ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 175 – 180

NATURAL AFFINORS ON HIGHER ORDER COTANGENT BUNDLE

Jan Kurek

Abstract. All natural affinors on the r-th order cotangent bundle $T^{r*}M$ are determined. Basic affinors of this type are the identity affinor id of $TT^{r*}M$ and the s-th power affinors $Q^s_M : TT^{r*}M \to VT^{r*}M$ with $s = 1, \dots, r$ defined by the s-th power transformations $A^{r,r}_s$ of $T^{r*}M$. An arbitrary natural affinor is a linear combination of the basic ones.

Recently, Kolář and Modugno have determined all natural affinors on cotangent bundle T^*M, which constitute a 2 parameter family linearly generated by the identity of TT^*M and by a natural affinor $Q_M : TT^*M \to VT^*M$, [1].

In this paper, we determine all natural affinors on the r-th order cotangent bundle $T^{r*}M$. We deduce that all natural affinors on the r-th cotangent bundle $T^{r*}M$ form a $(r+1)$-parameter family linearly generated by the identity affinor id of $TT^{r*}M$ and by the s-th power natural affinors $Q^s_M : TT^{r*}M \to VT^{r*}M$ with $s = 1, \dots, r$ defined by the s-th power natural transformations $A^{r,r}_s$ of $T^{r*}M$ into itself introduced in [3].

The author is grateful to Professor I. Kolář for suggesting the problem, valuable remarks and useful discussions.

1. Let M be a smooth n-dimensional manifold. Let $T^{r*}M = J^r(M, \mathbb{R})_0$ be the space of all r-jets from a manifold M into $\mathbb{R}$ with target 0. The vector bundle

$$\pi_M : T^{r*}M \to M \tag{1.1}$$

where π_M is the source jet projection, is called the r-th cotangent bundle of M. Let

$$p_M : TT^{r*}M \to T^{r*}M \tag{1.2}$$

$$q_M : T^{r*}M \to T^*M \tag{1.3}$$

1991 *Mathematics Subject Classification*: 58A20, 53A55.
Key words and phrases: higher order cotangent bundle, natural affinor.
Received May 4, 1991.

be the bundle projections. Let $\lambda_M : TT^{r*}M \to \mathbb{R}$ be the generalized Liouville form of $T^{r*}M$, [1], defined by

$$\lambda_M(X) = \langle q_M(p_M(X)), T\pi_M(X)\rangle . \tag{1.4}$$

Let $A_s^{r,r} : T^{r*}M \to T^{r*}M$ be the s-th power natural transformation of $T^{r*}M$, [3], defined by

$$A_s^{r,r} : j_x^r f \to j_x^r (f)^s \tag{1.5}$$

where $(f)^s$ denote the s-th power of f.

Since $\pi_M : T^{r*}M \to M$ is a vector bundle, we have an identification $VT^{r*}M = T^{r*}M \oplus T^{r*}M$. Then using this identification we can define natural affinors Q_M^s on $T^{r*}M$ with $s = 1, \dots, r$.

Definition 1. A natural affinor $Q_M^s : TT^{r*}M \to VT^{r*}M$ defined by

$$Q_M^s(X) = (p_M(X), \lambda_M(X) A_s^{r,r}(p_M(X))) \tag{1.6}$$

is called the s-th power natural affinor.

If (x^i) are some local coordinates on M and $(u_1, u_2, \dots, u_r) := (u_i, u_{i_1 i_2}, \dots, u_{i_1 \dots i_r})$ are the induced fibre coordinates on $T^{r*}M$ (symmetric in all indices), the coordinate expressions of the s-th power natural affinors Q_M^s with $s = 1, \dots, r$ are

$$\begin{aligned} Q_M^s &= u_{i_1} \dots u_{i_s} u_j \frac{\partial}{\partial u_{i_1 \dots i_s}} \otimes dx^j \\ &+ \frac{(s+1)!}{(s-1)!2!} u_{(i_1} \dots u_{i_{s-1}} u_{i_s i_{s+1})} u_j \frac{\partial}{\partial u_{i_1 \dots i_{s+1}}} \otimes dx^j \\ &+ \dots + \frac{r!}{(s-1)!(r-s+1)!} u_{(i_1} \dots u_{i_{s-1}} u_{i_s \dots i_r)} u_j \frac{\partial}{\partial u_{i_1 \dots i_r}} \otimes dx^j \end{aligned} \tag{1.7}$$

where $(i_1 \dots i_r)$ denote the symmetrization.

The identity id of $TT^{r*}M$ constitute a trivial natural affinor on $T^{r*}M$.

2. In this part we determine, by induction with respect to the order r, all natural affinors on $T^{r*}M$.

Theorem 1. *All natural affinors $F^r : TT^{r*}M \to TT^{r*}M$ defined on the r-th cotangent bundle $T^{r*}M$ constitute the $(r+1)$ parameter family of the form*

$$F^r = k_0 \, id + k_1 Q_M^1 + \dots + k_r Q_M^r \tag{2.1}$$

with any real parameters $k_0, k_1, \dots, k_r \in \mathbb{R}$.

Proof. The r-th cotangent bundle functor T^{r*} is defined on the category $\mathcal{M}f_n$ of n dimensional smooth manifolds with values in the category $\mathcal{VB}$ of vector bundles and is of the order r. Since the tangent functor T is of the order 1, the superposition

TT^{r*} is of the $(r+1)$-th order functor. Then, its standard fibre $T(T^{r*}\mathbb{R}^n)_0$ is a G_n^{r+1}-space, where G_n^{r+1} means a group of all invertible $(r+1)$ jets of $\mathbb{R}^n$ into $\mathbb{R}^n$ with source and target at 0.

Natural affinors $F^r : TT^{r*}M \to TT^{r*}M$ are in bijection with G_n^{r+1}-equivariant maps of the standard fibres $F^r : T(T^{r*}\mathbb{R}^n)_0 \to T(T^{r*}\mathbb{R}^n)_0$. Let tilda $\tilde{a} = a^{-1}$ denote the coordinates of the inverse element in G_n^{r+1}. If $(x^i, u_i, u_{i_1i_2}, \dots, u_{i_1\dots i_r})$ are the fibre coordinates on $T^{r*}M$, then we have the induced coordinates on $TT^{r*}M$ of the form

$$Y^i = dx^i,\ W_i = du_i,\ W_{i_1i_2} = du_{i_1i_2}, \dots, W_{i_1\dots i_r} = du_{i_1\dots i_r}\,. \tag{2.2}$$

The action of an element $(a^i_j,\, a^i_{j_1j_2}, \dots, a^i_{j_1\dots j_{r+1}}) \in G_n^{r+1}$ on $(u_i, u_{i_1i_2}, \dots$ $\dots, u_{i_1\dots i_r}, Y^i, W_i, W_{i_1i_2}, \dots, W_{i_1\dots i_r}) \in T(T^{r*}\mathbb{R}^n)_0$ is of the form

$$\begin{aligned}
&\bar{u}_i = u_j\ \tilde{a}^j_i\\
&\bar{u}_{i_1i_2} = u_{j_1j_2}\,\tilde{a}^{j_1}_{i_1}\,\tilde{a}^{j_2}_{i_2} + u_{j_1}\,\tilde{a}^{j_1}_{i_1i_2}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\bar{u}_{i_1\dots i_r} = u_{j_1\dots j_r}\,\tilde{a}^{j_1}_{i_1}\dots\tilde{a}^{j_r}_{i_r}\\
&+ u_{j_1\dots j_{r-1}}\frac{r!}{(r-2)!2!}\,\tilde{a}^{j_1}_{(i_1}\dots\tilde{a}^{j_{r-2}}_{i_{r-2}}\,\tilde{a}^{j_{r-1}}_{i_{r-1}i_r)} + \dots +\\
&+ u_{j_1j_2}\left[\frac{r!}{(r-1)!1!}\,\tilde{a}^{j_1}_{(i_1}\,\tilde{a}^{j_2}_{i_2\dots i_r)} + \dots\right] + u_j\,\tilde{a}^j_{i_1\dots i_r}\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\bar{Y}^i = a^i_j\,Y^j\\
&\bar{W}_i = W_j\,\tilde{a}^j_i + u_j\,\tilde{a}^j_{ik}\,a^k_l\,Y^l\\
&\bar{W}_{i_1i_2} = W_{j_1j_2}\,\tilde{a}^{j_1}_{i_1}\,\tilde{a}^{j_2}_{i_2} + W_{j_1}\,\tilde{a}^{j_1}_{i_1i_2} + u_{j_1j_2}\,\tilde{a}^{j_1}_{i_1k}\,\tilde{a}^{j_2}_{i_2}\,a^k_l Y^l\\
&+ u_{j_1j_2}\,\tilde{a}^{j_1}_{i_1}\,\tilde{a}^{j_2}_{i_2k}a^k_l Y^l + u_{j_1}\,\tilde{a}^{j_1}_{i_1i_2k}a^k_l Y^l\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\bar{W}_{i_1\dots i_r} = W_{j_1\dots j_r}\,\tilde{a}^{j_1}_{i_1}\dots\tilde{a}^{j_r}_{i_r}\\
&+ W_{j_1\dots j_{r-1}}\frac{r!}{(r-2)!2!}\,\tilde{a}^{j_1}_{(i_1}\dots\tilde{a}^{j_{r-2}}_{i_{r-2}}\,\tilde{a}^{j_{r-1}}_{i_{r-1}i_r)} + \dots +\\
&+ W_{j_1j_2}\left[\frac{r!}{(r-1)!1!}\,\tilde{a}^{j_1}_{(i_1}\,\tilde{a}^{j_2}_{i_2\dots i_r)} + \dots\right] + W_j\,\tilde{a}^j_{i_1\dots i_r}\\
&+ u_{j_1\dots j_r}\left[\tilde{a}^{j_1}_{i_1k}\dots\tilde{a}^{j_r}_{i_r}\,a^k_l\,Y^l + \dots\right]\\
&+ u_{j_1\dots j_{r-1}}\left[\frac{r!}{(r-2)!2!}\,\tilde{a}^{j_1}_{(i_1k}\dots\tilde{a}^{j_{r-2}}_{i_{r-2}}\,\tilde{a}^{j_{r-1}}_{i_{r-1}i_r)}a^k_l\,Y^l + \dots\right]\\
&+ \dots + u_{j_1j_2}\left[\frac{r!}{(r-1)!1!}\,\tilde{a}^{j_1}_{(i_1k}\,\tilde{a}^{j_2}_{i_2\dots i_r)}\,a^k_l\,Y^l\right.\\
&\left.+\frac{r!}{(r-1)!1!}\,\tilde{a}^{j_1}_{(i_1}\,\tilde{a}^{j_2}_{i_2\dots i_r)k}\,a^k_l\,Y^l + \dots\right] + u_j\,\tilde{a}^j_{i_1\dots i_rk}a^k_l\,Y^l\ .
\end{aligned} \tag{2.3}$$

I. In the first induction step, we consider the case $r = 2$. Any G_n^3-equivariant map $F^2 : T(T^{2*}\mathbb{R}^n)_0 \to T(T^{2*}\mathbb{R}^n)_0$ in the induced coordinates $(u_i, u_{ij}, Y^i, W_i, W_{ij})$ on $T(T^{2*}\mathbb{R}^n)_0$ is of the form

$$
\begin{aligned}
\bar{Y}^i &= F^i_j(u_1, u_2)Y^j + F^{ij}(u_1, u_2)W_j + F^{ijk}(u_1, u_2)W_{jk} \\
\bar{W}_i &= F_{ij}(u_1, u_2)Y^j + F_i^j(u_1, u_2)W_j + F_i^{jk}(u_1, u_2)W_{jk} \\
\bar{W}_{ij} &= F_{ijk}(u_1, u_2)Y^k + F^k_{ij}(u_1, u_2)W_k + F^{kl}_{ij}(u_1, u_2)W_{kl} \ .
\end{aligned} \tag{2.4}
$$

Considering the equivariancy of the map F^2 with respect to homotheties : $a^i_j = = k\ \delta^i_j,\ a^i_{jk} = 0,\ a^i_{jkl} = 0$ in G^3_n, we obtain a homogeneity condition

$$
\begin{aligned}
&F^i_j(u_1, u_2) = F^i_j(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&F^{ij}(u_1, u_2) = \frac{1}{k^2}\, F^{ij}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&F^{ijk}(u_1, u_2) = \frac{1}{k^3}\, F^{ijk}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&\frac{1}{k^2}\, F_{ij}(u_1, u_2) = F_{ij}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&F^j_i(u_1, u_2) = F^j_i(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&F^{jk}_i(u_1, u_2) = \frac{1}{k} F^{jk}_i(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&\frac{1}{k^3} F_{ijk}(u_1, u_2) = F_{ijk}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&\frac{1}{k} F^k_{ij}(u_1, u_2) = F^k_{ij}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \\
&F^{kl}_{ij}(u_1, u_2) = F^{kl}_{ij}(\frac{1}{k}\, u_1, \frac{1}{k^2}\, u_2) \ .
\end{aligned} \tag{2.5}
$$

Using the homogeneous function theorem and the invariant tensor theorem, [2], we obtain the map F^2 in the form

$$
\begin{aligned}
\bar{Y}^i &= k_0 Y^i \\
\bar{W}_i &= (k_1 u_i u_k + k_3 u_{ik})\, Y^k + k_4 W_i \\
\bar{W}_{ij} &= (k_2 u_i u_j u_k + k_5 u_{ij} u_k + k_6 u_{(i} u_{j)k})\, Y^k \\
&\quad + k_7 \delta^k_{(i} u_{j)} W_k + k_8 W_{ij}
\end{aligned} \tag{2.6}
$$

with any real parameters $k_0, k_1, \dots, k_8 \in \mathbb{R}$.

The equivariancy of the map F^2 of the form (2,6) with respect to the kernel of the projection $G^3_n \to G^1_n : a^i_j = \delta^i_j$ and a^i_{jk}, a^i_{jkl} are arbitrary, gives the relations for parameters:

$$
k_3 = 0,\ k_4 = k_0,\ k_5 = k_1,\ k_6 = 0,\ k_7 = 0,\ k_8 = k_0 \ . \tag{2.7}
$$

This gives 3 parameter family of natural affinors F^2 of the form

$$\begin{aligned}\bar{Y}^i &= k_0 Y^i \\ \bar{W}_i &= k_1 u_i u_k Y^k + k_0 W_i \\ \bar{W}_{ij} &= k_2 u_i u_j u_k Y^k + k_1 u_{ij} u_k Y^k + k_0 W_{ij}\end{aligned} \tag{2.8}$$

with any real parameters $k_0,\ k_1,\ k_2 \in \mathbb{R}$. Hence, the natural affinors F^2 may be expressed in the form $F^2 = k_0 \text{ id } + k_1 Q^1_M + k_2 Q^2_M$.

II. In the induction step for $(r-1)$, we assume that natural affinors F^{r-1} on $(T^{(r-1)*}\mathbb{R}^n)_0$ constitute the r parameter family of the form

$$F^{r-1} = k_0 \text{ id } + k_1 Q^1_M + \cdots + k_{r-1} Q^{r-1}_M \tag{2.9}$$

with any real parameters $k_0,\ k_1, \ldots, k_{r-1} \in \mathbb{R}$.

Consider any natural affinor F^r on $(T^{r*}\mathbb{R}^n)_0$. Using the homogeneous function theorem, we deduce that its first r components are the components of F^{r-1}. Let the last $(r+1)$-th component be

$$\begin{aligned}\bar{W}_{i_1\ldots i_r} = {} & F_{i_1\ldots i_r j}(u_1,\ldots,u_r)\, Y^j + F^j_{i_1\ldots i_r}(u_1,\ldots,u_r) W_j \\ & + \cdots + F^{j_1\ldots j_r}_{i_1\ldots i_r}(u_1,\ldots,u_r)\, W_{j_1\ldots j_r}\ .\end{aligned} \tag{2.10}$$

Considering the equivariancy of F^r with respect to the homotheties
$a^i_j = k\ \delta^i_j,\ a^i_{j_1 j_2} = 0, \ldots, a^i_{j_1\ldots j_{r+1}} = 0$ in G^{r+1}_n, we obtain the homogeneity conditions

$$\begin{aligned}&\frac{1}{k^{r+1}} F_{i_1\ldots i_r j}(u_1,\ldots,u_r) = F_{i_1\ldots i_r j}(\frac{1}{k}\, u_1, \ldots, \frac{1}{k^r}\, u_r) \\ &\frac{1}{k^{r-1}} F^j_{i_1\ldots i_r}(u_1,\ldots,u_r) = F^j_{i_1\ldots i_r}(\frac{1}{k}\, u_1, \ldots, \frac{1}{k^r}\, u_r) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\frac{1}{k}\, F^{j_1\ldots j_{r-1}}_{i_1\ldots i_r}(u_1,\ldots,u_r) = F^{j_1\ldots j_{r-1}}_{i_1\ldots i_r}(\frac{1}{k}\, u_1, \ldots, \frac{1}{k^r}\, u_r) \\ &F^{j_1\ldots j_r}_{i_1\ldots i_r}(u_1,\ldots,u_r) = F^{j_1\ldots j_r}_{i_1\ldots i_r}(\frac{1}{k}\, u_1, \ldots, \frac{1}{k^r}\, u_r)\ .\end{aligned} \tag{2.11}$$

By the homogeneous function theorem and by the invariant tensor theorem [2], we deduce that the $(r+1)$-th component of F^r is of the form

$$\begin{aligned}\bar{W}_{i_1\ldots i_r} = {} & \big[h^0_r u_{i_1}\ldots u_{i_r} u_j + h^0_{r-1,1} u_{(i_1}\ldots u_{i_{r-2}} u_{i_{r-1} i_r)} u_j \\ & + h^0_{r-1,2} u_{(i_1}\ldots u_{i_{r-2}} u_{i_{r-1}} u_{i_r)j} + \cdots + \\ & + h^0_1 u_{i_1\ldots i_r} u_j\big]\ Y^j \\ & + \Big[h^1_{r-1} \delta^{j_1}_{(i_1}\ u_{i_2}\ldots u_{i_r)} + \cdots + h^1_1\, \delta^{j_1}_{(i_1}\ u_{i_2\ldots i_r)}\Big]\ W_{j_1} \\ & + \cdots + h^{r-1}_1\ \delta^{j_1}_{(i_1}\ldots \delta^{j_{r-1}}_{i_{r-1}}\ u_{i_r)}\, W_{j_1\ldots j_{r-1}} \\ & + h^r_0\, \delta^{j_1}_{(i_1}\ldots \delta^{j_r}_{i_r)}\, W_{j_1\ldots j_r}\ ,\end{aligned} \tag{2.12}$$

with any real parameters $h_r^0,\ h_{r-1,1}^0,\ h_{r-1,2}^0,\dots,h_1^0,\ h_{r-1}^1,\dots,h_1^1,\dots,h_1^{r-1},\ h_0^r\in$ $\in\mathbb{R}$.

The equivariancy of F^r with respect to the kernel of the projection $G_n^{r+1}\to G_n^1$: $a_j^i=\delta_j^i$ and $a_{j_1j_2}^i,\dots,a_{j_1\cdots j_{r+1}}^i$ are arbitrary, gives the relations for parameters:

$$
\begin{aligned}
&h_{r-1,1}^0=k_{r-1},\ h_{r-1,2}^0=0,\dots,h_1^0=k_1\\
&h_{r-1}^1=0,\dots,h_1^1=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&h_1^{r-1}=0,\ h_0^r=k_0
\end{aligned}
\tag{2.13}
$$

If we put $h_r^0=k_r$, this gives the $(r+1)$ parameter family of the natural affinors F^r on $(T^{r*}\mathbb{R}^n)_0$ of the desired form (2,1), $F^r=k_0\,\mathrm{id}+k_1Q_M^1+\dots+k_{r-1}Q_M^{r-1}+k_rQ_M^r$ with any real parameters $k_0,\ k_1,\dots,k_{r-1},k_r\in\mathbb{R}$. This proves our theorem. □

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Jan Kurek
Institute of Mathematics
Maria Curie Sklodovska University
Plac Marii Curie Skłodowskiej 1
20031 Lublin, Poland