

$$\nabla_X Y = \tilde{\nabla}_X Y + T_X Y,$$
$$\nabla_X N = T(N)X + \perp_X N,$$

for all X and Y tangent to P and all N normal to P, and

$$T(u)_{ij} = g(T(u)E_i, E_j)(m),$$
$$\perp (u)_{ia} = g(\perp_{E_i} E_a, E_n)(m).$$

$T_X Y = T(X,Y)$ is the second fundamental form operator of P and $T(N)$ is the shape operator of P with respect to N. Further, $\perp_X N = \nabla^{\perp}_X N$ where $\nabla^{\perp}$ is the normal connection along P.

Using these formulas (together with the Gauss lemma), one immediately sees that the components of the metric tensor g with respect to the Fermi coordinate system are given by

$$\begin{aligned} g_{ij}(p) &= ({}^t DD)_{ij}(r), \\ g_{ia}(p) &= \frac{1}{r}({}^t DD)_{ia}(r), \\ g_{ab}(p) &= \frac{1}{r^2}({}^t DD)_{ab}(r), \\ g_{in}(p) &= g_{an}(p) = 0, \\ g_{nn}(p) &= 1, \end{aligned} \tag{16}$$

where $i,j = 1,\dots,q$ and $a,b = q+1,\dots,n-1$. Further, using the Taylor expansion for D together with the Jacobi equation (14) and the initial conditions (15), one obtains the following power series expansions for the components of g:

$$\begin{aligned} g_{ij}(p) &= g(E_i,E_j)(m) + 2rg(TE_i,E_j)(m) + O(r^2), \\ g_{ia}(p) &= -rg({}^t \perp E_i, E_a)(m) - \frac{2}{3}r^2 g(RE_i,E_a)(m) + O(r^3), \\ g_{ab}(p) &= g(E_a,E_b)(m) - \frac{1}{3}r^2 g(RE_a,E_b)(m) + O(r^3). \end{aligned} \tag{17}$$

Next, if we suppose that M carries an almost contact metric structure, we can in a similar way derive a power series expansion for the Sasaki form ϕ. Up to order two in r, this expansion for ϕ_{ij}, $i,j = 1,\dots,q$ is given by

$$\begin{aligned} \phi_{ij}(p) &= g(E_i,\varphi E_j)(m) \\ &\quad + r\{g(E_i,(\nabla_u\varphi)E_j) + \sum_{k=1}^{q} T(u)_{ki} g(E_k,\varphi E_j) \\ &\quad + \sum_{k=1}^{q} T(u)_{kj} g(E_i,\varphi E_k) - \sum_{a=q+1}^{2n} \perp (u)_{ia} g(E_a,\varphi E_j) \\ &\quad - \sum_{a=q+1}^{2n} \perp (u)_{ja} g(E_i,\varphi E_a)\}(m) + O(r^2). \end{aligned} \tag{18}$$

The other components can be computed in the same way, but as we do not need them here, we skip the explicit expressions. Note that, analogously, the expressions for the components of φ can be obtained by using the expansions for ϕ and g, together with the fact that

$$\varphi_\alpha^\beta = -\phi_{\alpha\gamma} g^{\gamma\beta},$$

but we omit the explicit expressions again.

Finally, we state the following results for later use.

Lemma 3. *Let M^{2n+1} be a Sasakian manifold with structure tensors (ξ, η, φ, g) and with constant holomorphic sectional curvature $c \neq 1$. Further, let P be a submanifold of M. Then the reflection ψ_P is isometric if and only if P is either a totally geodesic invariant submanifold or a totally geodesic anti-invariant submanifold with ξ normal to P and $\dim P = n$.*

For the proof of this theorem we refer to [3]. In the case where $c = 1$, (7) implies that M is a manifold of constant curvature (a real space form) and in this case the following result holds (see for example [6], [12], [13]):

Lemma 4. *Let M be a real space form and suppose that P is a submanifold of M. Then ψ_P is isometric if and only if P is a totally geodesic submanifold of M.*

4. ϕ-PRESERVING REFLECTIONS

In this section, we start our study of the relation between some particular properties of the reflection with respect to a submanifold and the geometry of that submanifold. As a first step, we will consider the submanifolds P of an almost contact metric manifold M such that the reflection ψ_P is ϕ-preserving. The following theorem gives necessary conditions for the submanifold in order to have such a ϕ-preserving reflection.

Theorem 5. *Let M be an almost contact metric manifold with structure tensors (ξ, η, φ, g) and Sasaki form ϕ. If the reflection ψ_P with respect to a submanifold P in M is ϕ-preserving, then P is either an invariant submanifold of M, or it satisfies the following conditions:*

1. *$\varphi(T_m P) \subset T_m P$ for all $m \in P$;*
2. *ξ is normal to P everywhere.*

Proof. Suppose that P is a submanifold in M such that the reflection ψ_P with respect to P is ϕ-preserving. Note that, for $m \in P$, $X \in T_m P$ if and only if $\psi_{P\star}(m)X = X$, and $X \in T_m^\perp P$ if and only if $\psi_{P\star}(m)X = -X$. Hence we have for all $m \in P$, all $X \in T_m P$ and all $Y \in T_m^\perp P$,

$$g(X, \varphi Y) = g(\psi_{P\star}X, \varphi\psi_{P\star}Y) = g(X, -\varphi Y) = -g(X, \varphi Y). \tag{19}$$

So, $g(X, \varphi Y) = 0$ for all $X \in T_m P$ and all $Y \in T_m^\perp P$ and consequently, $\varphi(T_m P) \subset T_m P$.

Further, (8) implies that

$$g(\psi_{P\star}\xi, \varphi\psi_{P\star}Y) = g(\xi, \varphi Y) = 0$$

for all $Y \in T_m M$ and all $m \in P$. Hence, $\psi_{P\star}\xi \sim \xi$ and since $\psi_{P\star}^2 = I$, we see that $\psi_{P\star}\xi = \pm\xi$. Consequently, at every point m, the characteristic vector field ξ is either tangent to P or orthogonal to P. The differentiability of ξ then implies that ξ is either tangent to P at every point, or normal to P at every point. This proves the theorem. □

In what follows, we will use the power series expansion (18) to derive additional information concerning the submanifold P. Therefore, we choose a Fermi coordinate system $(x^1, \ldots, x^{2n+1})$ as described in Section 3. (Note that, for such a coordinate system $\psi_{P\star}(\frac{\partial}{\partial x^i})(p) = \frac{\partial}{\partial x^i}(\psi_P(p))$ for all $i = 1, \ldots, q$.) Since ψ_P is ϕ-preserving, this implies that

$$\phi_{ij}(p) = \phi_{ij}(\psi_P(p))$$

for all $i, j = 1, \ldots, q$ and the power series (18) then yields

$$\begin{aligned} &g(E_i, (\nabla_u\varphi)E_j) + \sum_{k=1}^{q} T(u)_{ki}g(E_k, \varphi E_j) \\ &+ \sum_{k=1}^{q} T(u)_{kj}g(E_i, \varphi E_k) - \sum_{a=q+1}^{2n} \perp(u)_{ia}g(E_a, \varphi E_j) \\ &- \sum_{a=q+1}^{2n} \perp(u)_{ja}g(E_i, \varphi E_a) = 0 \end{aligned} \tag{20}$$

for all $u \in T_m^\perp P$ and all $i, j = 1, \ldots, q$. As we know from Theorem 5 that $\varphi(T_m P) \subset T_m P$, $g(E_a, \varphi E_j) = 0$ and (20) yields

$$g(E_i, (\nabla_u\varphi)E_j) + \sum_{k=1}^{q} T(u)_{ki}g(E_k, \varphi E_j) - \sum_{k=1}^{q} T(u)_{kj}g(E_k, \varphi E_i) = 0. \tag{21}$$

Hence, for all $X, Y \in T_m P$ and all $u \in T_m^\perp P$, we must have

$$g(X, (\nabla_u\varphi)Y) + g(T(u)X, \varphi Y) - g(T(u)Y, \varphi X) = 0$$

and substituting X by φX, this yields

$$\begin{aligned} &g(\varphi X, (\nabla_u\varphi)Y) + g(T(u)\varphi X, \varphi Y) \\ &+ g(T(u)X, Y) - \eta(X)g(T(u)Y, \xi) = 0. \end{aligned} \tag{22}$$

We will now consider the consequences of (22) in the two occurring cases: ξ normal to P and ξ tangent to P.

Case 1: *ξ is normal to P*

In this case, $\eta(X) = 0$ for all $X \in T_m P$ and (22) becomes

$$g(\varphi X, (\nabla_u \varphi)Y) + g(T(u)\varphi X, \varphi Y) + g(T(u)X, Y) = 0. \tag{23}$$

Putting $X = Y$ in (23), we get

$$g(\varphi X, (\nabla_u \varphi)X) + g(T(u)\varphi X, \varphi X) + g(T(u)X, X) = 0 \tag{24}$$

and since

$$g(\varphi X, (\nabla_u \varphi)X) = 0,$$

(24) yields

$$g(T(u)\varphi X, \varphi X) + g(T(u)X, X) = 0 \tag{25}$$

for all $m \in P$ and all $X \in T_m P$.

Finally, we choose a basis for $T_m P$ of the form $E_1, \varphi E_1, E_2, \varphi E_2, \ldots, E_s, \varphi E_s$, where $2s = \dim P$. Then one sees from (25) that, for all $i = 1, \ldots, s$,

$$g(T(u)E_i, E_i) + g(T(u)\varphi E_i, \varphi E_i) = 0$$

and, after summation over all i,

$$\mathrm{tr} T(u) = 0 \tag{26}$$

for all $u \in T_m^\perp P$ and for all $m \in P$. Hence, P is a minimal submanifold.

Case 2 : *ξ is tangent to P*

In this case,

$$g(T(u)X, X) + g(T(u)\varphi X, \varphi X) = 0 \tag{27}$$

for all $X \in T_m P$ orthogonal to ξ. Taking an orthonormal basis for $T_m P$ of the form $\xi, E_1, \varphi E_1, \ldots, E_s, \varphi E_s$, where $\dim P = 2s + 1$, it is easy to see from (27) that

$$\mathrm{tr} T(u) = g(\nabla_\xi u, \xi) = -g(\nabla_\xi \xi, u) \tag{28}$$

which is not necessarily zero. Hence, P is not necessarily minimal.

However, if M carries a contact metric structure or, more generally, an almost contact metric structure such that the integral curves of ξ are geodesics, it is obvious that $\nabla_\xi \xi = 0$, and in this case P is again a minimal submanifold.

Hence, we proved

Theorem 6. *Let M be an almost contact metric manifold with structure tensors (ξ, η, φ, g) and suppose that the integral curves of the characteristic vector field are geodesics. If the reflection ψ_P with respect to a submanifold P in M is ϕ-preserving, then P is a minimal submanifold.*

We remark that, if M carries a K-contact structure, Lemma 1 implies that ξ must be tangent to P at every point. Indeed, if ξ is normal to P at every point, the submanifold P is anti-invariant, which contradicts the theorem. Hence, we can state the following

Corollary 7. *Let M be a K-contact manifold with structure tensors (ξ, η, φ, g) and Sasaki form ϕ. If the reflection ψ_P with respect to a submanifold P in M is ϕ-preserving, then P is an invariant minimal submanifold of M.*

Finally, we will now consider the case where the ambient manifold is a Sasakian space form. In this case, one can explicitly solve the Jacobi differential equation (see for example [2]), and hence compute explicit expressions for the basic vector fields of a Fermi coordinate system. Using this, one can give complete expressions for the components of the Sasaki form ϕ and prove the following

Theorem 8. *Let $M^{2n+1}(c)$ be a Sasakian space form with structure tensors (ξ, η, φ, g) and Sasaki form ϕ. Then the reflection ψ_P with respect to an invariant (minimal) submanifold P in M is always ϕ-preserving.*

Proof. Let P be a q-dimensional invariant submanifold in M and let m be a point in P. (We note that an invariant submanifold in a Sasakian manifold is automatically minimal [15].) Further, let γ be a geodesic orthogonal to P, emanating from m with unit velocity vector field u. Finally, we choose an orthonormal frame field at m adapted to our constructions:

$$E_1, \ldots, E_{q-1}, E_q = \xi \in T_m P, E_{q+1} = \varphi u, E_{q+2}, \ldots, E_{2n}, E_{2n+1} = u \in T_m^{\perp} P,$$

and we may always suppose that $E_{q+1}, \ldots, E_{2n+1}$ are parallel with respect to $\nabla^{\perp}$ at the fixed point m, i.e. $\perp = 0$ at m. Next, we construct the frame field $\{F_1, \ldots, F_{2n+1}\}$ along γ by parallel translation of $\{E_1(m), \ldots, E_{2n+1}(m)\}$ with respect to ∇. Following the technique described in Section 3, we now compute the Jacobi vector fields $Y_1, \ldots, Y_{2n}$ along γ, satisfying the following initial conditions:

$$\begin{aligned} Y_i(0) &= E_i(m), \\ Y_i'(0) &= \sum_{j=1}^{p} T(u)_{ji} E_j(m), \\ Y_a(0) &= 0, \\ Y_a'(0) &= E_a(m), \end{aligned}$$

for $i = 1, \ldots, q$ and $a = q+1, \ldots, 2n$. As in [2], we have to perform the computations in three different cases, namely $c + 3 > 0$, $c + 3 = 0$ and $c + 3 < 0$. In what follows, we describe the results of these long computations and refer to [2] for more details.

Case 1 : $c+3=0$

$$\begin{aligned} Y_j(s) &= \sum_{i=1}^{p-1} T(u)_{ij} s F_i(s) + F_j(s), \quad j=1,\dots,q-1; \\ Y_q(s) &= (\beta_q(s)\sin s + \gamma_q(s)\cos s)F_q(s) + (\beta_q(s)\cos s - \gamma_q(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_q(s) = s, \quad \gamma_q(s) = 1 - s^2$;

$$\begin{aligned} Y_{q+1}(s) = {} & (\beta_{q+1}(s)\sin s + \gamma_{q+1}(s)\cos s)F_q(s) \\ & + (\beta_{q+1}(s)\cos s - \gamma_{q+1}(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_{q+1}(s) = s, \quad \gamma_{q+1}(s) = -s^2$;

$$Y_a(s) = sF_a(s), \quad a = q+2,\dots,2n.$$

Case 2 : $c+3>0$

Putting $k=\sqrt{c+3}$, we obtain

$$\begin{aligned} Y_j(s) &= \sum_{i=1}^{q-1} \frac{2}{k} T(u)_{ij} \sin\frac{ks}{2} F_i(s) + \cos\frac{ks}{2} F_j(s), \quad j=1,\dots,q-1; \\ Y_q(s) &= (\beta_q(s)\sin s + \gamma_q(s)\cos s)F_q(s) + (\beta_q(s)\cos s - \gamma_q(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_q(s) = \frac{1}{k}\sin ks, \quad \gamma_q(s) = \frac{2}{k^2}(\cos ks - 1) + 1$;

$$\begin{aligned} Y_{q+1}(s) = {} & (\beta_{q+1}(s)\sin s + \gamma_{q+1}(s)\cos s)F_q(s) \\ & + (\beta_{q+1}(s)\cos s - \gamma_{q+1}(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_{q+1}(s) = \frac{1}{k}\sin ks, \quad \gamma_{q+1}(s) = \frac{2}{k^2}(\cos ks - 1)$;

$$Y_a(s) = \frac{2}{k}\sin\frac{ks}{2} F_a(s), \quad a = q+2,\dots,2n.$$

Case 3 : $c+3<0$

Putting $k=\sqrt{-(c+3)}$, we obtain

$$\begin{aligned} Y_j(s) &= \sum_{i=1}^{q-1} \frac{2}{k} T(u)_{ij} \sinh\frac{ks}{2} F_i(s) + \cosh\frac{ks}{2} F_j(s), \quad j=1,\dots,q-1; \\ Y_q(s) &= (\beta_q(s)\sin s + \gamma_q(s)\cos s)F_q(s) + (\beta_q(s)\cos s - \gamma_q(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_q(s) = \frac{1}{k}\sinh ks, \quad \gamma_q(s) = -\frac{2}{k^2}(\cosh ks - 1) + 1;$

$$\begin{aligned} Y_{q+1}(s) &= (\beta_{q+1}(s)\sin s + \gamma_{q+1}(s)\cos s)F_q(s) \\ &\quad + (\beta_{q+1}(s)\cos s - \gamma_{q+1}(s)\sin s)F_{q+1}(s), \end{aligned}$$

where $\beta_{q+1}(s) = \frac{1}{k}\sinh ks, \quad \gamma_{q+1}(s) = -\frac{2}{k^2}(\cosh ks - 1);$

$$Y_a(s) = \frac{2}{k}\sinh\frac{ks}{2}F_a(s), \quad a = q+2, \dots, 2n.$$

Using these solutions of the Jacobi differential equation, (12) and (13) immediately yield expressions for the basic vector fields of the Fermi coordinate system along γ with respect to the chosen frame field $\{E_1, \dots, E_{2n+1}\}$ at m. With these expressions it is then possible to compute the components of the Sasaki form ϕ. Taking into account that

$$\begin{aligned} F_q &= (\cos s)\xi + (\sin s)\varphi u, \\ F_{q+1} &= -(\sin s)\xi + (\cos s)\varphi u, \\ \varphi F_q &= -(\sin s)u, \\ \varphi F_{q+1} &= -(\cos s)u, \end{aligned}$$

together with the fact that

$$\nabla_u\{g(F_i, \varphi F_a)\} = 0,$$

and hence $g(F_i, \varphi F_a) = 0$ everywhere along γ, the computed expressions are easily seen to satisfy

$$\begin{aligned} \phi_{jl}(s) &= \phi_{jl}(-s), & \phi_{qq+1}(s) &= -\phi_{qq+1}(-s), \\ \phi_{jq}(s) &= \phi_{jq}(-s), & \phi_{qa}(s) &= -\phi_{qa}(-s), \\ \phi_{jq+1}(s) &= -\phi_{jq+1}(-s), & \phi_{q+1q+1}(s) &= \phi_{q+1q+1}(-s), \\ \phi_{ja}(s) &= -\phi_{ja}(-s), & \phi_{q+1a}(s) &= \phi_{q+1a}(-s), \\ \phi_{qq}(s) &= -\phi_{qq}(-s), & \phi_{ab}(s) &= \phi_{ab}(-s), \end{aligned}$$

everywhere along γ. Since m and u are arbitrary, this proves the theorem. □

5. φ-PRESERVING REFLECTIONS

In this section we will study the submanifolds of almost contact metric manifolds admitting φ-preserving reflections. Again, we will start our study by constructing a list of necessary conditions for the submanifold P in order to admit φ-preserving reflections. Using the same technique as in the previous section, we prove the following

Theorem 9. *Let M be an almost contact metric manifold and suppose P is a submanifold in M. If the reflection ψ_P with respect to P in M is φ-preserving, then*

1. *ξ is everywhere tangent to P or everywhere normal to P;*
2. *$\varphi(T_mP) \subset T_mP$ for all $m \in P$.*

Proof. Take $m \in P$ and choose $X \in T_mP$. Since ψ_P is φ-preserving, we get

$$\psi_{P\star}(\varphi X) = \varphi X \tag{29}$$

and hence $\varphi X \in T_mP$. On the other hand, if $Y \in T_m^{\perp}P$, then

$$\psi_{P\star}(\varphi Y) = \varphi(-Y) = -\varphi Y, \tag{30}$$

and so $\varphi Y \in T_m^{\perp}P$. Finally, putting $X = \xi$ in (9) yields

$$\varphi(\psi_{P\star}\xi) = 0, \tag{31}$$

showing that $\xi \in T_mP$ for all $m \in P$ or $\xi \in T_m^{\perp}P$ for all $m \in P$. This proves the theorem. □

As before, we note that if M carries a K-contact metric structure, ξ is necessarily tangent to P.

Next, suppose that M carries a contact metric structure (ξ, η, φ, g) and let ψ_P be a φ-preserving reflection. Then we know from Lemma 2 that there exists a *positive* constant α such that

$$\psi_{P\star}\xi = \alpha\xi, \quad \psi_P^{\star}\eta = \alpha\eta, \quad (\psi_P^{\star}g)(X,Y) = \alpha g(X,Y) + \alpha(\alpha-1)\eta(X)\eta(Y).$$

As $\psi_P^2 = I$ and as α must be positive, $\alpha = 1$. Consequently, ψ_P is isometric and ϕ-preserving, and ξ is tangent to P ($\psi_{P\star}\xi = \xi$). Because ψ_P is isometric, the submanifold P must be a totally geodesic submanifold (see for example [8]). Hence, we can state the following

Theorem 10. *Let M be a contact metric manifold and P a submanifold in M such that the reflection ψ_P is φ-preserving. Then P is a totally geodesic invariant submanifold of M and ψ_P is isometric and ϕ-preserving.*

As in the case of ϕ-preserving reflections, we are again able to prove the following converse of Theorem 10 in the case where M is a Sasakian space form.

Theorem 11. *Let M be a Sasakian space form and suppose that P is a totally geodesic, invariant submanifold in M. Then the reflection ψ_P with respect to P is φ-preserving.*

Proof. Let P be a submanifold satisfying the hypothesis. Then Theorem 8 implies that the reflection ψ_P with respect to P must be ϕ-preserving. Further, from Lemma 3 and Lemma 4 we see that ψ_P must be isometric. Combining these two facts yields that ψ_P is φ-preserving. □

6. A special example

In this last section, we describe a special example of a submanifold in an almost contact metric space: a certain two-dimensional unit sphere in a five-dimensional unit sphere, equipped with its non-standard structure (see for example [1], [10]). This example shows that Lemma 1 cannot be generalized to almost contact metric manifolds, and it will also provide an example of the case "ξ orthogonal to P" arising in the proof of Theorem 5 and Theorem 6.

First, we describe the non-standard structure on the five-dimensional unit sphere $S^5(1)$. Therefore, we consider $\mathbf{R}^7$ with its Euclidean metric, and we identify all tangent spaces with the space of imaginary Cayley numbers. We can then define a two-fold vector cross-product $\times$: the product of two vectors X and Y is the imaginary part of their product as Cayley numbers (see for example [7]). On the six-dimensional unit sphere $S^6(1)$, (with equation $\sum_{i=1}^{7}(x_i)^2 = 1$) we denote by $N_1(x) = \sum_{i=1}^{7} x_i E_i$ the unit outer normal to $S^6(1)$. Then we define an almost complex structure J on $S^6(1)$ by

$$JX = N_1 \times X \tag{32}$$

for all X tangent to $S^6(1)$. The induced metric is compatible with J and makes S^6 into a (nearly Kähler) almost Hermitian manifold. Now, consider $S^5(1)$ as the totally geodesic hypersurface of $S^6(1)$ given by $x_7 = 0$. We take $N_2 = E_7$ as unit normal and define an almost contact metric structure by

$$\xi = -JN_2, \quad JX = \varphi X + \eta(X)N_2, \tag{33}$$

together with the induced metric g. Finally, let $S^2(1)$ be the submanifold of $S^5(1)$ defined by $x_1 = x_3 = x_5 = 0$. Then ξ is normal to $S^2(1)$ everywhere. Moreover, it can be shown that $\varphi(T_m S^2(1)) \subset T_m S^2(1)$ for all $m \in S^2(1)$. Further, the reflection $\psi_{S^2(1)}$ with respect to $S^2(1)$ (in $S^5(1)$) is φ-preserving (see for example [5]). As $S^2(1)$ is a totally geodesic submanifold in a space of constant curvature, we know from Lemma 4 that the reflection $\psi_{S^2(1)}$ is isometric, and hence also ϕ-preserving.

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