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SPRAYS AND HOMOGENEOUS CONNECTIONS ON R × TM

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Abstract. The homogeneity properties of two different families of geometric objects playing a crutial role in the non-autonomous first-order dynamics - semisprays and dynamical connections on $R \times TM$ - are studied. A natural correspondence between sprays and a special class of homogeneous connections is presented.

1. Introduction

The importance of the homogeneity of second-order differential equation fields (briefly semisprays) and of related connections on TM is well known (e.g. [7], [16], [3], [10], [9], [15], [1], [4] etc.). Namely, if we take an arbitrary semispray ζ on TM, then $\Gamma = -\partial_\zeta J$ (∂_ζ is Lie derivative, J is the canonical almost tangent structure on TM (see (5)) is a connection in the sense of Grifone . However, its paths are not generally just the paths of ζ, because ζ need not be the associated semispray to Γ. A homogeneous semispray is called a spray and then $\Gamma = -\partial_\zeta J$ is the unique homogeneous connection without torsion, now with the same paths [3],[4]. The homogeneity requirement on a regular lagrangian guarantees the associated Lagrange vector field to be a spray, which consequently leads to the geometrical characterization of the related regular autonomous dynamics. These considerations are naturally extended to $T^k M = J^k_0(R \times M, \pi, R)$.

In addition, a canonical connection whose paths are the solutions of the Euler-Lagrange equations for only regular lagrangian are constructed in [4].

The situation on $R \times TM$ was studied by de León and Rodrigues. They have shown in [6] that for any semispray on $R \times TM$ there is the so-called dynamical connection with the same paths (related papers are [2], [5]). However, the role of the homogeneity was not yet (as far as we know) studied.

Our approach to the regular (generally higher-order) dynamics on an arbitrary fibred manifold with a one-dimensional base developed in [18] and [17] allows us to present the following considerations. Remark that some of them are closely related to the geometrical structures on $R \times TM$, which admit the possibility of

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their natural generalization to $R \times T^k M$ but not to an arbitrary first prolongation $J^1\pi$ of π with a one-dimensional base X. On the opposite, many of the used notions are the special cases of those defined on an arbitrary manifold (e.g. [12], [14], [11], [13]).

In Sec. 2 we give a survey of basic structures and notions related to the geometry of $R \times TM$ (see [6], [2], [8]) within the context of [18], [17]. The main results can be found in Sec. 3, where we study the notions like a spray, tension, strong torsion and the relation sprays $\longleftrightarrow$ homogeneous connections. Finally we mention the importance of the homogeneity for regular lagrangians.

We use the following standard notation throughout the work : (Y, π, X) is a fibred manifold with the total space Y and the base X, $\mathcal{F}(Y)$ denotes the set of locally defined real functions on Y , $S_U(\pi)$ is the set of smooth local sections of π defined on U , $\Lambda^r Y$ is the so-called r–fold alternating product of π and [,] means the Frölicher-Nijenhuis bracket of the tangent valued forms.

All manifolds and mappings are supposed smooth and the summation convention is used as far as possible.

2. Geometrical structures on $R \times TM$

In what follows, we consider the trivial bundle $(R \times M, \pi, R)$ with $\pi = pr_1$, where M is an arbitrary m–dimensional manifold. We suppose t to be the canonical global coordinate on R ; $\psi = (t, q^\sigma)$ is then a fibre chart for any local coordinate system $\varphi = (q^\sigma)$, $1 \le \sigma \le m$, on M . Thus a section $\gamma \in S_U(\pi)$ has a form

$$\gamma(t) \;=\; (t, c(t)) \tag{1}$$

where $c : U \longrightarrow M$ is a differentiable curve. The first jet prolongation $J^1\pi$ of π can be naturally identified with $R \times TM$ and the fibration

$$\pi_{1,0} \;: R \times TM \longrightarrow R \times M$$

is obviously a vector bundle. The local coordinates on $J^1\pi$ associated to $\psi = (t, q^\sigma)$ on $V \subset R \times M$ are

$$\psi_1 \;=\; (t, q^\sigma, q^\sigma_{(1)}).$$

If $\overline{\psi} = (t, \overline{q}^\lambda)$ are some other coordinates on $\overline{V} \subset R \times M$ and $V \cap \overline{V} \neq \emptyset$, then

$$\overline{\psi} \circ \psi^{-1}(t, q^\sigma) \;=\; (t, \overline{q}^\lambda(q^\sigma)) \tag{2}$$

and consequently

$$\overline{q}^\lambda_{(1)} = \frac{\partial \overline{q}^\lambda}{\partial q^\sigma} q^\sigma_{(1)} \tag{3}$$

on $\pi^{-1}_{1,0}(V \cap \overline{V}) \subset R \times TM$. Due to the product structure $V_{\pi_1} = R \times TTM$ and $V_{\pi_{1,0}} = < \partial/\partial q^\sigma_{(1)} >$. From (2) and (3) it holds

$$\frac{\partial}{\partial q^\sigma} \;=\; \frac{\partial \overline{q}^\lambda}{\partial q^\sigma}\,\frac{\partial}{\partial \overline{q}^\lambda} \;+\; \frac{\partial \overline{q}^\lambda_{(1)}}{\partial q^\sigma}\,\frac{\partial}{\partial \overline{q}^\lambda_{(1)}}$$

$$\frac{\partial}{\partial q^\sigma_{(1)}} \;=\; \frac{\partial \overline{q}^\lambda}{\partial q^\sigma}\,\frac{\partial}{\partial \overline{q}^\lambda_{(1)}}$$

and

$$dq^\sigma = \frac{\partial q^\sigma}{\partial \overline{q}^\lambda}\, d\overline{q}^\lambda$$

$$dq^\sigma_{(1)} = \frac{\partial q^\sigma_{(1)}}{\partial \overline{q}^\lambda}\, d\overline{q}^\lambda + \frac{\partial q^\sigma}{\partial \overline{q}^\lambda}\, d\overline{q}^\lambda_{(1)}$$

on $T(R \times TM)$ and $T^*(R \times TM)$ respectively.

A *tangent valued r–form* on $J^1\pi$ is (in accordance with [12]) a section of the bundle

$$TJ^1\pi \otimes \Lambda^r T^* J^1\pi \longrightarrow J^1\pi \quad .$$

Tangent valued 1–forms, called also *affinors*, are tensors of type (1,1) on $J^1\pi$ i.e. endomorphisms on $TJ^1\pi$; especially $\pi_{1,0}$–vertical affinors are called *soldering forms*. They are locally expressed by

$$\varphi = \varphi^\sigma \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt + \varphi^\sigma_j \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^j \tag{4}$$

with $\varphi^\sigma, \varphi^\sigma_j \in \mathcal{F}(J^1\pi)$. In terms of natural bundles and operators it can be shown [8] that there is an essential subset (more precisely a linear subspace) of the so-called *natural affinors*. Any such natural affinor has a form

$$\alpha\, I_{TM} + \beta\, J + \gamma\, I_R + \delta\, C \otimes dt \quad ,$$

where

$$I_{TM} = \frac{\partial}{\partial q^\sigma} \otimes dq^\sigma + \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^\sigma_{(1)}$$

and

$$J = \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^\sigma \tag{5}$$

are the unique two natural affinors on TM ;

$$C = q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma_{(1)}}$$

is the *Liouville vector field* on TM and $\alpha, \beta, \gamma, \delta \in \mathcal{F}(R)$. The most important natural soldering form is the endomorphism

$$S = J - C \otimes dt$$

locally given by

$$S = \frac{\partial}{\partial q^\sigma_{(1)}} \otimes \omega^\sigma \quad , \tag{6}$$

where

$$\omega^\sigma = dq^\sigma - q^\sigma_{(1)}\, dt \tag{7}$$

are the well-known canonical contact forms. Obviously rank $S = \dim V_{\pi_{1,0}} = m$ and $S^2 = 0$. If we put

$$\overline{J} = S + (C + \frac{\partial}{\partial t}) \otimes dt \ ,$$

it is easy to see that $R \times TM$ is endowed with a particular case of the so-called *almost stable tangent structure*, which means a triple $(\overline{J}, \frac{\partial}{\partial t}, dt)$ satisfying

$$i_{\frac{\partial}{\partial t}} dt = 1 \ , \ \overline{J}^2 = \frac{\partial}{\partial t} \otimes dt \ , \ \ \mathrm{rank}\ \overline{J} = m + 1 \ .$$

This structure may be used for example to an intrinsic description of the inverse problem (see [5]).

A distinguished vector field on $J^1\pi = R \times TM$ is a (global) *semispray* which can be characterized by means of any of the following conditions:

(i)

$$\zeta \ = \ \frac{\partial}{\partial t} \ + \ q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma} \ + \ \zeta^\sigma_{(1)} \frac{\partial}{\partial q^\sigma_{(1)}}$$

in any fibre coordinates, where $\zeta^\sigma_{(1)} \in \mathcal{F}(J^1\pi)$;

(ii)

$$T\pi_{1,0} \circ \zeta \ = \ \rho_{1,0}$$

where $\rho_{1,0} \ : \ J^1\pi \longrightarrow T(R \times M)$ is a canonical injective mapping (in fact, a vector field along $\pi_{1,0}$, called *total derivative with respect to* t), defined for any given 1-jet $J^1_t\gamma \in J^1\pi$ by

$$\rho_{1,0}(J^1_t\gamma) = \left\{ \frac{d}{ds}\gamma(t+s) \right\}_0 \quad ;$$

(iii)

$$\rho_{1,0} \circ \alpha \ = \ \frac{d}{ds}(\pi_{1,0} \circ \alpha)$$

for any integral curve α of ζ ;

(iv)

$$S\zeta = 0 \ \wedge \ J\zeta = C \, ;$$

(v)

$$\omega^\sigma(\zeta) = 0 \ \wedge \ dt(\zeta) = 1$$

for ω^σ given by (7) and $1 \le \sigma \le m$.

A section $\gamma \in S_U(\pi)$ given by (1) is called a *path* of the semispray ζ if and only if any of the following conditions holds :

(i)

$$\frac{d^2c^\sigma}{dt^2} \ = \ \zeta^\sigma(t, c, \frac{dc}{dt})$$

on U for any fibre coordinates, $1 \leq \sigma \leq m$;

(ii) $J^1\gamma$ is an integral curve of ζ ;

(iii) $J^1\gamma$ is an integral mapping of the so-called (one-dimensional) *semispray distribution* $\Delta_0^1[\zeta]$, generated by ζ ;

(iv)

$$\zeta \circ J^1\gamma \ = \ \rho_{2,1} \circ J^2\gamma$$

on U , where $\rho_{2,1} \ : \ J^2\pi = R \times T^2M \longrightarrow T(R \times TM)$ is analogously to $\rho_{1,0}$ defined by

$$\rho_{2,1}(J_t^2\gamma) = \left\{ \frac{d}{ds} J_{t+s}^1\gamma \right\}_0 \ .$$

The one-dimensional semispray distribution $\Delta_0^1[\zeta]$ on $R \times TM$, spanned by ζ, can be naturally identified with the *connection* Γ *of order 2* on π by

$$\Gamma^\sigma_{(2)} = \zeta^\sigma_{(1)} \ . \tag{8}$$

Any such a connection is a section $\Gamma : J^1\pi \longrightarrow J^2\pi$ locally given by

$$(t, q^\sigma, q^\sigma_{(1)}, q^\sigma_{(2)}) \circ \Gamma = (t, q^\sigma, q^\sigma_{(1)}, \Gamma^\sigma_{(2)})$$

for $\Gamma^\sigma_{(2)} \in \mathcal{F}(J^1\pi)$, characterized among others uniquely by its *horizontal form*

$$h_\Gamma \ = \left(\frac{\partial}{\partial t} \ + q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma} \ + \ \Gamma^\sigma_{(2)} \frac{\partial}{\partial q^\sigma_{(1)}} \right) \otimes dt \ .$$

The *path* (or *integral section*) of Γ is a section $\gamma \in S_U(\pi)$ such that

$$J^2\gamma = \Gamma \circ J^1\gamma$$

on U . It is easy to see that Γ and ζ identified by (8) (ζ is called *associated* to Γ) have the same paths and

$$h_\Gamma \ = \ \zeta \otimes dt \ .$$

Furthermore, there is also another kind of connections closely related to the given semispray.

Let Γ_d be a connection on $\pi_{1,0}$, i.e. a section

$$\Gamma_d \ : J^1\pi \longrightarrow J^1\pi_{1,0}$$

locally given by

$$(t, q^\sigma, q^\sigma_{(1)}, q^\sigma_{(1,0)}, q^\sigma_{(1,0)\lambda}) \circ \Gamma_d = (t, q^\sigma, q^\sigma_{(1)}, \Gamma^\sigma, \Gamma^\sigma_\lambda) \ ,$$

where $\Gamma^\sigma, \Gamma^\sigma_\lambda \in \mathcal{F}(J^1\pi)$. The horizontal form of Γ_d is

$$h_{\Gamma_d} \ = \ \left(\frac{\partial}{\partial t} + \Gamma^\sigma \frac{\partial}{\partial q^\sigma_{(1)}} \right) \otimes dt \ + \ \frac{\partial}{\partial q^\sigma} \otimes dq^\sigma \ + \ \Gamma^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^\lambda \tag{9}$$

and the $(m+1)$-dimensional $\pi_{1,0}$-horizontal subbundle Im $h_{\Gamma_d} =: H_{\Gamma_d}$ is locally generated by the vector fields

$$\frac{\partial}{\partial t} + \Gamma^\sigma \frac{\partial}{\partial q^\sigma_{(1)}} \; ; \; \frac{\partial}{\partial q^\lambda} + \Gamma^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}}$$

or equivalently by the forms

$$\psi^\sigma_{(2)} = dq^\sigma_{(1)} - \Gamma^\sigma \, dt - \Gamma^\sigma_\lambda \, dq^\lambda \quad .$$

A section $\gamma \in S_U(\pi)$ is called a *(dynamical) path* of Γ_d if $J^1\gamma$ is horizontal with respect to Γ_d, which means

$$T(J^1_\gamma) \subset H_{\Gamma_d} \quad .$$

An endomorphism F on $TJ^1\pi = R \times TM$ is called an *f(3,-1) structure* on $R \times TM$ if $F^3 - F = 0$. A special class of such structures is generated by the conditions

$$JF_d = SF_d = S \; ; \; F_d S = -S \; ; \; F_d J = -J \quad . \tag{10}$$

Any endomorphism F_d given by (10), called *dynamical f(3,-1) structure*, is locally expressed by

$$F_d = \left(F^\sigma \frac{\partial}{\partial q^\sigma_{(1)}} - q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma} \right) \otimes dt + \\ + F^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^\lambda + \frac{\partial}{\partial q^\sigma} \otimes dq^\sigma - \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^\sigma_{(1)} \; ,$$

where F^σ , $F^\sigma_\lambda \in \mathcal{F}(J^1\pi)$. Thus F_d generates (by means of its eigenspaces) a direct sum decomposition

$$T(R \times TM) = V_{\pi_{1,0}} \oplus H_{F_d} \oplus \mathrm{Im}\,(F_d^2 - I) \; ,$$

where $V_{\pi_{1,0}} = \mathrm{Im}\,(F_d^2 - F_d)$ and $H_{F_d} = \mathrm{Im}\,(F_d^2 + F_d)$ is called a *strong horizontal subbundle* ($\dim H_{F_d} = m$). It is generated by the vector fields

$$\frac{\partial}{\partial q^\lambda} + \frac{1}{2} F^\sigma_\lambda \frac{\partial}{\partial q^\sigma_{(1)}} \quad . \tag{11}$$

Im $(F_d^2 - I)$ is generated by the semisprays

$$\frac{\partial}{\partial t} + q^\sigma_{(1)} \frac{\partial}{\partial q^\sigma} + (F^\sigma + F^\sigma_\lambda q^\lambda_{(1)}) \frac{\partial}{\partial q^\sigma_{(1)}} \quad . \tag{12}$$

The generators (11) and (12) constitute a *weak horizontal subbundle*

$$H'_{F_d} = \mathrm{Im}\,(F_d^2 - I) \oplus H_{F_d} \quad .$$

There is a bijective correspondence between dynamical f(3,-1) structures on $R\times TM$ and connections on $\pi_{1,0}$, arranged by means of their horizontal subbundles; thus F_d and Γ_d are called *associated* if

$$H_{\Gamma_d} = H'_{F_d} \quad .$$

The local expression of this correspondence is

$$F^\sigma = \Gamma^\sigma - \Gamma^\sigma_\lambda \, q^\lambda_{(1)} \; ; \; F^\sigma_\lambda = 2\,\Gamma^\sigma_\lambda \quad ,$$

or

$$\Gamma^\sigma = F^\sigma + \frac{1}{2}\, F^\sigma_\lambda \, q^\lambda_{(1)} \; ; \; \Gamma^\sigma_\lambda = \frac{1}{2}\, F^\sigma_\lambda \quad .$$

This is the reason for connections on $\pi_{1,0}$ to be also called *dynamical connections* on $R\times TM$.

A connection Γ of order 2 on π is called *associated* to a dynamical connection Γ_d if

$$\Gamma^\sigma_{(2)} = \Gamma^\sigma + \Gamma^\sigma_\lambda \, q^\lambda_{(1)} = F^\sigma + F^\sigma_\lambda \, q^\lambda_{(1)} \quad .$$

It is so if and only if

$$\Delta^1_0[\Gamma] \;=\; \mathrm{Im}\, h_\Gamma \subset H_{\Gamma_d} \quad .$$

Thus if we take an arbitrary connection Γ of order 2 on π and any dynamical connection Γ_d on $R\times TM$ such that Γ is associated to Γ_d , then both connections have the same (dynamical) paths and in addition

$$\mathrm{Im}\,(F^2_d - I) = \Delta^1_0[\Gamma] \quad .$$

Consequently, there is the whole family of dynamical connections Γ_d on $R\times TM$ with the same paths for any semispray ζ on $R\times TM$. The dynamical f(3,-1) structure F_d associated to any such Γ_d generates a direct sum decomposition

$$T(R\times TM) \;=\; V_{\pi_{1,0}} \oplus \Delta^1_0[\zeta] \oplus H_{F_d} \quad .$$

However, there is a canonical choice of such a dynamical connection Γ_d . Using the natural soldering form S given by (6), one can construct a dynamical f(3,-1) structure

$$F_d \;=\; -\partial_\zeta S \quad ,$$

locally given by

$$F^\sigma_\lambda = \frac{\partial \zeta^\sigma_{(1)}}{\partial q^\lambda_{(1)}} \quad , \quad F^\sigma = \zeta^\sigma_{(1)} - \frac{\partial \zeta^\sigma_{(1)}}{\partial q^\lambda_{(1)}}\, q^\lambda_{(1)} \quad .$$

The associated dynamical connection Γ_d has the components

$$\Gamma^\sigma_\lambda = \frac{1}{2}\,\frac{\partial \zeta^\sigma_{(1)}}{\partial q^\lambda_{(1)}} \quad , \quad \Gamma^\sigma = \zeta^\sigma_{(1)} \;-\; \frac{1}{2}\,\frac{\partial \zeta^\sigma_{(1)}}{\partial q^\lambda_{(1)}}\, q^\lambda_{(1)} \quad . \tag{13}$$

This Γ_d will be called *natural dynamical connection* associated to ζ .

3. Sprays and homogeneous connections

Let Γ_d be a dynamical connection on $R \times TM$, φ an arbitrary soldering form on $R \times TM$. The *(weak) torsion* of Γ_d *of type* φ is

$$\tau_\varphi \;=\; [h_{\Gamma_d}, \varphi] \;\; .$$

Following (4) and (9), this tangent valued 2-form can be expressed by

$$\tau_\varphi = \left(\frac{\partial \varphi^\sigma_j}{\partial q^i} + \Gamma^\lambda_i \,\frac{\partial \varphi^\sigma_j}{\partial q^\lambda_{(1)}} - \frac{\partial \Gamma^\sigma_i}{\partial q^\lambda_{(1)}}\,\varphi^\lambda_j \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^i \wedge dq^j +$$
$$+ \left(\frac{\partial \varphi^\sigma_j}{\partial t} - \frac{\partial \varphi^\sigma}{\partial q^j} + \Gamma^\lambda \,\frac{\partial \varphi^\sigma_j}{\partial q^\lambda_{(1)}} - \Gamma^\lambda_j \,\frac{\partial \varphi^\sigma}{\partial q^\lambda_{(1)}} - \frac{\partial \Gamma^\sigma}{\partial q^\lambda_{(1)}}\,\varphi^\lambda_j + \frac{\partial \Gamma^\sigma_j}{\partial q^\lambda_{(1)}}\,\varphi^\lambda \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt \wedge dq^j \;\; .$$

The weak torsion of Γ_d of type S (briefly *weak torsion*) is then

$$\tau_S = \left(- \frac{\partial \Gamma^\sigma_i}{\partial q^j_{(1)}} \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^i \wedge dq^j +$$
$$+ \left(\Gamma^\sigma_j - \frac{\partial \Gamma^\sigma}{\partial q^j_{(1)}} - \frac{\partial \Gamma^\sigma_j}{\partial q^\lambda_{(1)}}\, q^\lambda_{(1)} \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt \wedge dq^j \;\; .$$

Let ζ be an arbitrary semispray on $R \times TM$. Then the contraction of τ_S by ζ is

$$i_\zeta \tau_S = \left(\frac{\partial \Gamma^\sigma}{\partial q^j_{(1)}}\, q^j_{(1)} + \frac{\partial \Gamma^\sigma_i}{\partial q^\lambda_{(1)}}\, q^\lambda_{(1)} q^i_{(1)} - \Gamma^\sigma_i \, q^i_{(1)} \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt +$$
$$+ \left(\Gamma^\sigma_j - \frac{\partial \Gamma^\sigma_i}{\partial q^j_{(1)}}\, q^i_{(1)} - \frac{\partial \Gamma^\sigma}{\partial q^j_{(1)}} \right) \,\frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^j \;\; . \tag{14}$$

A *tension* of a dynamical connection Γ_d is the soldering form

$$H \;=\; -[C, h_{\Gamma_d}] = -\partial_C h_{\Gamma_d} \;\; ,$$

which locally means

$$H \;=\; \left(\Gamma^\sigma - \frac{\partial \Gamma^\sigma}{\partial q^j_{(1)}}\, q^j_{(1)} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt + \left(\Gamma^\sigma_i - \frac{\partial \Gamma^\sigma_i}{\partial q^j_{(1)}}\, q^j_{(1)} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^i \;\; . \tag{15}$$

Definition 1. A dynamical connection Γ_d on $R \times TM$ is called *homogeneous* if its tension vanishes.

By means of (15) it means that the components Γ^σ and Γ^σ_λ of Γ_d are homogeneous of order one in $q^j_{(1)}$. Consequently we denote

$$\Gamma^\sigma_{ij} = \frac{\partial \Gamma^\sigma_i}{\partial q^j_{(1)}} \ .$$

The *strong torsion* of Γ_d will be the soldering form

$$T = i_\zeta \tau_S - H$$

where $i_\zeta \tau_S$ and H are defined by (14) and (15).

All the previous objects are of the particular meaning in the case of the natural dynamical connection associated to the semispray ζ on $R \times TM$. Owing to (13) it holds

$$\begin{aligned} \tau_S &= \left(-\frac{1}{2} \frac{\partial^2 \zeta^\sigma_{(1)}}{\partial q^i_{(1)} \partial q^j_{(1)}} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^i \wedge dq^j \\ i_\zeta \tau_S &= 0 \ , \end{aligned}$$

and

$$\begin{aligned} H = &\left(\zeta^\sigma_{(1)} + \frac{1}{2} \frac{\partial^2 \zeta^\sigma_{(1)}}{\partial q^i_{(1)} \partial q^j_{(1)}} q^i_{(1)} q^j_{(1)} - \frac{\partial \zeta^\sigma_{(1)}}{\partial q^i_{(1)}} q^i_{(1)} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dt + \\ &+ \frac{1}{2} \left(\frac{\partial \zeta^\sigma_{(1)}}{\partial q^j_{(1)}} - \frac{\partial^2 \zeta^\sigma_{(1)}}{\partial q^i_{(1)} \partial q^j_{(1)}} q^i_{(1)} \right) \frac{\partial}{\partial q^\sigma_{(1)}} \otimes dq^j_{(1)} \ . \end{aligned} \tag{16}$$

Consequently

$$T = -H \ .$$

Definition 2. A semispray ζ on $R \times TM$ is called a *spray*, if $\zeta^\sigma_{(1)}$ are homogeneous functions of order two in $q^j_{(1)}$, which means

$$\frac{\partial \zeta^\sigma_{(1)}}{\partial q^j_{(1)}} q^j_{(1)} = 2\, \zeta^\sigma_{(1)}. \tag{17}$$

Immediately we have

Proposition 1. *The natural dynamical connection Γ_d associated to ζ is homogeneous if and only if ζ is a spray.*

Notice that for the above mentioned homogeneous Γ_d it holds :

$$\Gamma^\sigma_{ij} = \frac{1}{2} \frac{\partial^2 \zeta^\sigma_{(1)}}{\partial q^i_{(1)} \partial q^j_{(1)}}$$

and

$$\Gamma^\sigma = 0 \ . \tag{18}$$

Proposition 2. *Let Γ_d be an arbitrary homogeneous dynamical connection. Then its associated semispray ζ given by*

$$\zeta^\sigma_{(1)} = \Gamma^\sigma + \Gamma^\sigma_j \, q^j_{(1)}$$

is a spray if and only if

$$\Gamma^\sigma = 0 \ .$$

Proof. By the coordinate relations. □

Corollary 1. *There is a bijective correspondence between the set of all sprays on $R \times TM$ and the set of all homogeneous connections on $R \times TM$ whose components satisfy (18) .*

Proof. By the previous two propositions, this correspondence identifies spray ζ with its associated natural dynamical connection Γ_d , which is the unique homogeneous dynamical connection with the same paths whose strong torsion vanishes. □

Finally we note the corresponding direct sum decomposition generated by an arbitrary spray. The generators of the weak horizontal subbundle H_{Γ_d} are

$$\frac{\partial}{\partial t} \ ; \ \frac{\partial}{\partial q^i} + \frac{1}{2} \, \Gamma^\sigma_{ij} q^j_{(1)} \, \frac{\partial}{\partial q^\sigma_{(1)}} \ ,$$

where the latter ones are the generators of the strong horizontal subbundle H_{F_d}. In particular, for an autonomous case on $R \times TM$ (i.e. $\zeta^\sigma_{(1)}$ depend on $q^\lambda, q^\lambda_{(1)}$ only) we obtain nothing else than a theory concerning the "graphs" of geodesics of homogeneous (resp. linear) connections on TM . Then the following assertion is not much surprising.

A lagrangian $\lambda = L \, dt$ on $R \times TM$ is called *homogeneous* if L is homogeneous of order two in $q^\lambda_{(1)}$. Its *Lagrange vector field* is the solution of the so-called *characteristic equation* (see [6], [18]).

Proposition 3. *Let a lagrangian $\lambda = L\colon dt$ on $R \times TM$ be regular and homogeneous. Then its Lagrange vector field ζ is a spray if and only if L depends on $q^\lambda, q^\lambda_{(1)}$ only.*

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