ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 205 – 213

EXISTENCE OF SOLUTIONS FOR HYPERBOLIC DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Abstract. In this paper we examine nonlinear hyperbolic inclusions in Banach spaces. With the aid of a compactness condition involving the ball measure of noncompactness we prove two existence theorems. The first for problems with convex valued orientor fields and the second for problems with nonconvex valued ones.

1. Introduction

In this paper we study the existence of solutions for hyperbolic differential inclusions (Darboux problems) defined in a separable Banach space. Using a compactness type condition involving the ball (Hausdorff) measure of noncompactness, we are able to obtain two existence theorems. One when the orientor field is convex valued and the other when it is nonconvex valued. The single valued finite dimensional version of the problem was considered by DeBlasi-Myjak [4], who also established the topological regularity of the solutions set. The single valued, infinite dimensional version of the problem was examined by Kubiaczyk [9], who proved a Kneser-type theorem for the solution set.

2. Preliminaries

Let (Ω, Σ) be a measurable space and V a separable Banach space. Throughout this paper we will be using the following notations:

$$P_{f(c)}(V) = \{A \subseteq V : \text{ nonempty, closed, (convex)}\}$$
$$\text{and } \; P_{(w)k(c)}(V) = \{A \subseteq V : \text{ nonempty, (weakly-) compact, (convex)}\}\,.$$

A multifunction $F : \Omega \to P_f(V)$ is said to be measurable if for all $y \in V$, the $\mathbf{R}_+$-valued function $\omega \to d(y, F(\omega)) = \inf\{\|y - x\| : x \in F(\omega)\}$ is measurable. In fact this definition of measurability of multifunctions is equivalent to saying that

1991 *Mathematics Subject Classification*: 34G20.
Key words and phrases: hyperbolic inclusion, measure of noncompactness, measurable multifunction, upper and lower semicontinuous multifunctions, fixed point.
Received August 27, 1991.

there exists a sequence $f_n : \Omega \to V$, $n \geq 1$, of measurable functions s.t. for all $\omega \in \Omega$, $F(\omega) = \overline{\{f_n(\omega)\}_{n\geq 1}}$ (see Wagner [14], theorem 4.2). The multifunction $F(\cdot)$ is said to be weakly (or scalarly) measurable, if for every $x^* \in V^*$, the $\bar{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$-valued function $\omega \to \sigma(x^*, F(\omega)) = \sup\{(x^*, x) : x \in F(\omega)\}$ is measurable. It is clear from the above definitions that measurability implies weak measurability. Indeed let $f_n : \Omega \to V$, $n \geq 1$, be measurable functions s.t. $F(\omega) = \overline{\{f_n(\omega)\}_{n\geq 1}}$. Then $\sigma(x^*, F(\omega)) = \sup_{n\geq 1}(x^*, f_n(\omega)) \Rightarrow \omega \to \sigma(x^*, F(\omega))$ is measurable $\Rightarrow F(\cdot)$ is weakly measurable. The converse is not in general true. However, if there is a σ-finite measure $\mu(\cdot)$ defined on Σ, Σ is μ-complete and $F(\cdot)$ is $P_{wkc}(V)$-valued, then weak measurability implies measurability. To see this, let $\{x_n^*\}_{n\geq 1}$ be a sequence which is dense in V^* for the Mackey topology $m(V^*, V)$. Such a sequence exists since V is separable (see Wilansky [15], p. 144). Because $F(\cdot)$ is $P_{wkc}(V)$-valued $\sigma(\cdot, F(\omega))$ is $m(V^*, V)$-continuous and so we have

$$F(\omega) = \bigcap_{n\geq 1} \{y \in V : (x_n^*, y) \leq \sigma(x_n^*, F(\omega))\}$$

$$\Rightarrow GrF = \{(\omega, y) \in \Omega \times V : y \in F(\omega)\} = \bigcap_{n\geq 1} \{(\omega, y) : (x_n^*, y) \leq \sigma(x_n^*, F(\omega))\} \in \Sigma \times B(V)$$

with $B(V)$ being the Borel σ-field of V. Since Σ is μ-complete, from theorem 4.2 of Wagner [4], we deduce that $F(\cdot)$ is indeed measurable.

Let $\mathcal{B}$ be the family of bounded subsets of V. Then the ball (Hausdorff) measure of noncompactness $\beta : \mathcal{B} \to \mathbf{R}_+$ is defined by

$$\beta(B) = \inf\{r > 0 : B \text{ can be covered finitely many balls of radius } r\}.$$

So a set $A \in \mathcal{B}$ is relatively compact if and only if $\beta(A) = 0$. For a detailed analysis of the properties of $\beta(\cdot)$ (and of more general measures of noncompactness), we refer to the book of Banas-Goebel [1].

Let Y, Z be Hausdorff topological spaces and $G : Y \to 2^Z \setminus \{\emptyset\}$ a multifunction. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if for every open set $U \subseteq Z$, we have $G^+(U) = \{y \in Y : G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y : G(y) \cap U \neq \emptyset\}$) is open in Y. For other equivalent definitions and for further properties we refer to the book by Klein-Thompson [8].

3. Existence theorems

Let $Q = [0, r] \times [0, r]$ and X a separable Banach space. Let B be the Banach space defined by $B = \{(\eta, \theta) \in C(T, X) \times C(T, X) : \eta(0) = \theta(0)\}$, where $T = [0, r]$ and with norm $\|(\eta, \theta)\| = \|\eta\|_{C(T,X)} + \|\theta\|_{C(T,X)}$.

Given $(\eta, \theta) \in B$, consider the following multivalued hyperbolic Cauchy problem (Darboux problem):

$$(*) \qquad \left\{ \begin{array}{l} \frac{\partial^2 v(x,y)}{\partial x\, \partial y} \in F(x, y, v(x, y)) a.e \\ v(x, 0) = \eta(x),\ v(0, y) = \theta(y)\,. \end{array} \right\}$$

By a solution of $(*)$, we mean a function $v(\cdot,\cdot) \in C(Q,X)$ s.t. there exists $f \in L^1(Q,X)$ for which we have

$$v(x,y) = z_0(x,y) + \int_0^z \int_0^y f(t,s)\, dt\, ds$$

for all $(x,y) \in Q$ and with $f(t,s) \in F(t,s,v(t,s)) a.e.$ and $z_0(x,y) = \eta(x) + \theta(y) - \eta(0)$.

Our first existence result deals with the case where the orientor field $F(t,x)$ is convex valued. So our hypothesis on $F(t,x)$ is the following:

$\underline{H(F)_1}$: $F : Q \times X \to P_{fc(X)}$ is a multifunction s.t.

($\underline{1}$) $(x,y,v) \to F(x,y,v)$ is weakly measurable,

($\underline{2}$) $v \to F(x,y,v)$ is u.s.c. from X into X_w (here X_w denotes the Banach space X equipped with the weak topology)

($\underline{3}$) $|F(x,y,v)| = \sup\{\|z\| : z \in F(x,y,v)\|] \le a(x,y) + b(x,y)\|v\| \quad a.e.$, with $a, b \in L^1_+(Q)$,

($\underline{4}$) for every $B \subseteq X$ nonempty and bounded set we have

$$\beta(F(x,y,B)) \le k(x,y)\beta(B)\, a.e.$$

with $k \in L^\infty_+(Q)$,

($\underline{5}$) for all $(x,y) \in Q$, $F(x,y,\cdot)$ maps bounded sets into relatively weakly compact sets.

Remark. Note that hypothesis $H(F)_1$ ($\underline{4}$) implies that the orientor field $F(x,y,\cdot)$ is $P_{kc}(X)$-valued for almost all $(x,y) \in Q$. To see this, let $B = \{v\}$. Then $\beta(B) = 0$ and so $\beta(F(x,y,v)) = 0$ for almost all $(x,y) \in Q \Rightarrow F(x,y,v) \in P_{kc}(X)$ for almost all $(x,y) \in Q$. Also note that hypothesis $H(F)_1$ ($\underline{5}$) is automatically satisfied if X is reflexive.

Theorem 3.1. *If hypothesis $H(F)_1$ holds, then $(*)$ admits a solution.*

Proof. First we will obtain an a priori bound for the solutions of problem $(*)$. So let $v(\cdot,\cdot) \in C(Q,X)$ be such a solution. We have

$$v(x,y) = z_0(x,y) + \int_0^x \int_0^y f(t,s)\, dt\, ds$$

for all $(x,y) \in Q$ and with $f \in L^1(Q,X)$, $f(t,s) \in F(t,s,v(t,s))\, a.e.$ Hence

$$\begin{aligned}
\|v(x,y)\| &\le \|z_0(x,y)\| + \int_0^x \int_0^y \|f(t,s)\|\, dt\, ds \\
&\le \|z_0(x,y)\| + \int_0^x \int_0^y (a(t,x) + b(t,s)\|v(t,s)\|)\, dt\, ds \\
&\le \|z_0\|_{C(Q,X)} + \|a\|_{L^1(Q)} + \int_0^x \int_0^y b(t,s)\|v(t,s)\|\, dt\, ds\,.
\end{aligned}$$

Invoking the Wendroff-Gronwall inequality (see for example Beckenback-Bellman [2]), we get that

$$\|v(x,y)\| \leq [\|z_0\|_{C(Q,X)} + \|a\|_{L^1(Q)}] \exp \|b\|_{L^1(Q)} = M_1 \, .$$

Then let $\hat{F} : Q \times X \to P_{fc}(X)$ be defined by

$$\hat{F}(x,y,v) = \begin{cases} F(x,y,v) \text{ if } \|v\| \leq M_1, \\ F(x,y,\frac{M_1 v}{\|v\|}) \text{ if } \|v\| > M_1 \, . \end{cases}$$

Note that $\hat{F}(x,y,v) = F(x,y,p_{M_1}(v))$, where $p_{M_1}(\cdot)$ is the M_1-radial retraction in X. Recalling that $p_{M_1}(\cdot)$ is Lipschitz continuous, we deduce that $(x,y,v) \to \hat{F}(x,y,v)$ is weakly measurable (see hypothesis $H(F)_1(\underline{1})$ and the definition of $\hat{F}$), while theorem 7.3.11, p. 87 of Klein-Thompson [8] tells us that $v \to \hat{F}(x,y,v)$ is u.s.c. from X into X_w. Also if $B \subseteq X$ is nonempty and bounded, then by using hypothesis $H(F)_1$ $(\underline{4})$ we have

$$\beta(\hat{F}(x,y,B)) = \beta(F(x,y,p_{M_1}(B))) \leq k(x,y)\beta(p_{M_1}(B))\text{a.e.}$$

Note that $p_{M_1}(B) \subseteq \overline{\text{conv}}\,(B \cup \{0\})$. Using the properties of $\beta(\cdot)$, we get:

$$\beta(p_{M_1}(B)) \leq \beta(\overline{\text{conv}}\,(B \cup \{0\})) \leq \beta(B).$$

Hence we have

$$\beta(\hat{F}(x,y,B)) \leq k(x,y)\beta(B) \ a.e.$$

Finally note that

$$|\hat{F}(x,y,v) = \sup\{\|z\| : z \in \hat{F}(x,y,v)\} \leq a(x,y) + b(x,y)M_1 = \varphi(x,y) \ a.e.$$

with $\varphi(\cdot,\cdot) \in L^1_+(Q)$. Let $W = \{v \in C(Q,X) : v(x,y) = z_0(x,y) + \int_0^x \int_0^y g(t,s)\,dt\,ds$, $\|g(t,s)\| \leq \varphi(t,s) \ a.e\}$. Then let $T : W \to 2^W$ be the multifunction defined by

$$T(v) = \{w \in C(Q,X) : w(x,y) = z_0(x,y) + \int_0^x \int_0^y f(t,s)\,dt\,ds,$$
$$f \in L^1(Q,X), \ f(t,s) \in \hat{F}(t,s,v(t,s)) \ a.e.\}.$$

Note that since $\hat{F}(t,s,v)$ is weakly measurable, $(t,s) \to \hat{F}(t,s,v(t,s))$ is weakly measurable on $O, T \times T$ with the Lebesgue σ-field, which is complete with respect to the Lebesgue measure on Q. So $(t,s) \to \hat{F}(t,s,v(t,s))$ is measurable (see section 2) and thus by Aumann's selection theorem (see Wagner [14], theorem 5.10) we get that there exist $f \in L^1(Q,X) s.t.\ f(t,s) \in \hat{F}(t,s,v(t,s)) \ a.e. \ \Rightarrow T(\cdot)$ has nonempty values. Furthermore since the set $S^1_{\hat{F}(\cdot,\cdot,v(\cdot,\cdot))} = \{g \in L^1(Q,X) : g(t,s) \in$

$\hat{F}(t,s,v(t,s))\} \in P_{wkc}(L^1(Q,X))$ (see [11], proposition 3.1), we can easily check that $T(\cdot,\cdot)$ has values in $P_{fc(C(Q,X))}$. Let $B \subseteq W$ be a nonempty set. We have:

$$\beta(T(B)(x,y)) \le \beta[\int_0^x \int_0^y f(t,s)\,dt\,ds : f \in S^1_{\hat{F}(\cdot,\cdot,v(\cdot,\cdot))}, \qquad v \in B]$$
$$\le \beta[\int_0^x \int_0^y \hat{F}(t,s,\overline{B(t,s)})\,dt\,ds]$$

where $\overline{B(t,s)} = \overline{\{v(t,s) : v \in B\}}$ and $\int_0^x \int_0^y \hat{F}(t,s,\overline{B(t,s)}\,dt\,ds =$
$= \{\int_0^x \int_0^y h(t,s)\,dt\,ds\colon h \in \mathrm{L}^1(Q,X),\ h(t,s) \in \hat{F}(t,s,\overline{B(t,s)})\,a.e.\}$. For every $x^* \in X^*$, we have

$$\sigma(x^*, \hat{F}(t,s,B(t.s)) = \sigma(x^*, \bigcup_{w \in \overline{B(t,s)}} \hat{F}(t,s,w)) = sup_{w \in \overline{B(t,s)}} \sigma(x^*, \hat{F}(t,s,w)).$$

Observe $(t,s,v) \to \sigma(x^*, \hat{F}(t,s,v))$ is measurable and clearly $(t,s) \to \overline{B(t,s)}$ is a graph measurable (i.e. $GrB(\cdot,\cdot) = \{(t,s,w) \in Q \times X : w \in \overline{B(t,s)}\} \in B(Q) \times B(X)$), with $B(Q)$ being the Borel σ-field of Q (note that $B(Q) = B(T) \times B(T)$) and $B(X)$ the Borel σ-field of X. So from theorem 6.1 of Kandilakis-Papageorgiou [7], we deduce that $(t,s) \to sup[\sigma(x^*, \hat{F}(t,s,w)) : w \in \overline{B(t,s)}]$ is Lebesgue measurable on Q (i.e. measurable for the completion of $B(Q) = B(T) \times B(T)$ with respect to the Lebesgue measure on $Q \subseteq \mathbf{R}^2$, which incidentally is bigger than $\overline{B(T)} \times \overline{B(T)}$, where $\overline{B(T)}$ =Lebesgue completion of $B(T)$; see Hewitt-Stromberg [5], p. 392). Hence $(t,s) \to \overline{\mathrm{conv}}\hat{F}(t,s,B(t,s)) = H(t,s) \in P_{wkc}(X)$ (see hypothesis $H(F)_1(\underline{5})$) is Lebesgue measurable on Q. Let $h_n : Q \to X, n \ge 1$, be Lebesgue measurable functions s.t. $H(t,s) = \overline{\{h_n(t,s)\}}_{n \ge 1}$ for all $(t,s) \in Q$ (see section 2). Then we have

$$\beta[\int_0^x \int_0^y \hat{F}(t,s))\,dt\,ds]$$
$$\le \beta[\int_0^x \int_0^y \overline{\{h_n(t,s)\}}_{n \ge 1}\,dt\,ds]$$
$$= \beta[\int_0^x \int_0^y \{h_n(t,s)\}_{n \ge 1}\,dt\,ds]$$

(see Kandilakis-Papageorgiou [6], theorem 3.1 and recall the properties of $\beta(\cdot)$)

$$\le \int_0^x \int_0^y \beta[h_n(t,s) : n \ge 1]\,dt\,ds \text{ (see Mönch [10], proposition 1.6)}$$
$$\le \int_0^x \int_0^y k(t,s)\beta\overline{(B(t,s))}\,dt\,ds = \int_0^x \int_0^y k(t,s)\beta(B(t,s))\,dt\,ds.$$

So we have

$$\beta(T(B))(x,y)) \le \int_0^x \int_0^y k(t,s)\beta(B(t,s))\,dt\,ds \le \int_0^x \int_0^y \|k\|_\infty \beta(B(t,s))\,dt\,ds.$$

Set $\psi(B) = \sup\limits_{(x,y)\in Q} [e^{-\lambda\|k\|_\infty^{1/2}(t+s)}\beta(B(t,s))]$ for every $B \subseteq W$ and with $\lambda > 0$. Since $W \subseteq C(T,X)$ is equicontinuous, bounded and exploiting the properties of $\beta(\cdot)$ and the Arzela-Ascoli theorem, we can easily check that $\psi(\cdot)$ is a sublinear measure of noncompactness in the sense of Banas-Goebel [1]. Then we have

$$\begin{aligned}\beta(T(B)(x,y)) &\leq \int_0^x \int_0^y \|k\|_\infty e^{\lambda\|k\|_\infty^{1/2}(t+s)} e^{-\lambda\|k\|_\infty^{1/2}(t+s)}\beta(B(t,s))\,dt\,ds \\ &\leq \int_0^x \int_0^y \|k\|_\infty e^{\lambda\|k\|_\infty^{1/2}(t+s)}\psi(B)\,dt\,ds \\ &\Rightarrow \beta(T(B)(x,y)) \leq \frac{\psi(\beta)}{\lambda^2}\, e^{\lambda\|k\|_\infty^{1/2}(x+y)} \\ &\Rightarrow \psi(T(B)) \leq \frac{1}{\lambda^2}\psi(B).\end{aligned}$$

Let $\lambda > 1$. Then we have that $T(\cdot)$ is a $\psi(\cdot)$-contraction.

Next we will show that the multifunction $T(\cdot)$ has a closed graph (i.e. $GrT = \{(v,w) \in W \times W : w \in T(v)\}$ is closed in $C(Q,X) \times C(Q,X)$). To this end let $(v_n, w_n) \in GrT$, $n \geq 1$ and assume that $(v_n, w_n) \to (v,w)$ in $W \times W \subseteq C(Q,X) \times C(Q,X)$. Then by definition, we have

$$w_n(x,y) = z_0(x,y) + \int_0^x \int_0^y g_n(t,s)\,dt\,ds$$

for all $(x,y) \in Q$ and with $g_n \in L^1(Q,X)$,$g_n(t,s) \in \hat{F}(t,s,v_n(t,s))$ $a.e.$ Since $\hat{F}(t,s,\cdot)$ is $u.s.c.$ from X into X_w, with values in $P_{kc}(X)$, using theorem 7.4.2, p. 90 of Klein-Thompson [8], we get that $(t.s) \to \overline{\text{conv}} \bigcup\limits_{n\geq 1} \hat{F}(t,s,v_n(t,s)) = G(t,s)$, is a measurable, $P_{wkc}(X)$-valued multifunction $s.t.$ $|G(t,s)| = \sup\{\|y\| : y \in G(t,s)\} \leq \varphi(t,s)$ $a.e.$ So from proposition 3.1 of [11], we have that $S_G^1 = \{g \in L^1(Q,X) : g(t,s) \in G(t,s)\ a.e.\}$ is weakly compact in the Lebesgue-Bochner space $L^1(Q,X)$. Thus by passing to a subsequence if necessary, we may assume that $g_n \xrightarrow{w} g$ in $L^1(Q,X)$. Invoking theorem 3.1 of [12], we get

$$\begin{aligned} g(t,s) &\in \overline{\text{conv}}\ w\text{-}\overline{\lim}\{g_n(t,s)\}_{n\geq 1} \\ &\subseteq \overline{\text{conv}}\ w\text{-}\overline{\lim}\hat{F}(t,s,v_n(t,s)) \\ &\subseteq \hat{F}(t,s,v(t,s))\ a.e.\end{aligned}$$

the last inclusion following from the fact that $\hat{F}(t,s,\cdot)$ is u.s.c. from X into X_w with values in $P_{fc}(X)$, and since $v_n \to v$ in $C(Q,X)$. So in the limit as $n \to \infty$, we have for all $(x,y) \in Q$

$$w(x,y) = z_0(x,y) + \int_0^x \int_0^y g(t,s)\,dt\,ds$$

with $g \in L^1(Q,X)$, $g(t,s) \in \hat{F}(t,s,v(t,s))$ *a.e.* So $(v,w) \in GrT \Rightarrow T(\cdot)$ has a closed graph in $W \times W \subseteq C(Q,X) \times C(Q,X)$. Apply theorem 4.1 of Tarafdar-Vyborny [13], to get $v \in T(v)$. As in the beginning of the proof using the definition of $\hat{F}(x,y,v)$ and the Wendroff-Gronwall inequality, we can check that $\|v\|_{C(Q,X)} \leq M_1 \Rightarrow \hat{F}(x,y,v(x,y)) = F(x,y,v(x,y))$, $(x,y) \in Q \Rightarrow v \in C(Q,X)$ is the desired solution of $(*)$. □

We can also prove a "nonconvex" analog of theorem 3.1. For this we will need the following hypothesis on the orientor field $F(x,y,v)$:

$\underline{H(F)_2}$: $F : Q \times X \to P_f(X)$ is a multifunction *s.t.*

($\underline{1}$) $(x,y,v) \to F(x,y,v)$ is measurable,

($\underline{2}$) $v \to F(x,y,v)$ is *l.s.c.*,

($\underline{3}$) $|F(x,y,v)| = \sup\{\|z\| : z \in F(x,y,v)\} \leq a(x,y) + b(x,y)\|v\|$ *a.e.* with $a, b \in L^1_+(Q)$,

($\underline{4}$) for all $B \subseteq X$ nonempty, bounded, we have

$$\beta(F(x,y,B)) \leq k(x,y)\beta(B) \ a.e.$$

with $k(\cdot,\cdot) \in L^\infty_+(Q)$,

($\underline{5}$) for all $(x,y) \in Q$, $F(x,y,\cdot)$ maps bounded sets into relatively weakly compact sets.

Remark. Again hypothesis $H(F)_2\underline{(4)}$ above implies that for almost all $(x,y) \in Q$, $F(x,y,\cdot)$ is $P_k(X)$-valued. Also hypothesis $H(F)_2\underline{(5)}$ is satisfies if X is reflexive.

Theorem 3.2. *If hypothesis $H(F)_2$ holds, then $(*)$ admits a solution.*

Proof. As in the proof of theorem 3.1, if $v \in C(Q,X)$ is a solution of $(*)$, then

$$\|v(x,y)\| \leq M_1$$

for all $(x,y) \in Q$. Again introduce $\hat{F}(x,y,v) = F(x,y,p_{M_1}(v))$ (note that theorem 7.3.11, p. 87 of Klein-Thompson [8] guarantees that $\hat{F}(x,y,\cdot)$ is l.s.c.) and let

$$W = \{w \in C(Q,X) : w(x,y) = z_0(x,y) + \int_0^x \int_0^y g(t,s)\,dt\,ds, (x,y) \in Q,$$
$$\|g(t,s)\| \leq \varphi(t,s) \ a.e.\}\ .$$

This is a nonempty, closed, bounded and equicontinuous subset of $C(Q,X)$. Let $\Gamma : W \to P_f(L^1(Q,X))$ be the multifunction defined by

$$\Gamma(w) = S^1_{\hat{F}(\cdot,\cdot,w(\cdot,\cdot))}\,.$$

From theorem 4.1 of [12], we know that $\Gamma(\cdot)$ is l.s.c. So we can apply theorem 3 of Bressan-Colombo [3] and get a continuous map $\gamma : W \to l^1(X)$ s.t. $\gamma(w) \in \Gamma(w)$ for all $w \in W$. Set

$$\mu(W)(x,y) = z_0(x,y) + \int_0^x \int_0^y \gamma(w)(t,s)\,dt\,ds\ .$$

Then $\mu : W \to W$ and is clearly continuous since $\gamma(\cdot)$ is. Let $B \subseteq C(Q,X)$ be nonempty, bounded and closed and let $\{v_n\}_{n\geq 1} \subseteq B$ s.t. $\overline{\{v_n\}_{n\geq 1}}^{C(Q,X)} = B$. We have

$$\begin{aligned}\beta(\mu(B)(x,y)) &\leq \beta[\int_0^x \int_0^y \gamma(B)(t,s)\,dt\,ds]\\ &\leq \beta[\int_0^x \int_0^y \overline{\gamma(\{v_n\}_{n\geq 1})(t,s)}\,dt\,ds]\\ &= \beta[\int_0^x \int_0^y \gamma(\{v_n\}_{n\geq 1})(t,s)\,dt\,ds]\end{aligned}$$

(as before by theorem 3.1 of [6] and the properties of $\beta(\cdot)$)

$$\leq \int_0^x \int_0^y \|k\|_\infty \beta(\{v_n(t,s)\}_{n\geq 1})\,dt\,ds \text{ (using proposition 1.6 of Mönch [10])}$$

$$= \int_0^x \int_0^y \|k\|_\infty \beta(B(t,s))\,dt\,ds.$$

As in the proof of theorem 3.1, introduce the sublinear measure of noncompactness $\psi(B) = \sup_{(x,y)\in Q} [e^{-\lambda\|k\|_\infty^{1/2}(t+s)}\beta(B(t,s))\}$ and establish that $\psi(\mu(B)) \leq \frac{1}{\lambda}\psi(B)$, $\lambda > 0$. So if we choose $\lambda > 1$, then $\mu(\cdot)$ is a ψ-contraction. Apply theorem 4.1 of Tarafdar-Vyborny [13] to get $v = \mu(v)$ for some $v \in W$. Then, through the definition of $\hat{F}$ and the Wendroff-Gronwall inequality, we can show that $\|v\|_{C(Q,X)} \leq M_1 \Rightarrow \hat{F}(x,y,v(x,y)) = F(x,y,v(x,y)) \Rightarrow v \in C(Q,X)$ solves problem $(*)$. $\square$

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