ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 215 – 220

ON GENERALIZATION OF INJECTIVITY

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Dedicated to Professor Carl Faith on his 65th birthday

Abstract. Characterizations of quasi-continuous modules and continuous modules are given. A non-trivial generalization of injectivity (distinct from p-injectivity) is considered.

Introduction

Various generalizations of injective modules are extensively studied since several years. Y. Utumi introduced continuous rings as a generalization of self-injective rings. The concepts of continuity and quasi-continuity was extended to modules by L. Jeremy, S. Mohamed and T. Bouhy, V. Goel and S. K. Jain. According to [6], the notion of quasi-continuous modules, which effectively extends that of continuous modules, appers now to be more fundamental. We here give new characteristic properties of continuous and quasi-continuous modules. A generalization of injectivity, distinct from p-injectivity, is also studied.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. Recall that (1) ${}_AM$ is injective iff for any left ideal I of A, every left A-homomorphism of I into M extends to A; (2) ${}_AM$ is defined as quasi-injective if for any left submodule N of M, every left A-homomorphism of N onto M extends to an endomorphism of ${}_AM$; (3) ${}_AM$ is continuous iff (a) every complement left submodule of M is a direct summand of ${}_AM$ and (b) every left submodule of M isomorphic to a direct summand of ${}_AM$ is a direct summand of ${}_AM$; (4) ${}_AM$ is quasi-continuous if every complement left submodule of M is a direct summand of ${}_AM$ and for any direct summands P, N of ${}_AM$ such that $P \cap N = o$, $P \oplus N$ is also a direct summand of ${}_AM$. It is well-known that injectivity $\Rightarrow$ quasi-injectivity$\Rightarrow$continuity$\Rightarrow$ quasi-continuity (cf. for example [6]).

In [6, Theorem 2.8], three characteristic properties of quasi-continuous modules are listed. We here give another characterization of quasi-continuous modules motivated by the definition of quasi-injective modules.

1991 *Mathematics Subject Classification*: 16D50, 16L60.
Key words and phrases: continuous modules, quasi-continuous modules, injective modules, quasi-Frobeniusean rings, m-injective rings.
Received October 14, 1991.

Theorem 1. *The following conditions are equivalent for a left A-module M:*

(1) ${}_AM$ is quasi-continuous;

(2) For any complement left submodule K of M, any relative complement C of K in M, any submodule N of M containing $K \oplus C$, every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$.

Proof. Assume (1). Let K be a complement left submodule of M, C a relative complement of K in M. Then $K \oplus C$ is essential in ${}_AM$. Let N be a submodule of M containing $K \oplus C$. Since ${}_AM$ is quasi-continuous, $K \oplus C$ is a direct summand of ${}_AM$. Therefore $K \oplus C = N = M$ (1) implies (2).

Assume (2). Let K be a non-zero complement left submodule of M. If C is a relative complement of K in ${}_AM$ (C exists by Zorn's Lemma), let $E = K \oplus C, p : E \to K$ the natural projection. The set of submodules S of M containing E such that p extends to a left A-homomorphism of S into K has, by Zorn's Lemma, a maximal member L. Let $q : L \to K$ be the extension of p to L. If $j : K \to M$ is the inclusion map, then $jq : L \to M$ and by hypothesis, jq extends to an endomorphism h of ${}_AM$. Suppose that $h(M) \nsubseteq K$. Since K is a relative complement of ${}_AC$ in ${}_AM$, then $(h(M) + K) \cap C \neq o$. If $o \neq c \in (h(M) + K) \cap C$, $c = h(m) + k$, $m \in M$, $k \in K$, we see that $F = \{u \in M \backslash h(u) \in E\}$ is a submodule of ${}_AM$ which strictly contains L (because $m \in F$, $m \notin L$). If $s : F \to E$ is defined by $s(u) = h(u)$ for all $u \in F$, then $ps : F \to K$ extends p to F, which contradicts the maximality of L. Thus $h(M) \subseteq K$ which implies that $h(M) = K$. Now $K \cap \ker h = o$ and if $b \in M$, $b = h(b) + (b - h(b)) \in K + \ker h$ which yields $M = K \oplus \ker h$. Since C is a relative complement of K in ${}_AM$, $h(C) = o$ and then $C = \ker h$. Thus $M = K \oplus C$, proving that any complement submodule of M is a direct summand. Now let D be a direct summand of ${}_AM$ such that $K \cap D = o$. The set of submodules of ${}_AM$ containing D and having zero intersection with K has a maximal member V which is a relative complement of K in ${}_AM$. We have, as above, $M = K \oplus V$. Since $D \subseteq V$, D is a direct summand of ${}_AM$, then $V = D \oplus U$ which yields $M = K \oplus D \oplus U$. This proves that (2) implies (1). □

Theorem 2. *The following conditions are equivalent for a left A-module M:*

(1) ${}_AM$ is continuous;

(2) For any isomorphic image K of a complement left submodule of M, any relative complement C of K in M, any submodule N of M containing $C \oplus K$, every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$;

(3) ${}_AM$ is quasi-continuous such that for any left submodule N of M which is isomorphic to a direct summand of ${}_AM$, every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$.

Proof. Assume (1). Let K be a non-zero isomorphic image of a complement left submodule of M, C a relative complement of K in M, N a submodule of M containing $C \oplus K$. Since ${}_AM$ is continuous, K and C are direct summands of ${}_AM$ and since $K \cap C = o$, then $K \oplus C$ is a direct summand of ${}_AM$. But $K \oplus C$ is essential in ${}_AM$ which implies that $K \oplus C = M$ and hence $N = M$. Thus (1) implies (2).

Assume (2). By Theorem 1, ${}_AM$ is quasi-continuous. Now let N be a submodule of ${}_AM$ isomorphic to a direct summand of ${}_AM$. Let ${}_AQ$ be a relative complement of ${}_AN$ in ${}_AM$. If $f : N \to M$ is a left A-homomorphism, $g : Q \oplus N \to N$ the natural projection, then $fg : Q \oplus N \to M$ and by hypothesis, fg extends to an endomorphism h of ${}_AM$. Clearly, h is an extension of f and hence (2) implies (3).

Assume (3). Let N be a submodule of ${}_AM$ which is isomorphic to Q, where $M = Q \oplus D$. If $j : N \to M$ is the inclusion map, $g : N \to Q$ an isomorphism, $i : Q \to M$ the natural injection, $p : M \to Q$ the natural projection, then $ig : N \to M$ extends to an endomorphism $h : M \to M$. For every $n \in N$, $hj(n) = ig(n)$ and $g^{-1}phj(n) = g^{-1}pig(n) = g^{-1}g(n) = n$. This shows that $k = g^{-1}ph : M \to N$ such that $kj =$ identity map on N. This proves that N is a direct summand of ${}_AM$. Since ${}_AM$ is quasi-continuous, then ${}_AM$ is continuous and therefore (3) implies (1). □

Since a left continuous left natural Noetherian ring is left Artinian, applying [10, Theorem 7.10] to Theorem 2, we get a new characteristic property of commutative quasi-Frobeniusean rings.

Corollary 3. *The following conditions are equivalent for a commutative ring A:*

(1) A is quasi-Frobeniusean;

(2) A is a Noetherian ring such that for any ideal I containing a non-zero isomorphic image of a complement ideal of A, every A-homomorphism of I into A extends to an endomorphism of A.

Recall that A is a left V-ring iff every simple left A-module is injective. V-rings, von Neumann regular rings and their generalizations have drawn the attention of many authors (cf. [2],[3], [5], [7], [9], [11]-[14]). We note that if A is semi-prime, then any simple left A-module N has the following property $(*)$: for any left ideal I of A, any left A-monomorphism of I into N extends to a left A-homomorphism of A into N. We call a left A-module M m-injective (mono-injective) if M has property $(*)$. A is called left m-injective if ${}_AA$ is m-injective.

It is clear that m-injectivity does not imply injectivity (otherwise, any semi-prime ring would be a left (and right) V-ring !). Note that continuous modules need not be m-injective.

Recall that a left A-module M is p-injective if, for any principal left ideal P of A, any left A-homomorphism of P into M extends to A. A is called left p-injective if ${}_AA$ is p-injective. Without the terminology, a theorem of M. Ikeda – T. Nakayama asserts that A is left p-injective if, and only if, every principal right ideal of A is a right annihilator. P-injective modules have been studied in connection with von Neumann regular rings, continuous and self-injective regular rings (cf. for example, [3], [7], [12] – [19]).

Note that m-injectivity does not imply p-injectivity (otherwise, any commutative semi-prime ring would always be regular !). Since any m-injective left ideal of A is a direct summand of ${}_AA$, then p-injectivity does not imply m-injectivity either (otherwise, any von Neumann regular ring would always be Artinian !).

Remark 1. If A is left m-injective , then $Z = J$ and A/J is von Neumann regular (cf. [2, Corollary 19.28]).

Remark 2. A left m-injective left Noetherian ring is left Artinian. Following [5], ${}_AM$ is called semi-simple if the intersection of all maximal left submodules of M is zero.

Remark 3. A is semi-simple Artinian iff every semi-simple left A-module is flat and m-injective.

Remark 4. A commutative ring A is semi-simple Artinian iff A is a semi-prime ring whose m-injective modules coincide with p-injectivity modules.

We now consider a particular case when m-injectivity implies injectivity.

Proposition 4. *Let A be a left m-injective ring containing an injective maximal left ideal K. Then A is left self-injective.*

Proof. $A = K \oplus U$, where $K = Ae$, $e = e^2 \in A$, $U = Au$, $u = 1 - e$. Then $uA = r(K)$. We show that uA is a minimal right ideal of A. Let $o \neq v \in uA$. Then $vA \subseteq uA$ and $l(u) \subseteq l(v)$. If $f : Au \to Av$ is the map defined by $f(au) = av$ for each $a \in A$, then f is an isomorphism (since Au is a minimal left ideal), and if $j : Au \to A$ is the inclusion map, we have a monomorphism $jf^{-1} : Av \to A$. Since ${}_AA$ is m-injective , there exists $y \in A$ such that $jf^{-1}(v) = vy$. Therefore $u = vy \in vA$ which yields $uA \subseteq vA$, whence $uA = vA$, proving that uA is a minimal right ideal of A. In the paper presented to the AMS meeting at OHIO (cf. Abstract American Mathematical Society, August 1990, Vol. 11 no 4 and Notices AMS 37(1990), no 6 (p. 707)), we proved that if A contains an injective maximal left ideal K such that $r(K)$ is a minimal right ideal, then A must be left self-injective. $\square$

Applying [2, Theorem 24.20], [4, Theorem] to Proposition 4, we get

Corollary 5. *If A contains an injective maximal left ideal, the following conditions are equivalent:*

(a) A is quasi–Frobeniusean;

(b) A is left m-injective satisfying the maximum condition on left annihilators;

(c) A is left m-injective satisfying the maximum condition on right annihilators;

(d) A is left m-injective satisfying the ascending chain condition on essential left ideals;

(e) A is left m-injective satisfying the ascending chain condition on essential right ideals.

Corollary 6. *If A contains an injective maximal left ideal, then A is left pseudo-Frobeniusean if, and only if, A is a left m-injective left Kasch ring.*

Corollary 7. *A is left self-injective regular with non-zero socle iff A is a left m-injective ring containing a non-singular injective maximal left ideal.*

Question. Is A semi-simple Artinian if A contains an injective maximal left ideal and every maximal left ideal of A is projective ?

We now give a nice result on annihilators.

Proposition 8. *Let A be a left and right m-injective ring. Then any minimal left (or right) ideal of A is an annihilator.*

Proof. Let $U = Au, u \in A$, be a minimal left ideal of A. The proof of Proposition 4 shows that uA is a minimal right ideal of A. Let $o \neq d \in l(r(Au))$. Then $r(u) = r(Au) = r(l(r(Au))) \subseteq r(d)$ and if $f : uA \to dA$ is defined by $f(ua) = da$ for all $a \in A$, then f is an isomorphism (because uA is minimal). If $j : dA \to A$ is the inclusion map, then $jf : uA \to A$ is a monomorphism and since A_A is m-injective, there exists $z \in A$ such that $d = jf(u) = zu \in Au$. Since $Au \subseteq l(r(Au))$, we have $Au = l(r(Au))$. Similarly, any minimal right ideal of A is a right annihilator. □

Theorem 9. *The following conditions are equivalent:*

(1) A is quasi–Frobeniusean;

(2) A is a left Noetherian, left p-injective, right m-injective ring;

(3) A is a left Noetherian, left m-injective, right p-injective ring;

(4) A is a left Noetherian, left and right m-injective ring;

(5) A is a left Noetherian, left m-injective ring whose minimal left ideals are left annihilators;

(6) A satisfies the maximum condition on left annihilators and for every left (right) ideal I of A containing a non-zero isomorphic image of a complement left (right) ideal of A, every left (right) A-homomorphism of I into A extends to an endomorphism of ${}_AA(A_A)$;

(7) A is a left and right p-injective ring whose left and right socles coincide and A satisfies the maximum condition on left annihilators and complement right ideals.

Proof. It is obvious that (1) implies (2), (4) and (6). □

Assume (2). Let $U = Au, u \in A$, be a minimal left ideal of A. Then $M = l(u)$ is a maximal left ideal. Let $o \neq v \in uA$. Since $vA \subseteq uA$, $M = l(v)$. Now $uA = r(l(uA))$, $vA = r(l(vA))$ which yield $uA = r(M) = r(l(vA)) = vA$, showing that uA is a minimal right ideal. The proof of Proposition 8 then shows that Au is a left annihilator. Since A satisfies the descending chain condition on right annihilators and A is left p-injective, then A is left perfect, whence A is left Artinian (in so far as A is left Noetherian). By [8, Proposition 1], (2) implies (3).

(3) implies (5) by Ikeda-Nakayama's theorem.

(4) implies (5) by [8, Proposition 1], Remark 1 and Proposition 8.

Assume (5). By Remark 2, A is left Artinian. Let $U = uA, u \in A$, be a minimal right ideal of A . Since A is left Artinian, Au contains a minimal left ideal $V = Av$, $v \in A$. Since $M = r(u)$ is a maximal right ideal of A , then $M = r(v)$. Now $Au \subseteq l(r(Au)) = l(M) = l(r(Av)) = Av$, which implies that $Au = Av$ is a minimal left ideal. The proof of Proposition 8 then shows that uA is a right annihilator. Thus (5) implies (1) by [8, Proposition 1].

(6) implies (7) by [1, Theorem 1] and Theorem 2.

Assume (7). Since A satisfies the minimum condition on right annihilators and every principal right ideal is a right annihilator, then A is left perfect. Since A satisfies the maximum condition on left annihilators, then Z is nilpotent. Since A

is left p-injective , $Z = J$ which implies that A is semi-primary. By [1, Lemma 6], A is right Artinian. Then (7) implies (1 by [8, Proposition 1].

Corollary 10. *If A is commutative, the following are equivalent:*
(a) A is quasi–Frobeniusean;
(b) A is a p-injective Goldie ring;
(c) A is a m-injective Noetherian ring.

We conclude with a connection between m-injective and continuity.

Remark 5. If A is a left m-injective left uniform ring, then A is a local left continuous ring.

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