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COMMUTATORS OF FLOWS AND FIELDS

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Abstract. The well known formula $[X,Y] = \frac{1}{2}\frac{\partial^2}{\partial t^2}|_0(\mathrm{Fl}^Y_{-t}\circ\mathrm{Fl}^X_{-t}\circ\mathrm{Fl}^Y_t\circ\mathrm{Fl}^X_t)$ for vector fields X, Y is generalized to arbitrary bracket expressions and arbitrary curves of local diffeomorphisms.

Let M be a smooth manifold. It is well known that for vector fields $X, Y \in \mathfrak{X}(M)$ we have

$$0 = \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^Y_{-t}\circ\mathrm{Fl}^X_{-t}\circ\mathrm{Fl}^Y_t\circ\mathrm{Fl}^X_t),$$
$$[X,Y] = \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}|_0(\mathrm{Fl}^Y_{-t}\circ\mathrm{Fl}^X_{-t}\circ\mathrm{Fl}^Y_t\circ\mathrm{Fl}^X_t).$$

We give the following generalization:

1. Theorem. *Let M be a manifold, let $\varphi^i : \mathbb{R}\times M \supset U_{\varphi^i} \to M$ be smooth mappings for $i = 1,\dots,k$ where each U_{φ^i} is an open neighborhood of $\{0\}\times M$ in $\mathbb{R}\times M$, such that each φ^i_t is a diffeomorphism on its domain, $\varphi^i_0 = Id_M$, and $\frac{\partial}{\partial t}\big|_0 \varphi^i_t = X_i \in \mathfrak{X}(M)$. We put $[\varphi^i_t,\varphi^j_t] := (\varphi^j_t)^{-1}\circ(\varphi^i_t)^{-1}\circ\varphi^j_t\circ\varphi^i_t$. Then for each formal bracket expression B of length k we have*

$$0 = \tfrac{\partial^\ell}{\partial t^\ell}|_0 B(\varphi^1_t,\dots,\varphi^k_t) \quad \text{for } 1\le \ell < k,$$
$$B(X_1,\dots,X_k) = \tfrac{1}{k!}\tfrac{\partial^k}{\partial t^k}|_0 B(\varphi^1_t,\dots,\varphi^k_t) \in \mathfrak{X}(M)$$

in the sense explained in 3 below.

In fact this theorem is a special case of the more general theorem 10 below. The somewhat unusual choice of the commutator of flows is explained by the fact that the bracket on the Lie algebra of the diffeomorphism group is the negative of the usual Lie bracket of vector fields.

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2. Lemma. *Let $c : \mathbb{R} \to M$ be a smooth curve. If $c(0) = x \in M$, $c'(0) = 0, \dots, c^{(k-1)}(0) = 0$, then $c^{(k)}(0)$ is a well defined tangent vector in $T_x M$ which is given by the derivation $f \mapsto (f \circ c)^{(k)}(0)$ at x.*

Proof. We have

$$\begin{aligned}((f.g) \circ c)^{(k)}(0) = ((f \circ c).(g \circ c))^{(k)}(0) &= \sum_{j=0}^{k} \tbinom{k}{j} (f \circ c)^{(j)}(0)(g \circ c)^{(k-j)}(0) \\ &= (f \circ c)^{(k)}(0)g(x) + f(x)(g \circ c)^{(k)}(0),\end{aligned}$$

since all other summands vanish: $(f \circ c)^{(j)}(0) = 0$ for $1 \le j < k$. □

3. Curves of local diffeomorphisms. Let $\varphi : \mathbb{R} \times M \supset U_\varphi \to M$ be a smooth mapping where U_φ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that each φ_t is a diffeomorphism on its domain and $\varphi_0 = Id_M$. We say that φ_t is a *curve of local diffeomorphisms* though Id_M.

From lemma 2 we see that if $\frac{\partial^j}{\partial t^j}|_0 \varphi_t = 0$ for all $1 \le j < k$, then $X := \frac{1}{k!} \frac{\partial^k}{\partial t^k}|_0 \varphi_t$ is a well defined vector field on M. We say that X is the *first non-vanishing derivative* at 0 of the curve φ_t of local diffeomorphisms. We may paraphrase this as $(\partial_t^k|_0 \varphi_t^*)f = k! \mathcal{L}_X f$.

4. Natural vector bundles. See [KMS, 6.14]. Let $\mathcal{M}f_m$ denote the category of all smooth m-dimensional manifolds and local diffeomorphisms between them. A *vector bundle functor* or *natural vector bundle* is a functor F which associates a vector bundle $(F(M), p_M, M)$ to each manifold M and a vector bundle homomorphism

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(N) \\ {\scriptstyle p_M}\downarrow & & \downarrow{\scriptstyle p_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

to each $f : M \to N$ in $\mathcal{M}f_m$, which covers f and is fiber wise a linear isomorphism. If f is the embedding of an open subset of N then this diagram turns out to be a pullback diagram. We also point out that $f \mapsto F(f)$ maps smoothly parameterized families to smoothly parameterized families, see [KMS, 14.8]. Assuming this property all vector bundle functors were classified by [T]: They correspond to linear representations of higher jet groups, they are associated vector bundles to higher order frame bundles, see also [KMS, 14.8].

Examples of vector bundle functors are tangent and cotangent bundles, tensor bundles, and also the trivial bundle $M \times \mathbb{R}$ which will give us theorem 1.

5. Pullback of sections. Let F be a vector bundle functor on $\mathcal{M}f_m$ as described in 4. Let M be an m-manifold and let φ_t be a curve of local diffeomorphisms through Id_M on M. Then the flow φ_t, for fixed t, is a diffeomorphism defined on an open subset U_{φ_t} of M. The mapping

$$\begin{array}{ccccc} F(M) & \longleftarrow & F(U_{\varphi_t}) & \xrightarrow{F(\varphi_t)} & F(M) \\ {\scriptstyle p_M}\downarrow & & \downarrow & & \downarrow{\scriptstyle p_M} \\ M & \longleftarrow & U_{\varphi_t} & \xrightarrow[\varphi_t]{} & M \end{array}$$

is then a vector bundle isomorphism.

We consider a section $s \in C^\infty(F(M))$ of the vector bundle $(F(M), p_M, M)$ and we define for $t \in \mathbb{R}$

$$\varphi_t^* s := F(\varphi_t^{-1}) \circ s \circ \varphi_t.$$

This is a local section of the bundle $F(M)$. For each $x \in M$ the value $(\varphi_t^* s)(x) \in F(M)_x := p_M^{-1}(x)$ is defined, if t is small enough. So in the vector space $F(M)_x$ the expression $\frac{d}{dt}|_0(\varphi_t^* s)(x)$ makes sense and therefore the section $\frac{d}{dt}|_0(\varphi_t)^* s$ is globally defined and is an element of $C^\infty(F(M))$. If $\varphi_t = \mathrm{Fl}_t^X$ is the flow of a vector field X on M this section

$$\mathcal{L}_X s := \tfrac{d}{dt}|_0(\mathrm{Fl}_t^X)^* s$$

is called the *Lie derivative* of s along X. It satisfies $\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}$, see [KMS, 6.20].

6. Lemma. *Let φ_t be a smooth curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X = \partial_t^k|_0\varphi_t$. Then for any vector bundle functor F and for any section $s \in C^\infty(F(M))$ we have the first non-vanishing derivative*

$$k!\mathcal{L}_X s = \partial_t^k|_0\varphi_t^* s.$$

Proof. This is again a local question, so let $x \in M$. We choose a complete Riemannian metric on M and we denote by U_k the open ball with radius $r_k > 0$ and center x for this metric, and let let $\overline{U_k}$ be its closure. Since $\varphi_0 = Id_M$ we may choose a chart $(U, u : U \to \mathbb{R}^m)$ of M with $x \in U$ and $u(U) = \mathbb{R}^m$, radii $r_0 > r_1 > r_2 > r_3 > r_4 > 0$ and $\varepsilon > 0$ such that the following hold: φ is defined and smooth on $((-2\varepsilon, 2\varepsilon) \times U_0)$, $\varphi([-\varepsilon, \varepsilon] \times \overline{U_1}) \subset U$, $\varphi((-\varepsilon, \varepsilon) \times U_2) \supset \overline{U_3}$, and $\varphi((-\varepsilon, \varepsilon) \times \overline{U_4}) \subset U_3$. Let $\mathcal{E}$ be the set of all $f \in C^\infty(U_1, U)$ such that $f|\overline{U_2}$ is a diffeomorphism onto its image, $f(U_2) \supset \overline{U_3}$, and $f(\overline{U_4}) \subset U_3$. Then via the linear isomorphism $u_* : C^\infty(U_1, U) \to C^\infty(U_1, \mathbb{R}^m)$ which we suppress from now on, the set $\mathcal{E}$ is an open subset of the Frechét space $C^\infty(U_1, \mathbb{R}^m)$ for the compact C^∞-topology, since the closures $\overline{U_k}$ are compact for each $r_k > 0$ by completeness of the metric.

By cartesian closedness [FK, 4.4.13] or [KMb, 1.8] the curve $\check{\varphi} : (-\varepsilon, \varepsilon) \to C^\infty(U_1, \mathbb{R}^m)$ is smooth and takes values in the open subset $\mathcal{E}$.

Claim. Let $L(C^\infty(F(M)), C^\infty(F(U_4)))$ denote the space of all bounded linear mappings between the convenient vector spaces indicated which are equipped with the compact C^∞-topology, and let $P : C^\infty(U_1,\mathbb{R}^m) \supset \mathcal{E} \to L(C^\infty(F(M)), C^\infty(F(U_4)))$ be the mapping given by $P(f)(s) = f^*s = F(f^{-1}) \circ s \circ f$. Then P is smooth.

First we check that P takes values in the space of bounded (i. e. smooth) linear mappings. We have to check that $P(f)$ maps smooth curves in $C^\infty(F(M))$ to smooth curves in $C^\infty(F(U_4))$. A curve $c : \mathbb{R} \to C^\infty(F(M))$ is smooth if and only if the canonically associated mapping $\check{c} : \mathbb{R} \times M \to F(M)$ is smooth, see [KMa,7.7.2]. But clearly $P(f)(c_t)(x) = (F((f|U_2)^{-1}|U_3) \circ c_t \circ f|U_4)(x)$ is smooth in $(t,x) \in \mathbb{R} \times U_4$.

Now we check that P itself is smooth, i.e. maps smooth curves in $\mathcal{E}$ to smooth curves in $L(C^\infty(F(M)), C^\infty(F(U_4)))$. So let $f : \mathbb{R} \to \mathcal{E} \subset C^\infty(U_1,\mathbb{R}^m)$ be smooth, by cartesian closedness this means that $\hat{f} : \mathbb{R} \times U_1 \to \mathbb{R}^m$ is smooth. By the finite dimensional implicit function theorem the mapping $(t,x) \mapsto f_t^{-1}(x)$ is also smooth for $(t,x) \in \mathbb{R} \times U_3$. But then for each section $s \in C^\infty(F(M))$ the mapping $(t,x) \mapsto (P(f_t)s)(x) = (F((f_t|U_2)^{-1}|U_3) \circ s \circ f_t|U_4)(x)$ is also smooth since F respects smoothly parameterized families.

By the smooth uniform boundedness principle [FK, remark on page 89, also 4.4.7], see also [KMb, 1.7.2], the assignment $t \mapsto P(f_t)$ is smooth as a mapping

$$(-\varepsilon,\varepsilon) \to L(C^\infty(F(M)), C^\infty(F(U_4)))$$

if and only if the composition

$$(-\varepsilon,\varepsilon) \to L(C^\infty(F(M)), C^\infty(F(U_4))) \xrightarrow{\mathrm{ev}_s} C^\infty(F(U_4))$$

is smooth for each $s \in C^\infty(F(M))$. We have already checked this condition, so the claim follows.

Now the smooth curve $\check{\varphi}$ takes values in $\mathcal{E}$, so we may compute for $1 \le \ell \le k$ as follows:

$$\partial_t^\ell|_0 \varphi_t^* s = \partial_t^\ell|_0 (\mathrm{ev}_s \circ P \circ \check{\varphi})(t) = d(\mathrm{ev}_s \circ P)(\varphi_0)(\partial_t^\ell|_0 \varphi_t) + 0$$

since each other term contains a derivative at 0 of φ_t of order less than ℓ which is 0, and thus we get $\partial_t^\ell|_0 \varphi_t^* s = 0$ for $\ell < k$ and

$$\begin{aligned}\partial_t^k|_0 \varphi_t^* s &= d(\mathrm{ev}_s \circ P)(Id_{U_2})(\partial_t^k|_0 \varphi_t)\\ &= d(\mathrm{ev}_s \circ P)(Id_{U_2})(k!X) = k! d(\mathrm{ev}_s \circ P)(Id_{U_2})(\partial_t|_0 \mathrm{Fl}_t^X)\\ &= k!\partial_t|_0(\mathrm{ev}_s \circ P \circ \mathrm{Fl}^X)(t) = k!\partial_t|_0 (\mathrm{Fl}_t^X)^* s = k! \mathcal{L}_X s. \ \square\end{aligned}$$

7. Lemma. *Let M be a smooth manifold and let F be a vector bundle functor on $\mathcal{M}f_m$. Let φ_t, ψ_t be curves of local diffeomorphisms through Id_M and let $s \in C^\infty(F(M))$ be a section of the vector bundle $F(M) \to M$. Then we have*

$$\partial_t^k|_0 (\varphi_t \circ \psi_t)^* s = \partial_t^k|_0 (\psi_t^* \varphi_t^*) s = \sum_{j=0}^k \tbinom{k}{j} (\partial_t^j|_0 \psi_t^*)(\partial_t^{k-j}|_0 \varphi_t^*) s.$$

Also the multinomial version of this formula holds:

$$\partial_t^k|_0(\varphi_t^1 \circ \ldots \circ \varphi_t^\ell)^* s = \sum_{j_1+\dots+j_\ell=k} \frac{k!}{j_1!\ldots j_\ell!}(\partial_t^{j_\ell}|_0(\varphi_t^\ell)^*)\ldots(\partial_t^{j_1}|_0(\varphi_t^1)^*)s.$$

Proof. We only prove the binomial version. The question is local on M, so let U be an open neighborhood of some point x in M such that φ is defined and smooth on $(-\varepsilon,\varepsilon)\times U$. From the claim in the proof of lemma 6 we know that $t\mapsto \varphi_t^*$ is am smooth curve in the convenient vector space $L(C^\infty(F(M)),C^\infty(F(U)))$ of all bounded linear mappings.

Now let $V\subset M$ be an open neighborhood of x such that ψ is defined on $(-\varepsilon,\varepsilon)\times V$ and $\psi((-\varepsilon,\varepsilon)\times V)\subseteq U$. By the arguments just given the mapping $t\mapsto \psi_t^*$ is a smooth mapping $(-\varepsilon,\varepsilon)\to L(C^\infty(F(U)),C^\infty(F(V)))$ also. Composition

$$\begin{aligned}L(C^\infty(F(M)),C^\infty(F(U)))\times L(C^\infty(F(U)),C^\infty(F(V)))\to\\ \to L(C^\infty(F(M)),C^\infty(F(V)))\end{aligned}$$

is smooth and bilinear, see [FK, 4.4.16] and we may just apply the Leibniz formula for higher derivatives of bilinear expressions of functions. We evaluate first at $s\in C^\infty(F(M))$ and then at $x\in M$ to obtain the formula □

8. Lemma. *Let φ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X=\partial_t^k|_0\varphi_t$. Then the inverse curve of local diffeomorphisms φ_t^{-1} has first non-vanishing derivative $-k!X=\partial_t^k|_0\varphi_t^{-1}$.*

Proof. For we have $\varphi_t^{-1}\circ\varphi_t=Id$, so by lemma 7 we get for $1\le j\le k$

$$\begin{aligned}0=\partial_t^j|_0(\varphi_t^{-1}\circ\varphi_t)f=\sum_{i=0}^{j}\tbinom{j}{i}(\partial_t^i|_0\varphi_t^*)(\partial_t^{j-i}(\varphi_t^{-1})^*)f=\\ =\partial_t^j|_0\varphi_t^*(\varphi_0^{-1})^*f+\varphi_0^*\partial_t^j|_0(\varphi_t^{-1})^*f,\end{aligned}$$

i.e. $\partial_t^j|_0\varphi_t^*f=-\partial_t^j|_0(\varphi_t^{-1})^*f$ as required. □

9. Lemma. *Let M be a manifold, let F be a vector bundle functor, let s be a smooth section of $F(M)$, let φ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $m!X=\partial_t^m|_0\varphi_t$, and let ψ_t be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $n!Y=\partial_t^n|_0\psi_t$.*

Then the curve of local sections $[\varphi_t,\psi_t]^*s=(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t\circ\varphi_t)^*s$ has first non-vanishing derivative

$$(m+n)!\mathcal{L}_{[X,Y]}s=\partial_t^{m+n}|_0[\varphi_t,\psi_t]^*s.$$

Proof. From lemmas 6 and 8 we have the following first non-vanishing derivatives

$$\begin{aligned}&m!\mathcal{L}_Xs=\partial_t^m|_0\varphi_t^*s, && n!\mathcal{L}_Ys=\partial_t^n|_0\psi_t^*s,\\ &m!\mathcal{L}_{-X}s=\partial_t^m|_0(\varphi_t^{-1})^*s, && n!\mathcal{L}_{-Y}s=\partial_t^n|_0(\psi_t^{-1})^*s.\end{aligned}\tag{1}$$

By the multinomial version of lemma 7 we have

$$\begin{aligned}
A_N s :&= \partial_t^N|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t\circ\varphi_t)^* s\\
&= \sum_{i+j+k+\ell=N}\frac{N!}{i!j!k!\ell!}(\partial_t^i|_0\varphi_t^*)(\partial_t^j|_0\psi_t^*)(\partial_t^k|_0(\varphi_t^{-1})^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)s.
\end{aligned}$$

Let us suppose that $1\le n\le m$, the case $m\le n$ is similar. If $N<n$ all summands are 0. If $N=n$ we have by lemma 8

$$A_N s = (\partial_t^n|_0\varphi_t^*)s + (\partial_t^n|_0\psi_t^*)s + (\partial_t^n|_0(\varphi_t^{-1})^*)s + (\partial_t^n|_0(\psi_t^{-1})^*)s = 0.$$

If $n<N\le m$ we have, using again lemma 8

$$\begin{aligned}
A_N s &= \sum_{j+\ell=N}\frac{N!}{j!\ell!}(\partial_t^j|_0\psi_t^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)s + \delta_N^m\left((\partial_t^m|_0\varphi_t^*)s + (\partial_t^m|_0(\varphi_t^{-1})^*)s\right)\\
&= (\partial_t^N|_0(\psi_t^{-1}\circ\psi_t)^*)s + 0 = 0.
\end{aligned}$$

Now we come to the difficult case $m,n<N\le m+n$.

$$\begin{aligned}
A_N s &= \partial_t^N|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t)^* s + \tbinom{N}{m}(\partial_t^m|_0\varphi_t^*)(\partial_t^{N-m}|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t)^*)s\\
&\quad + (\partial_t^N|_0\varphi_t^*)s,
\end{aligned}\tag{2}$$

by lemma 7, since all other terms vanish, see (4) below. By lemma 7 again we get:

$$\partial_t^N|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t)^* s = \sum_{j+k+\ell=N}\frac{N!}{j!k!\ell!}(\partial_t^j|_0\psi_t^*)(\partial_t^k|_0(\varphi_t^{-1})^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)s$$

$$\begin{aligned}
&= \sum_{j+\ell=N}\tbinom{N}{j}(\partial_t^j|_0\psi_t^*)(\partial_t^\ell|_0(\psi_t^{-1})^*)s + \tbinom{N}{m}(\partial_t^{N-m}|_0\psi_t^*)(\partial_t^m|_0(\varphi_t^{-1})^*)s\\
&\quad + \tbinom{N}{m}(\partial_t^m|_0(\varphi_t^{-1})^*)(\partial_t^{N-m}|_0(\psi_t^{-1})^*)s + \partial_t^N|_0(\varphi_t^{-1})^* s\\
&= 0 + \tbinom{N}{m}(\partial_t^{N-m}|_0\psi_t^*)m!\mathcal{L}_{-X}s + \tbinom{N}{m}m!\mathcal{L}_{-X}(\partial_t^{N-m}|_0(\psi_t^{-1})^*)s\\
&\quad + \partial_t^N|_0(\varphi_t^{-1})^* s, \qquad \text{using (1)}\\
&= \delta_{m+n}^N(m+n)!(\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)s + \partial_t^N|_0(\varphi_t^{-1})^* s\\
&= \delta_{m+n}^N(m+n)!\mathcal{L}_{[X,Y]}s + \partial_t^N|_0(\varphi_t^{-1})^* s
\end{aligned}\tag{3}$$

From the second expression in (3) one can also read off that

$$\partial_t^{N-m}|_0(\psi_t^{-1}\circ\varphi_t^{-1}\circ\psi_t)^* s = \partial_t^{N-m}|_0(\varphi_t^{-1})^* s.\tag{4}$$

If we put (3) and (4) into (2) we get, using lemmas 7 and 8 again, the final result which proves lemma 9:

$$\begin{aligned}
A_N s &= \delta_{m+n}^N(m+n)!\mathcal{L}_{[X,Y]}s + \partial_t^N|_0(\varphi_t^{-1})^* s\\
&\quad + \tbinom{N}{m}(\partial_t^m|_0\varphi_t^*)(\partial_t^{N-m}|_0(\varphi_t^{-1})^*)s + (\partial_t^N|_0\varphi_t^*)s\\
&= \delta_{m+n}^N(m+n)!\mathcal{L}_{[X,Y]}s + \partial_t^N|_0(\varphi_t^{-1}\circ\varphi_t)^* s\\
&= \delta_{m+n}^N(m+n)!\mathcal{L}_{[X,Y]}s + 0. \ \square
\end{aligned}$$

10. Theorem. *Let M be a manifold, let φ^i be smooth curves of local diffeomorphisms through Id_M for $i=1,\dots,j$ with non-vanishing first derivative $\partial_t^{k_i}|_0\varphi_t^i = k_i!X_i \in \mathfrak{X}(M)$. Let F be a vector bundle functor and let $s \in C^\infty(F(M))$ be a section. Then for each formal bracket expression B of length j we have*

$$\begin{aligned} 0 &= \tfrac{\partial^\ell}{\partial t^\ell}|_0 B(\varphi_t^1,\dots\varphi_t^k)^* s \quad \text{for } 1\le \ell < k,\\ \mathcal{L}_{B(X_1,\dots,X_k)} s &= \tfrac{1}{k!}\tfrac{\partial^k}{\partial t^k}|_0 B(\varphi_t^1,\dots\varphi_t^k)^* s \in C^\infty(F(M)), \end{aligned}$$

where $k = k_1 + \dots + k_j$.

Proof. Apply lemma 9 recursively. □

11. Proposition. *Let φ be a curve of local diffeomorphisms through Id_M with first non-vanishing derivative $k!X = \partial_t^k|_0\varphi_t$. Then the curve of local vector fields $(\partial_t\varphi_t)\circ\varphi_t^{-1}$ has as first non-vanishing derivative*

$$k!X = \partial_t^{k-1}|_0\left((\partial_t\varphi_t)\circ\varphi_t^{-1}\right).$$

Proof. Using lemma 7 for $f \in C^\infty(M,\mathbb{R})$ we have for $1 \le \ell < k$:

$$\begin{aligned} \partial_t^{\ell-1}|_0((\partial_t\varphi_t)\circ\varphi_t^{-1})f &= \partial_t^{\ell-1}|_0(\varphi_t^{-1})^*\partial_t\varphi_t^* f\\ &= \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(\partial_t^j|_0(\varphi_t^{-1})^*)(\partial_t^{\ell-j}|_0\varphi_t^*)f\\ &= (\varphi_0^{-1})^*(\partial_t^\ell|_0\varphi_t^*)f + 0 = \delta_\ell^k k!\mathcal{L}_X f. \ \square \end{aligned}$$

12. Corollary. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $g, h \in G$ we consider the group commutator $[g,h] = ghg^{-1}h^{-1}$. Then for any bracket expression B of length k and $X_i \in \mathfrak{g}$ we have*

$$\begin{aligned} k!B(X_1,\dots,X_k) &= \partial_t^k|_0 B(\exp tX_1,\dots,\exp tX_k)\\ &= \partial_t^{k-1}|_0\left(T\lambda_{B(\exp tX_1,\dots,\exp tX_k)^{-1}}(\partial_t B(\exp tX_1,\dots,\exp tX_k))\right), \end{aligned}$$

where λ_g denotes left translation by g.

The first equation is a generalization of the well known 'Trotter product formula', i. e. the case of $B = [\ \ ,\ \ ]$.

Proof. The flow of the left invariant vector field L_X corresponding to $X \in \mathfrak{g}$ is the right translation $\rho_{\exp tX}$ by $\exp tX$, so we just apply theorem 1 to get

$$\begin{aligned} k!B(L_{X_1},\dots,L_{X_k}) &= \partial_t^k|_0 B(\rho_{\exp tX_1},\dots,\rho_{\exp tX_k})\\ &= \partial_t^k|_0\rho(B(\exp tX_1,\dots,\exp tX_k)), \end{aligned} \tag{1}$$

where in the first line the commutator of flows is applied, and in the second line the group commutator with reversed order. Evaluating both sides at $e \in G$ gives the first formula. From (1) and proposition 11 we get

$$\begin{aligned} &k!B(L_{X_1}, \dots, L_{X_k}) = \\ &\quad = \partial_t^{k-1}|_0 \left((\partial_t \rho(B(\exp tX_1, \dots, \exp tX_k))) \circ \rho(B(\exp tX_1, \dots, \exp tX_k))^{-1} \right). \end{aligned}$$

We evaluate this at $e \in G$ and get

$$\begin{aligned} &k!B(X_1, \dots, X_k) = \\ &\quad = \partial_t^{k-1}|_0 \left((\partial_t \rho(B(\exp tX_1, \dots, \exp tX_k)))(B(\exp tX_1, \dots, \exp tX_k)^{-1}) \right) \\ &\quad = \partial_t^{k-1}|_0 \left(T\lambda_{B(\exp tX_1, \dots, \exp tX_k)^{-1}}(\partial_t B(\exp tX_1, \dots, \exp tX_k)) \right). \ \square \end{aligned}$$

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