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ON A GENERAL SOLUTION OF FINITE ORDER DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

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Abstract. In the present paper we give new formulas for a general solution of the linear difference equation of finite order with constant complex coefficients without necessity of solving the characteristic equation

Introduction. In this paper we deal with the following difference equation of order m:

$$x_{n+m} = \sum_{r=1}^{m} a_r x_{n+m-r} \tag{1}$$

with constant complex coefficients $a_1, \dots, a_m$. Our Theorem gives a simple formula for the general solution depending only on the coefficients $a_1, \dots, a_m$. We do not have to solve the characteristic equation as it is usually done (cf for instance [1], [2]) and, in general, it is often impossible to find the exact solutions of it.

To formulate our Theorem we adopt the following convention:

$$(-1)! \cdot 0 = 1 \ .$$

Theorem. *Let $x_0, \dots, x_{m-1}$ be arbitrary complex numbers, let $h_1, \dots, h_m$ be nonnegative integers. The general solution of equation (1) is of the form (2):*

$$x_n = \sum_{l=0}^{m-1} \left[\sum_{1h_1+\dots+mh_m=n-l} \frac{(h_1+\dots+h_m-1)!(h_{m-l}+\dots+h_m)}{h_1! \cdot \ldots \cdot h_m!} \prod_{i=1}^{m} a_i^{h_i} \right] x_l$$

for $n = 0, 1, \dots$.

Proof of the Theorem. The proof is by induction with respect to n.

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In the first part we show that formula (2) holds for $n = 0, \dots, m-1$. Let us take $l \in \{0, \dots, m-1\}$ and consider three cases depending on l is smaller, greater than n or equal to n.

In the case $l < n$ we have $1h_1 + \dots + mh_m = n - l > 0$, and therefore $h_1 + \dots + h_m - 1 \geq 0$. Let us note that $h_{m-l} + \dots + h_m = 0$. In fact, if $h_{m-l} + \dots + h_m > 0$ then would exist $i \in \{m-l, \dots, m\}$ such that $h_i > 0$. Consequently we would have $1h_1 + \dots + mh_m \geq ih_i \geq (m-l) > n-l$; which is a contradiction. Therefore we have:

$$\frac{(h_1 + \dots + h_m - 1)!(h_{m-l} + \dots + h_m)}{h_1! \cdot \ldots \cdot h_m!} = 0 \ .$$

In the case $l > n$ we have $1h_1 + \dots + mh_m = n - l < 0$. Since the set of all such sequences $(h_1, \dots, h_m)$ is empty, the sum over this set of indices is 0.

For $l = n$ we have $1h_1 + \dots + mh_m = n - l = 0$. Consequently $h_1 = \dots = h_m = 0$ and, applying our convention, we get:

$$\frac{(h_1 + \dots + h_m - 1)!(h_{m-l} + \dots + h_m)}{h_1! \cdot \ldots \cdot h_m!} = \frac{(-1)!0}{1 \cdot \ldots \cdot 1} = 1 \ .$$

Summing up this three cases we can observe that formula (2) holds for $n = 0, \dots, m-1$.

Now, for an inductive step, we assume that Theorem is true for m consecutive indices $n, \dots, n+m-1$.

Substituting the right hand side of formula (2) into equality (1) (we change simultaneously n for $n+m-r$ in formula (2)) and changing the order of sumation we get:

$$\begin{aligned} x_{n+m} &= \sum_{r=1}^{m} a_r \sum_{l=0}^{m-1} \Bigg[\sum_{1h_1+\dots+mh_m=n-l} \cdot \\ &\cdot \frac{(h_1 + \dots + h_m - 1)!(h_{m-l} + \dots + h_m)}{h_1! \cdot \ldots \cdot h_m!} \prod_{i=1}^{m} a_i^{h_i} \Bigg] x_l = \\ &= \sum_{l=0}^{m-1} \Bigg[\sum_{r=1}^{m} a_r \sum_{1h_1+\dots+mh_m=n+m-r-l} \cdot \\ &\cdot \frac{(h_1 + \dots + h_m - 1)!(h_{m-l} + \dots + h_m)}{h_1! \cdot \ldots \cdot h_m!} \prod_{i=1}^{m} a_i^{h_i} \Bigg] x_l \ . \end{aligned} \tag{3}$$

Let us fix $l \in \{0, \dots, m-1\}$, and consider the coefficient standing before x_l. Performing the indicated operations we obtain that this coefficient is equal to:

$$\sum_{1g_1+\dots+mg_m=n+m-l} c_{g_1,\dots,g_m} \prod_{i=1}^{m} a_i^{g_i} \ . \tag{4}$$

where every $c_{g_1,\dots,g_m}$ is uniquely defined coefficient. We will determine the value of it depending on $g_1,\dots,g_m$. Let us fix $g_1,\dots,g_m$.

Since we get $\prod_{i=1}^m a_i^{g_i}$ as a product of admissible a_r (i.e., such that $g_r > 0$) and suitable uniquely defined $\prod_{i=1}^m a_i^{h_i}$:

$$g_i = \begin{cases} h_i & \text{for} \quad i = 1,\dots,r-1,r+1,\dots,m, \\ h_i + 1 & \text{for} \quad i = r. \end{cases} \tag{5}$$

Therefore to obtain the coefficient $c_{g_1,\dots,g_m}$ it is enough to add all the coefficients of the form:

$$\frac{(h_1+\dots+h_m-1)!(h_{m-l}+\dots+h_m)}{h_1!\cdot\ldots\cdot h_m!}$$

standing before admissible $\prod_{i=1}^m a_i^{h_i}$.

Let us put:

$$P(i,j) := \{r : g_r > 0\} \cap \{i,\dots,j\}\ .$$

We have:

$$\begin{aligned} c_{g_1,\dots,g_m} &= \sum_{P(1,m)} \frac{(h_1+\dots+h_m-1)!(h_{m-l}+\dots+h_m)}{h_1!\cdot\ldots\cdot h_m!} = \\ &= \sum_{P(1,m-l-1)} \frac{(h_1+\dots+h_m-1)!(h_{m-l}+\dots+h_m)}{h_1!\cdot\ldots\cdot h_m!} + \\ &+ \sum_{P(m-l,m)} \frac{(h_1+\dots+h_m-1)!(h_{m-l}+\dots+h_m)}{h_1!\cdot\ldots\cdot h_m!}\ . \end{aligned}$$

Because if $r \in \{i,\dots,j\} - P(i,j)$ then $g_r = 0$, it follows that $\sum_{P(i,j)} g_r = \sum_{r=i}^{j} g_r$. Applying formula (5), hence we get:

$$\begin{aligned} c_{g_1,\dots,g_m} &= \sum_{P(1,m-l-1)} \frac{(g_1+\dots+(g_r-1)+\dots+g_m-1)!(g_{m-l}+\dots+g_m)}{g_1!\cdot\ldots\cdot(g_r-1)!\cdot\ldots\cdot g_m!} + \\ &+ \sum_{P(m-l,m)} \frac{(g_1+\dots+(g_r-1)+\dots+g_m-1)!(g_{m-l}+\dots+(g_r-1)+\dots+g_m)}{g_1!\cdot\ldots\cdot(g_r-1)!\cdot\ldots\cdot g_m!} = \\ &= \sum_{P(1,m-l-1)} g_r \frac{(g_1+\dots+g_m-1)!(g_{m-l}+\dots+g_m)}{(g_1+\dots+g_m-1)\cdot g_1!\cdot\ldots\cdot g_m!} + \\ &+ \sum_{P(m-l,m)} g_r \frac{(g_1+\dots+g_m-1)!(g_{m-l}+\dots+g_m-1)}{(g_1+\dots+g_m-1)\cdot g_1!\cdot\ldots\cdot g_m!} = \\ &= \left[\sum_{r=1}^{m-l-1} \frac{g_r}{(g_1+\dots+g_m-1)} + \sum_{r=m-l}^{m} \frac{g_r(g_{m-l}+\dots+g_m-1)}{(g_1+\dots+g_m-1)(g_{m-l}+\dots+g_m)}\right]\cdot \\ &\cdot \left[\frac{(g_1+\dots+g_m-1)!(g_{m-l}+\dots+g_m)}{g_1!\cdot\ldots\cdot g_m!}\right] = \end{aligned}$$

$$= \left[\frac{(g_1 + \cdots + g_{m-l-1})}{(g_1 + \cdots + g_m - 1)} + \frac{(g_{m-l} + \cdots + g_m)(g_{m-l} + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1)(g_{m-l} + \cdots + g_m)}\right] \cdot$$
$$\cdot \left[\frac{(g_1 + \cdots + g_m - 1)!(g_{m-l} + \cdots + g_m)}{g_1! \cdot \ldots \cdot g_m!}\right] =$$
$$= \left[\frac{(g_1 + \cdots + g_{m-l-1}) + (g_{m-l} + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1)}\right] \cdot$$
$$\cdot \left[\frac{(g_1 + \cdots + g_m - 1)!(g_{m-l} + \cdots + g_m)}{g_1! \ldots g_m!}\right] =$$
$$= \frac{(g_1 + \cdots + g_m - 1)!(g_{m-l} + \cdots + g_m)}{g_1! \cdot \ldots \cdot g_m!} .$$

Inserting this into (4) and then into (3) we obtain:

$$x_{n+m} = \sum_{l=0}^{m-1} \left[\sum_{1g_1 + \cdots + mg_m = n-l} \frac{(g_1 + \cdots + g_m - 1)!(g_{m-l} + \cdots + g_m)}{g_1! \cdot \ldots \cdot g_m!} \prod_{i=1}^{m} a_i^{g_i} \right] x_l .$$

Now induction concludes the proof. □

Remark 1. It may be interesting to note here that formula (2) can be written as follows:

$$x_n = \sum_{l=0}^{m-1} \left[\sum_{\{k_1, \ldots, k_s \in \{1, \ldots, m\} : k_1 + \cdots + k_s = n-l,\ k_s \geq m-l,\ s \in \mathbf{N}\}} \prod_{i=1}^{s} a_{k_i} \right] x_l .$$

Remark 2. Theorem can be proved by some combinatorial reasoning.

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References

[1] Gelfond, A.O., *Isčislenije kanečnych raznostěj*, Moskva, 1952.
[2] Levy, H., Lessman, F., *Finite difference equations*, London, 1959.

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