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ON A NEW FAMILY OF HOMOGENEOUS EINSTEIN MANIFOLDS

E. D. Rodionov

Abstract. We show that there exists exactly one homothety class of invariant Einstein metrics on each space $[SU(2)]^{S+1}/T^S$ defined below.

In this paper we study a special family of homogeneous spaces $M_\iota^{2S+3} = SU(2) \times \cdots \times SU(2)/T^S (s \geq 1)$, where a maximal torus $T_{\max}$ of $SU(2) \times \cdots \times SU(2)(S+1 \text{ times})$ is decomposed into the direct product $T_{\max} = T^S \times S^1$ with S^1 as the subgroup of all product matrices of the form:

$$\begin{bmatrix} e^{2\pi it} & 0 \\ 0 & e^{-2\pi it} \end{bmatrix} \times \begin{bmatrix} e^{2\pi i\iota_1 t} & 0 \\ 0 & e^{-2\pi i\iota_1 t} \end{bmatrix} \times \cdots \times \begin{bmatrix} e^{2\pi i\iota_s t} & 0 \\ 0 & e^{-2\pi i\iota_S t} \end{bmatrix}$$

$$t \in R,\ \iota = (1, \iota_1, \ldots, \iota_S): \ \iota_1, \ldots, \iota_S \in Q$$

We prove the existence, up to a homothety, of a unique invariant Einstein metric on each $M_\iota^{2S+3}(|\iota_i| \neq 0, 1; i \in \{1, \ldots, S\})$.

Let us remark that every space M_ι^{2S+3} appears naturally as the underlying manifold of a globally φ-symmetric Sasakian structure $(M, g, \varphi, \xi, \eta)$ which fibers over a Hermitian symmetric space $[SU(2)]^{S+1}/T_{\max} = CP^1(\lambda_1) \times \cdots \times CP^1(\lambda_{S+1})$ with the convenient holomorphic curvatures $\lambda_1, \ldots, \lambda_{S+1}$ (cf. [T],[K - W],[J - K]). Here $\pi : M_\iota^{2S+3} \xrightarrow{S^1} CP^1(\lambda_1) \times \cdots \times CP^1(\lambda_{S+1})$ is a Riemannian submersion. Nevertheles, the corresponding Riemannian-Sasaki metric g is always different from the Einstein metric constructed in this paper.

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1. Preliminaries

Let $su(2)$ denote the Lie algebra of $SU(2)$ provided by the scalar product $B(x,y) = -1/2\mathrm{Re}\, tr\, xy$. We consider an orthonormal basis $\{e_1, e_2, e_3\}$ of $su(2)$ such that $[e_1,e_2] = e_3$, $[e_1,e_3] = -e_2$, $[e_2,e_3] = e_1$. The Lie algebra s^1 of S^1 is of the form $s^1 = R{\cdot}W = R{\cdot}(e_1, \iota_1 e_1, \ldots, \iota_s e_1)$. Put $G = SU(2)\times\cdots\times SU(2)(S{+}1$ times). Consider the scalar product on $g = \mathcal{SU}(2) \oplus \cdots \oplus \mathcal{SU}(2)$ given by $B/g \times g = B_{\mathcal{SU}(2)} + \cdots + B_{\mathcal{SU}(2)}$. Then the Lie algebra h of the Lie group $T^S = H$ has the following form:

$$h = \{Y = (\beta_0 e_1, \beta_1 e_1, \ldots, \beta_S e_1) \in g : \ B(Y,W) = 0, \beta_0, \ldots, \beta_S \in R\}\,, \text{or}$$
$$h = \{Y = (\beta_0 e_1, \beta_1 e_1, \ldots, \beta_S e_1) \in g : \ \beta_0 + \iota_1\beta_1 + \ldots \iota_S\beta_S = 0, \beta_0, \ldots, \beta_S \in R\}\,.$$

Further, we have a B-orthogonal decomposition $g = h \oplus p_0 \oplus p_1 \oplus \cdots \oplus p_{S+1}$, where $p_0 = R\cdot W$, $p_i = R\cdot(0,\ldots,\overset{i}{e_2},\ldots,0) + R\cdot(0,\ldots,\overset{i}{e_3},\ldots,0)$, $\ i = 1,\ldots,S+1$. Morever, $p_0, p_1, \ldots, p_{S+1}$ are irreducible invariant subspaces w. r. to the adjoint representation $ad\,h$ on $p = p_0 \oplus p_1 \oplus \cdots \oplus p_{S+1}$ and $p_0 \not\simeq p_i$, $i \in \{1,\ldots,S+1\}$ w.r. to this representation.

Lemma 1.1. *Under the assumption $|\iota_l| \neq 0,1$, $l \in \{1,\ldots,s\}$, we have $p_i \not\simeq p_j$ for $i \neq j$ with respect to the adjoint representation of h on p.*

Proof. Suppose that there exists an isomorphism $\varphi : p_i \to p_j$ such that $ad\,V \circ \varphi = \varphi \circ ad\,V$ for every $V \in h$. Put $A = (0,\ldots,\overset{i}{e_2},\ldots,0)$, $B = (0,\ldots,\overset{i}{e_3},\ldots,0) \in p_i$. Then we can write $\varphi(A) = (0,\ldots,\overset{j}{x},\ldots,0)$, $\varphi(B) = (0,\ldots,\overset{j}{y},\ldots,0)$, where $x, y \in \mathrm{span}(e_2,e_3)$. For $V = (\beta_0 e_1, \beta_1 e_1, \ldots, \beta_S e_1) \in h$ we have :

$$[V,\varphi(A)] = \varphi([V,A]) = \beta_i\varphi(B), \text{ and also}$$
$$[V,\varphi(B)] = \varphi([V,B]) = -\beta_i\varphi(A).$$

Hence we get $\beta_j[e_1,x] = \beta_i y, \beta_j[e_1,y] = \beta_i x$. Further, since the Lie bracket $[x,y]$ on $\mathcal{SU}(2)$ coincides with the usual vector cross-product, we obtain immediately; $|\beta_j|\cdot\|x\| = |\beta_i|\cdot\|y\|$ and $|\beta_j|\cdot\|y\| = |\beta_i|\cdot\|x\|$. Hence the equality $|\beta_i| = |\beta_j|$ holds. But $|\iota_l| \neq 0,1$ for $l \in \{1`,\ldots,s\}$, therefore there exists an element $V = (\beta_0 e_1, \beta_1 e_1, \ldots, \beta_S e_S) \in h$ such that $\beta_0 + \iota_1\beta_1 + \cdots + \iota_S\beta_S = 0$ and $|\beta_i| \neq |\beta_j|$, which is a contradiction. □

Corollary 1.1. *For $|\iota_l| \neq 0,1$, $l \in \{1,\ldots,s\}$, every $Ad(H)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on p has, up to a constant factor, the following form:*

$$\langle\cdot,\cdot\rangle/_{p\times p} = B \restriction p_0 + y_1 B \restriction p_1 + \cdots + y_{S+1} B \restriction p_{S+1} \tag{1.1}$$

where $y_1, \ldots, y_{S+1} \in R^+$.

The proof follows from the Lemma 1.1 and the Schur's lemma.

We constract a scalar product $(\cdot,\cdot)$ on g by setting $(\cdot,\cdot) = \langle\cdot,\cdot\rangle/_{p\times p} + B/_{h\times h}$. Further, we consider the following $(\cdot,\cdot)$ -orthonormal basis of p:

$$E_0 = W/\|W\| = 1/\|W\| \cdot (e_1, \iota_1 e_1, \ldots, \iota_S e_1) \in p_0,$$
$$E_{2i-1} = 1/\sqrt{y}_i \cdot (0, \ldots, \overset{i}{e_2}, \ldots, 0),\ E_{2i} = 1/\sqrt{y}_i (0, \ldots, \overset{i}{e_3}, \ldots, 0) \in p_i;$$
$$i = 1, \ldots, S+1.$$

By the direct calculation we obtain

$$(1.2) \qquad \begin{cases} [E_0, E_{2i-1}] & = \iota_{i-1}/\|W\| \cdot E_{2i},\ [E_0, E_{2i}] = -\iota_{i-1}/\|W\| \cdot E_{2i-1}, \\ [E_{2i-1}, E_{2i}] & = \iota_{i-1}/y_i \|W\| \cdot E_0 + [E_{2i-1}, E_{2i}]_h \end{cases}$$

$$(1.3) \qquad \begin{cases} [p_i, p_j] & = 0,\ i \neq j,\ i,j \in \{1, \ldots, S+1\} \\ [p_0, p_0] & = 0,\ [p_0, p_i] = p_i,\ [p_i, p_i] \subseteq p_0 \oplus h \end{cases}$$

2. The sectional curvatures of M_ι^{2S+3}

In this part we shall use the notation of the previous part. First, we see that every G-invariant Riemannian metric on M_ι^{2S+3}, for $|\iota_l| \neq 0, 1,\ l \in \{1, \ldots, S\}$, is determined, up to a homothety, by an $Ad(H)$-invariant scalar product of the form (1.1). Further, the sectional curvatures of such metric can be calculated by means of the standard formula (see [B]):

$$(2.1) \qquad \begin{aligned} &\langle R(X,Y)X, Y\rangle = -3/4\|[X,Y]_p\|^2 - 1/2\langle [X,[X,Y]g]_p, Y\rangle - \\ &-1/2\langle [Y,[Y,X]_g]_p, X\rangle + \|U(X,Y)\|^2 - \langle u(X,X), u(Y,Y)\rangle \end{aligned}$$

where $X, Y \in p$, $\langle\cdot,\cdot\rangle$ is the corresponding scalar product on p and the mapping $u : p \times p \to p$ is defined by the formula:

$$(2.2) \qquad 2\langle u(X,Y), Z\rangle = \langle [Z,X]_p, Y\rangle + \langle [Z,Y]_p, X\rangle$$

for all $Z \in p$.

Lemma 2.1. *For an $Ad(H)$-invariant scalar product $\langle\cdot,\cdot\rangle$ of the form (1.1), the following formulas are true:*

$$u(X,Y) = (y_i - 1)/2y_i[X,Y], \text{ where } X \in p_0,\ Y \in p_i,$$
$$u(p_0, p_0) = u(p_i, p_i) = u(p_i, p_j) = 0, \text{ where } i,j \in \{1, \ldots, S+1\}$$

Proof. We shall use Formula (1.3) and (2.2). Let $X \in p_0, Y \in p_i$. If $Z \in p_0$, then $[Z,X]_p = 0$, $[Z,Y]_p \in p_i$ and hence $\langle u(X,Y),\ Z\rangle = 0$. Further, if $Z \in p_j$, $j \neq i$, then $[Z,X]_p \in p_j$, $[Z,Y]_p = 0$ and also $\langle u(X,Y),\ Z\rangle = 0$. Therefore $u(X,Y) \in p_i$. Finally, let $Z \in p_i$, then $[Z,X]_p \in p_i, [Z,Y]_p \in p_0$ and we have

$$2y_i B(u(X,Y), Z) = y_i B([Z,X]_p, Y) + B([Z,Y]_p, X).$$

But

$$\begin{aligned} &y_i B([Z,X]_p,Y)+B([Z,Y]_p,X)= \\ =&y_i B([Z,X],Y)+B([Z,Y],X)= \\ =&y_i B([X,Y],Z)-B([X,Y],Z), \end{aligned}$$

since B is $Ad(G)$-invariant. Hence $u(X,Y)=(y_i-1)/2y_i[X,Y]$ for $X\in p_0, Y\in p_i$. The other formulas of Lemma 2.1 can be proved similarly. □

Lemma 2.2. *For the sectional curvatures of M_ι^{2S+3} we have*

$$\begin{aligned} &K_\sigma(E_0,E_{2i-1})=K_\sigma(E_0,E_{2i})=(\iota_{i-1}/\|W\|2y_i)^2, \\ &K_\sigma(E_{2i-1},E_{2i})=1/y_i-3\iota_{i-1}^2/4y_i^2\|W\|^2, \\ &K_\sigma(X,Y)=0,\ X\in p_i,\ Y\in p_j,\ i\neq j,\ i,j\in 1,\dots,S+1. \end{aligned}$$

Proof. Let us calculate $K_\sigma(E_0,E_{2i-1})$. From the formulas (2.1), (1.2) and Lemma 2.1 we get

$$\begin{aligned} K_\sigma(E_0,E_{2i-1}) &= -\frac{3}{4}\|\frac{\iota_{i-1}}{\|W\|}E_{2i}\|^2-\frac{1}{2}\langle[E_0,\frac{\iota_{i-1}}{\|W\|}E_{2i}]_p,E_{2i-1}\rangle+ \\ &+\frac{1}{2}\langle[E_{2i-1},\frac{\iota_{i-1}}{\|W\|}E_{2i}]_p,E_0\rangle+\|\frac{(y_i-1)}{2y_i}\cdot\frac{\iota_{i-1}}{\|W\|}\cdot E_{2i}\|^2= \\ =-\frac{3\iota_{i-1}^2}{4\|W\|^2}&+\frac{\iota_{i-1}^2}{2\|W\|^2}+\frac{\iota_{i-1}^2}{2\|W\|^2y_i}+\frac{\iota_{i-1}^2}{\|W\|^2}\cdot\frac{(y_i-1)^2}{4y_i^2}=(\frac{\iota_{i-1}}{2\|W\|y_i})^2. \end{aligned}$$

The other sectional curvatures are calculated analogously. □

Corollary 2.1. *For the Ricci curvatures of M_ι^{2S+3} we have*

$$textricc(E_0)=\frac{1}{2}\sum_{i=1}^{s+1}(\frac{\iota_{i-1}}{\|W\|y_i})^2;\ ricc(E_{2i-1})=ricc(E_{2i})=\frac{1}{y_i}-\frac{\iota_{i-1}^2}{2y_i^2\|W\|^2}.$$

The proof follows from Lemma 2.2 by a straightforward compatation.

3. Invariant Einstein metrics on M_ι^{2S+3}

We start with

Lemma 3.1. *Let $\langle\cdot,\cdot\rangle$ be an $Ad(H)$ -invariant scalar product on p of the form (1.1). Then the invariant Einstein metrics on $M_\iota^{2S+3}(|\iota_l|\neq 0,1,\ l\in 1,\dots,S)$ are defined by the following system of quadratic equations*

$$\left\{\begin{aligned} \alpha_i Z_i^2-Z_i &= \alpha_j Z_j^2-Z_j \\ \textstyle\sum_{i=1}^{S+1}\alpha_i Z_i^2 &= Z_j-\alpha_j Z_j^2,\quad i,j\in 1,\dots,S+1 \end{aligned}\right. \tag{3.1}$$

where $\alpha_i = \iota_{i-1}^2/2\|W\|^2$, $Z_i = 1/y_i$.

Proof. Since $H = T^S$ acts transitively on $p_0, p_1, \dots, p_{S+1}$ and preserves the Ricci curvature, then $\langle\cdot,\cdot\rangle$ is Einsteinian iff $\mathrm{ricc}(E_0) = \mathrm{ricc}(E_{2i}) = \mathrm{ricc}(E_{2j})$ for all $i, j \in 1, \dots, S$. Our lemma follows immediately from this fact and Corollary 2.1. □

Further, it is obvious that (3.1) is equivalent to the formulas

$$\begin{cases} \alpha_i Z_i^2 - Z_i + A = 0, & i \in 1, \dots, S+1, \\ \sum_{i=1}^{S+1} \alpha_i Z_i^2 = A & (\alpha_i > 0,\ Z_i > 0,\ A > 0) \end{cases}$$

or

$$\begin{cases} Z_i^{\pm} = (1 \pm \sqrt{1 - 4\alpha_i A})/2\alpha_i\ , & i = 1, \dots, S+1, \\ \sum_{i=1}^{S+1} \alpha_i Z_i^2 = A \qquad (0 < \alpha_i,\ 0 < Z_i, & 0 < A \le 1/4\alpha_i) \end{cases} \tag{3.2}$$

where A is an auxiliary parameter.

But $\alpha_i (Z_i^+)^2 \ge A$ for all $i \in 1, \dots, S+1$, under the assumption $0 < \alpha_i$, $0 < A \le 1/4\alpha_i$. In fact, we have the folowing sequence of inequalities:

$$\begin{gathered} \sqrt{1 - 4\alpha_i A} \ge 0 \ge 4\alpha_i A - 1\ , \\ 1 + 2\sqrt{1 - 4\alpha_i A} + (1 - 4\alpha_i A) \ge 4\alpha_i A\ , \\ (1 + \sqrt{1 - 4\alpha_i A})^2 \ge 4\alpha_i A \ge 0\ , \\ 1 + \sqrt{1 - 4\alpha_i A} \ge 2\alpha_i \sqrt{A/A_i}\ , \end{gathered}$$

i. e., $Z_i^+ \ge \sqrt{A/\alpha_i}$ and hence $\alpha_i (Z_i^+)^2 \ge A$. On the other hand, $\alpha_j (Z_j^-)^2 > 0$ for all $j = 1, \dots, S+1$. Because $S + 1 \ge 2$, the second equality of (3.2) shows that no Z_i^+ can be a part of any solution $(Z_1, \dots, Z_{S+1})$ of (3.1). Hence (3.1), or (3.2), reduces to the formulas

$$\begin{cases} Z_i = (1 - \sqrt{1 - 4\alpha_i A})/2\alpha_i, \quad i = 1, \dots, S+1, \\ \sum_{i=1}^{S+1} Z_i = (S+2)A, \quad (0 < \alpha_i,\ 0 < Z_i, \qquad 0 < A \le 1/4\alpha_i) \end{cases} \tag{3.3}$$

Finally, from (3.3) we see that the problem to find all invariant Einstein metrics of M_ι^{2S+3} is equivalent to the problem of finding all positive zeros of the function:

$$f(A) = \sum_{i=1}^{S+1} (1 - \sqrt{1 - 4\alpha_i A})/2\alpha_i - (S+2)A\ ,$$

under the assumptions $0 < \alpha_i$, $0 < A \le 1/4\alpha_{\max}$, where

$$\alpha_{\max} = \max\{\alpha_1, \dots, \alpha_{S+1}\}\,.$$

Lemma 3.2. *The function $f(A)$ has exactly one zero in the interval* $]0,1/4\alpha_{\max}[$.

Proof. In the first place we have

$$f'(A)=\sum_{i=1}^{S+1}1/\sqrt{1-4\alpha_iA}-(S+2),$$

$$f''(A)=\sum_{i=1}^{S+1}2\alpha_i/\sqrt{(1-4\alpha_iA)^3}>0 \quad \text{for} \quad A\in]0,1/4\alpha_{\max}[.$$

Further, $f'(A)=-1$ and also

$$\lim_{A\to 1/4\alpha_{\max}-0}f'(A)=\lim_{A\to 1/4\alpha_{\max}-0}(\frac{1}{\sqrt{1-4\alpha_{\max}A}}-(S+2))=+\infty.$$

Finally, $f(0)=0$. Let us prove that $f(1/4\alpha_{\max})>0$. Indeed, we have

$$\begin{aligned}
f(1/4\alpha_{\max})&=\sum_{i=1}^{S+1}\frac{1-\sqrt{1-\alpha_i/\alpha_{\max}}}{2\alpha_i}-\frac{(S+2)}{4\alpha_{\max}}=\\
&=\sum_{i=1}^{S+1}\frac{1-1+\alpha_i/\alpha_{\max}}{2\alpha_i(1+\sqrt{1-\alpha_i/\alpha_{\max}})}-\frac{(S+2)}{4\alpha_{\max}}=\\
&=\sum_{i=1}^{S+1}\frac{1}{2\alpha_{\max}(1+\sqrt{1-\alpha_i/\alpha_{\max}})}-\frac{(S+2)}{4\alpha_{\max}}=\\
&=\sum_{\substack{i=1\\ i\neq\max}}^{S+1}\frac{1}{2\alpha_{\max}(1+\sqrt{1-\alpha_i/\alpha_{\max}})}+\frac{1}{2\alpha_{\max}}-\frac{S}{4\alpha_{\max}}-\frac{1}{2\alpha_{\max}}>\\
&>\frac{S}{4\alpha_{\max}}-\frac{S}{4\alpha_{\max}}=0\ .
\end{aligned}$$

The assertion of our Lemma follows now from the above fact above the behavior of $f(A)$, $f'(A)$, $f''(A)$ in the segment $[0,1/4\alpha_{\max}]$. □

Hence we obtain the folowing.

Theorem 3.1. *For each $\iota=(1,\iota_1,\dots,\iota_S)$ $(\iota_i\in Q,|\iota_i|\neq 0,1,\ i\in 1,\dots,S)$ there exists, up to a homothety, a unique invariant Einstein metric on M_ι^{2S+3} $(S\geq 1)$.*

Remark. For $S=1$ the solution can be calculated from a cubic equation (see the previous paper [R], which was devoted to this special case).

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E. D. Rodionov
Altai State University
Dimitrov Street 66
656099 Barnaul, USSR