ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 181 – 186

CHARACTERIZATION OF GLOBALLY LIPSCHITZIAN COMPOSITION OPERATORS IN THE BANACH SPACE $BV_p^2[a,b]$

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Abstract. We give a characterization of the globally Lipschitzian composition operators acting in the space $BV_p^2[a,b]$

Introduction. In [8], F. Riesz introduced the so-called Riesz class $A_p = A_p[a,b]$ $(1 \le p < \infty)$ in the following way: The function $u \in A_p[a,b]$ if u is absolutely continuous on $[a,b]$ and $u' \in L_p[a,b]$. In the same paper he proved that the function $u \in A_p[a,b]$ $(1 < p < \infty)$ if and only if u has bounded p-variation on $[a,b]$. Moreover the p-variation of the function u is given by

$$V_p(u;\,[a,b]) = \|u'\|^p_{L_p[a,b]}\ .$$

Recently N. Merentes [7] proved an analogous result for the class $A_p^2[a,b]$ $(1 < p < \infty)$ of functions u such that u' is absolutely continuous on $[a,b]$ and $u'' \in L_p[a,b]$. More precisely, let $1 < p < \infty$. The function $u \in A_p^2[a,b]$ if and only if u has bounded $(p,2)$-variation on $[a,b]$ and the $(p,2)$-variation of u is given by

$$V_{(p,2)}(u;\,[a,b]) = \|u''\|^p_{L_p[a,b]}\ .$$

In [6] N. Merentes, making use of an idea of the first author (cf. [3]), applied the Riesz result to deduce a characterization of functions f generating a composition operators F satisfying a global Lipschitz condition on the space $BV_p[a,b]$. In the present paper we will apply the characterization of the class $A_p^2[a,b]$ to deduce an analogous result for the space $BV_p^2[a,b]$.

1991 *Mathematics Subject Classification*: 47H30, 39B22.
Key words and phrases: Riesz space, bounded p-variation, globally Lipschitzian composition operator.
Received June 7, 1991.
*Research supported by Fundación Gran Mariscal de Ayacucho and Consejo de Desarrollo de la Universidad Central de Venezuela

1. Let u be a function on $[a,b]$. For a given partition $\pi : a = t_0 < \cdots < t_m = b$ of $[a,b]$, let

$$\sigma_p(u;\pi) := \sum_{j=1}^{m} \frac{|u(t_j) - u(t_{j-1})|^p}{|t_j - t_{j-1}|^{p-1}} \quad (1 \le p < \infty) .$$

The number

$$V_p(u) = V_p(u;\,[a,b]) := \sup_{\pi} \sigma_p(u;\pi) , \tag{1}$$

where the supremum is taken over all partitions π of $[a,b]$ is called the p-variation of the function u on $[a,b]$. By $BV_p[a,b]$ we shall denote the Banach space of all functions u for which the norm

$$\|u\|_p := |u(a)| + (V_p(u;[a,b]))^{1/p} \tag{2}$$

is finite.

The space $BV_1[a,b]$ is simply denoted by $BV[a,b]$ and is referred to as the space of functions of bounded variation.

It is easy to see that every function $u \in BV_p[a,b]$ is continuous on $[a,b]$ provided $1 < p < \infty$. More precisely, the following embeddings

$$BV_p[a,b] \hookrightarrow AC[a,b] \hookrightarrow BV[a,b] , \quad (1 < p < \infty) \tag{3}$$

hold, where $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$, equipped with either the $BV[a,b]$-norm (2) or the norm

$$\|u\|_{AC[a,b]} := |u(a)| + \|u'\|_{L_1[a,b]} .$$

Lemma 1. *(F. Riesz [8]). Let $1 < p < 5$. The function $u \in A_p[a,b]$ if and only if $V_p(u;\,[a,b]) < \infty$. Moreover*

$$V_p(u;\,[a,b]) = \|u'\|^p_{L_p[a,b]} .$$

Let

$$\sigma_{(p,2)}(u;\pi) :=$$
$$\sum_{j=1}^{m-1} \left| \frac{u(t_{j+1}) - u(t_j)}{t_{j+1} - t_j} - \frac{u(t_j) - u(t_{j-1})}{t_j - t_{j-1}} \right|^p \frac{1}{|t_{j+1} - t_{j-1}|^{p-1}} .$$

The number

$$V_{(p,2)}(u;\,[a,b]) := \sup_{\pi} \sigma_{(p,2)}(u;\pi) , \tag{4}$$

where the supremum is taken over all partitions π of $[a,b]$, is called the $(p,2)$-variation of the function u on $[a,b]$. By $BV_{(p,2)}[a,b]$ we shall denote the Banach space of all functions u for which the norm

$$\|u\|_{(p,2)} := |u(a)| + |u'(a)| + (V_{(p,2)}(u;\ [a,b]))^{p/1} \tag{5}$$

is finite.

The space $BV_{(1,2)}[a,b]$ is simple denoted by $BC[a,b]$ and is referred to as the space of functions of bounded second-variation

Remark 1. The notion of second-variation was introduced in [10] by de la Vallée Poussin.

The following result is well-known (cf. [9]).

Lemma 2. *The function $u \in BC[a,b]$; i.e., the function u has bounded second-variation if and only if the function u is representable in the form $u = v_1 - v_2$ where v_1 and v_2 are convex and $v'_{1,+}(a)$, $v'_{1,-}(b)$, $v'_{2,+}(a)$ and $v'_{2,-}(b)$ are all finite.*

Recently N. Merentes [7] proved the following characterization of the class $A^2_p[a,b]$.

Lemma 3. *Let $1 < p < \infty$. The function $u \in A^2_p[a,b]$ if and only if $V_{(p,2)}(u;\ [a,b]) < \infty$. Moreover*

$$V_{(p,2)}(u;\ [a,b]) = \|u''\|^p_{L_p[a,b]}\ .$$

It is easy to see the following embedding

$$BV_{(p,2)}[a,b] \hookrightarrow BV_p[a,b]\ , \quad (1 < p < \infty)$$

holds, i.e., there exists a constant $K > 0$ such that

$$\|u\|_p \le K\|u\|_{(p,2)}\ , \quad (u \in BV_{(p,2)}[a,b])\ . \tag{6}$$

2. Denote by $\mathcal{F}[a,b]$ the class of all functions $u : [a,b] \to \mathbf{R}$. Let $f : [a,b] \times \mathbf{R} \to \mathbf{R}$. The composition operator $F : \mathcal{F}[a,b] \to \mathcal{F}[a,b]$ is defined by

$$(Fu)(t) := f(t, u(t))\ , \quad (u \in \mathcal{F}[a,b],\ t \in [a,b])\ .$$

Theorem. *Let $1 < p < \infty$ and $f : [a,b] \times \mathbf{R} \to \mathbf{R}$. The composition operator F maps the space $BV_{(p,2)}[a,b]$ into itself and satisfies the global Lipschitz condition; i.e., there exists a constant $L > 0$ such that*

$$\|Fu - FV\|_{(p,2)} \le L\|u - v\|_{(p,2)}\ ,\ (u, v \in BV_{(p,2)}[a,b]) \tag{7}$$

if and only if there exist functions $g, h \in BV_{(p,2)}[a,b]$ such that

$$f(t,x) = g(t)x + h(t) \quad (t \in [a,b],\ x \in \mathbf{R})\ . \tag{8}$$

Proof. Suppose that the operator F satisfies the Lipschitz condition (7). Since the embedding (6) holds, we have

$$\begin{aligned} &\|Fu - Fv\|_p \le K\|Fu - Fv\|_{(p,2)} \le KL\|u - v\|_{(p,2)} \\ &\qquad (u, v \in BV_{(p,2)}[a,b]) \ . \end{aligned} \tag{9}$$

Fix $t, \bar{t} \in [a,b]$, $(t < \bar{t})$ and $x_1, \bar{x}_1, x_2, \bar{x}_2 \in \mathbf{R}$, and define two functions $u_i : [a,b] \to \mathbf{R}$ by

$$\begin{aligned} u_i(s) := \left(\frac{\bar{x}_i - x_i}{\bar{t} - t}\right) \Bigg[(s-a)^2 + \left(1 - \frac{(\bar{t}-a)^2 - (t-a)^2}{\bar{t} - t}\right)(s-a) - \\ -(t-a) - \left(1 - \frac{(\bar{t}-a)^2 - (t-a)}{\bar{t}-t}\right)(\bar{t}-a)\Bigg] + \bar{x}_i \quad (i = 1,2). \end{aligned}$$

Obviously, the functions $u_i \in BV_{(p,2)}[a,b]$, $(i = 1,2)$, and the relations

$$(u_1 - u_2)(a) = |\bar{x}_1 - \bar{x}_2|, \quad (u_1 - u_2)'(a) = 0$$

and

$$\|(u_1 - u_2)'\|^p_{L_p[a,b]} = \left(\frac{2^p|\bar{x}_1 - \bar{x}_2 - x_1 + x_2|^p}{|t - \bar{t}|^{p-1}}\right)^{1/p}$$

hold.

The inequalities (9) give for the above functions

$$\|Fu_1 - Fu_2\|_p \le M\left(|\bar{x}_1 - x_2| + \left(\frac{2^p|\bar{x}_1 - \bar{x}_2 - x_1 + x_2|^p}{|\bar{t} - t|^{p-1}}\right)^{p/1}\right) \tag{10}$$

where $M := KL$.

Since $u_i(t) = x_i$ and $u_i(\bar{t}) = \bar{x}_i$, $(i = 1,2)$, then by definition of the norm $\|\cdot\|_p$ we obtain

$$\begin{aligned} &\left(\frac{|f(\bar{t},\bar{x}_1) - f(\bar{t},\bar{x}_2) - f(t,x_1) + f(t,x_2)|^p}{|\bar{t} - t|^{p-1}}\right) \le \\ &\le 2^{p-1}M^p\left(|x_1 - x_2|^p + \frac{2^p|\bar{x}_1 - \bar{x}_2 - x_1 + x_2|^p}{|\bar{t} - t|^{p-1}}\right) \end{aligned}$$

which can be rewritten in the form

$$\begin{aligned} &|f(\bar{t},\bar{x}_1) - f(\bar{t},\bar{x}_2) - f(t,x_1) + f(t,x_2)|^p \le \\ &\le 2^{p-1}M^p(|x_1 - x_2|^p|\bar{t} - t|^{p-1} + 2^p|\bar{x}_1 - \bar{x}_2 - x_1 + x_2|^p) \ . \end{aligned} \tag{11}$$

For every fixed $x \in \mathbf{R}$ the constant function $u_0(t) = x$ belongs to $BV_{(p,2)}[a,b]$, whence the function $f(\cdot, x)$ belongs to $BV_{(p,2)}[a,b]$, consequently the function

$f(\cdot, x)$ is continuous on $[a,b]$. Therefore letting $\bar{t} \to t$ in the inequality (11), we get

$$|f(t,\bar{x}_1) - f(t,\bar{x}_2) - f(t,x_1) + f(t,x_2)| \leq \leq 4M|\bar{x}_1 - \bar{x}_2 - x_1 + x_2| \tag{12}$$

for all $t \in [a,b]$ and $x_1, \bar{x}_1, x_2, \bar{x}_2 \in \mathbf{R}$.
Let us fix $t \in [a,b]$ and define the function $P_t : \mathbf{R} \to \mathbf{R}$ by the following formula

$$P_t(x) := f(t,x) - f(t,0) \quad (x \in \mathbf{R}) . \tag{13}$$

Setting $x_1 := w + z$, $x_2 := w$, $\bar{x}_1 := z$ and $\bar{x}_2 := 0$ in the inequality (12) we get

$$f(t, w+z) = f(t,w) + f(t,z) - f(t,0)$$

which, using (13), can be written in the following form

$$P_t(w+z) = P_f(w) + P_t(z) , \quad (w, z \in \mathbf{R}) .$$

Setting $\bar{x}_1 = \bar{x}_2 = 0$ in the inequality (12) we get

$$|P_t(x_1) - P_t(x_2)| \leq 4M|x_1 - x_2| , \quad (x_1, x_2 \in \mathbf{R}) .$$

Thus the function P_t is additive and continuous. Consequently it is linear and, therefore there exists a function $g : [a,b] \to \mathbf{R}$ such that

$$P_t(x) = g(t)x , \quad (x \in \mathbf{R}) .$$

Define the function $h : [a,b] \to \mathbf{R}$ by

$$h(t) := f(t,0) , \quad (t \in [a,b]) .$$

It follows from (13) that

$$f(t,x) = g(t)x + h(t) , \quad (t \in [a,b], x \in \mathbf{R}) .$$

Since the composition operator F maps the space $BV_{(p,2)}[a,b]$ into $BV_{(p,2)}[a,b]$, then the function $h(\cdot) = f(\cdot, 0)$ belongs to $BV_{(p,2)}[a,b]$, and the function $g(\cdot) = f(\cdot,1) - f(\cdot,0)$ also belongs to $BV_{(p,2)}[a,b]$. Thus the function f has the above form with $g, h \in BV_{(p,2)}[a,b]$.

Now suppose that the function f has the form (8); i.e., $f(t,x) = g(t)x + h(t)$, where $g, h \in BV_{(p,2)}[a,b]$.

Since $BV_{(p,2)}[a,b]$ is an algebra, we obtain

$$\|Fu - Fv\|_{(p,2)} \leq \|g\|_{(p,2)} \|u - v\|_{(p,2)} \quad (u, v \in BV_{(p,2)}[a,b]) ,$$

consequently the composition operator F generated by the function f maps the space $BV_{(p,2)}[a,b]$ into itself and satisfies the global Lipschitz condition (7).

Remark 2. A similar problem has been investigated in [3], [2], [4], [1], [5], [6] in the spaces: Lip$[a,b]$, Lip$^\alpha[a,b]$, $C^r[a,b]$, Lip $C^r[a,b]$, $BV[a,b]$ and $BV_\varphi[a,b]$.

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