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THE STRUCTURE TENSOR AND FIRST ORDER NATURAL DIFFERENTIAL OPERATORS

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Abstract. The notion of a structure tensor of section of first order natural bundles with homogeneous standard fibre is introduced. Properties of the structure tensor operator are studied. The universal factorization property of the structure tensor operator is proved and used for classification of first order $*$-natural differential operators $\underline{D} : \underline{T \times T} \to \underline{T}$ for $n \geq 3$.

Introduction

In this paper, in analogy to the structure tensor of a G-structure, we introduce the notion of a structure tensor for sections of first order natural bundles with homogeneous standard fibre. In this approach, the structure tensor turns out to be a natural differential operator of order ≤ 1. We prove that it has the following universal factorization property: first order natural differential operators $\underline{D} : \underline{F} \to \underline{G}$, where ord $(F) = 1$, ord $(G) \leq 1$ and F has homogeneous fibre, are compositions of the structure tensor and natural transformations (that is, operators of order zero). Therefore the classification of such n.d. operators can be reduced to the classification of L^1_n-equivariant maps $K : HF_0 \to G_0$, where HF is the bundle of structure tensors. We give explicit formulae for structure tensors of some natural bundles. As an example of the application of this results, we give the classification of first order $*$-n.d. operators $\underline{D} : \underline{T \times T} \to \underline{T}$ for $n \geq 3$. All facts conserning natural bundles, needed in this paper, can be found in [12] (see also [4], [6], [11]). Information on G-structures and the structure tensor can be found for example in [1], [3]. Another approach to the structure tensor (of order k) and its relation to natural bundles is presented in [8].

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1. Preliminaries

We introduce here some basic notations and definitions. We will assume, if not stated otherwise, that all manifolds, bundles and maps are smooth. For a category $\mathcal{C}$ the class of morphism of $\mathcal{C}$ will be denoted by $\mathrm{Mor}(\mathcal{C})$ and the class of objects of $\mathcal{C}$ by $\mathrm{Ob}(\mathcal{C})$. We will often write $D \in \mathcal{C}$ instead of $D \in \mathrm{Ob}(\mathcal{C})$. The identity functor on $\mathcal{C}$ will be denoted by $\iota_{\mathcal{C}}$ and ι_D will denote the identity morphism on D ($D \in \mathrm{Ob}(\mathcal{C})$). $\mathrm{Hom}_{\mathcal{C}}(D, E)$ or $\mathrm{Hom}(D, E)$ will denote the set of morphisms from D to E, where $D, E \in \mathrm{Ob}(\mathcal{C})$.

Let $\mathcal{M}_n$ denote the category of n-dimensional manifolds and embeddings, and let $\mathcal{F}_n$ be the category of libre bundles over objects of $\mathcal{M}_n$ and fibred maps which are diffeomorphisms on fibres and cover morphisms of $\mathcal{M}_n$. The projection functor $\mathcal{F}_n \to \mathcal{M}_n$ will be denoted by π.

1.1. Definition. A natural bundle is a covariant functor $F : \mathcal{M}_n \to \mathcal{F}_n$ such that $\pi \circ F = \iota_{\mathcal{M}_n}$ and F is regular: if $f : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$ is smooth, then $F(f) : \mathbf{R}^k \times F(\mathbf{R}^n) \to F(\mathbf{R}^n)$ is smooth, where $F(f)(t, \cdot) = F(f(t, \cdot))$, $t \in \mathbf{R}^k$. For $M \in \mathcal{M}_n$ π_F or π will denote the bundle projection $F(M) \to M$.

The category of natural bundles over n-dimensional manifolds with natural transformation of functors as morphisms will be denoted by $\mathcal{NF}_n$. Then $f \in \mathrm{Hom}_{\mathcal{NF}_n}(F, G)$ iff $f = \{f(M)\}_{M \in \mathcal{M}_n}, \forall M \in \mathcal{M}_n,\ f(M) \in \mathrm{Mor}(\mathcal{F}_n)$ and

$$(1.1) \qquad \forall \phi \in \mathrm{Mor}(\mathcal{M}_n),\ \phi : M \to N,\ G(\phi) \circ f(M) = f(N) \circ F(\phi)\ .$$

Let $F \in \mathcal{NF}_n$. The smallest integer k, such that $F(f)(x)$ depends only on $j^k_x f$, for every $f \in \mathrm{Mor}(\mathcal{N}F_n)$, is called the order of F. We will write $k = \mathrm{ord}\ (F)$.

1.2. Remark. It is known (see for example [4], [12]), that every natural bundle F has finite order and can be obtained as a bundle associated to the k-frame bundle, with the standard fibre $F_0 = F(\mathbf{R}^n)_0$. More precisely, let $L^k_n = \{j^k_0\varphi : \varphi \in \mathrm{Diff}(\mathbf{R}^n, \mathbf{R}^n), \varphi(0) = 0\}$ be the k-th order differential group and let $F^k(M) = \{j^k_0 f | f \in \mathrm{Diff}(\mathbf{R}^n, M)\}$ denote the k-frame bundle on $M \in \mathcal{M}_n$. Then F^k is a principal fibre bundle with the right action of L^k_n given by composition of jets. The functor F determines a left action $\rho : L^k_n \times F_0 \to F_0, \rho(j^k_0\varphi, x) = F(\varphi)(x)$. Let $F_\rho(M)$ be the quotient $(F^k \times F_0)/L^k_n$ where L^k_n acts on $F^k \times F_0$ on the right in the following way:

$$(1.2) \qquad (F^k(M) \times F_0) \times L^k_n \ni ((h, y), a) \to (ha, \rho(a^{-1}, y)) \in F^k(M) \times F_0\ .$$

Then F_ρ is a natural functor, and is naturally equivalent (or isomorphic in the category $\mathcal{NF}_n$) to F. Further on we will identify F_ρ with F.

Note that if $f \in \mathrm{Hom}_{\mathcal{NF}_n}(F, G)$ then $f_0 : F_0 \to G_0, f_0 = f(\mathbf{R}^n)|_{F_0}$ is L^k_n-equivariant. On the other hand L^k_n-equivariant maps between L^k_n-manifolds give rise to natural transformations of associated functors (see [6]). One can prove that, in fact, the category of L^k_n-manifolds with L^k_n-equivariant maps is equivalent to the full subcategory of $\mathcal{NF}_n$, whose objects are functors of order $\leq k$.

Natural differential operators

Let $F \in \mathcal{NF}_n$. $\underline{F}(M)$ will denote the set of local sections of $F(M) \to M$, and $J^rF \in \mathcal{NF}_n$ the r-jet prolongation of $F(J^0F = F)$. If $G \in \mathcal{NF}_n$ then the formula $\underline{D} : \underline{F} \to \underline{G}$ will mean that $\underline{D} = \{\underline{D}(M)\}_{M \in \mathcal{M}_n}$ is a family of maps $\underline{D}(M) : \underline{F}(M) \to \underline{G}(M)$. For $N \in \mathrm{Hom}(F, G), \underline{N} : \underline{F} \to \underline{G}$ will be defined so that $\underline{N}(M) : \underline{F}(M) \ni \sigma \to N \circ \sigma \in \underline{G}(M)$.

1.3. Definition. $\underline{D} : \underline{F} \to \underline{G}$ is a natural differential operator if and only if

$$\exists k \ \in \mathbf{N}\, \exists D^k \in \mathrm{Hom}(J^kF, G) : \underline{D} = \underline{D^k} \circ j^k \tag{1.3}$$

where $j^k : \underline{F} \to \underline{J^kF}, j^k(M)(\sigma) = j^k\sigma$. The smallest k satisfying (1.3) will be called the order of $\underline{D}$ (we will write $k = \mathrm{ord}\ (\underline{D})$).

1.4. Remark. Let $\varphi \in \mathrm{Hom}_{\mathcal{M}_n}(M, N), \sigma \in \underline{F}(M)$. We put

$$\varphi_*\sigma = F(\varphi) \circ \sigma \circ \varphi^{-1} \in \underline{F}(N).$$

It is not difficult to check that if $\underline{D} : \underline{F} \to \underline{G}$ is a n.d. operator, then

$$\forall \varphi \in \mathrm{Mor}(\mathcal{M}_n), \varphi : M \to N \ \forall \sigma \in \underline{F}(M),\ D(\varphi_*\sigma) = \varphi_*D(\sigma)\,. \tag{1.4}$$

Natural differential operators are often defined as families $\underline{D} : \underline{F} \to \underline{G}$ which satisfy (1.4) (throughout this paper such families will be called $*$-natural differential operators). If $\underline{D} : \underline{F} \to \underline{G}$ is a $*$-n.d. operator then for $\sigma \in \underline{F}(M)$ and $x \in \mathrm{dom}(\sigma)$, $\underline{D}(\sigma)$ depends only on $j^\infty_x\sigma$ (see [12]). If $k = \mathrm{ord}\ (\underline{D}) < \infty$ then $\underline{D} = \underline{D^k} \circ j^k$ where $D^k : J^kF \to G$ fulfils the condition (1.1) but $D^k(M)$ is not necessarily continuous (see[2]).

Natural affine bundles

Some natural bundles (for example jet bundles) have an additional affine structure. We use them often in this paper, and to give our statements in a more compact form, we introduce the notion of a natural affine bundle.

We will denote by $\mathcal{AF}_n$ (resp. $\mathcal{V}\ \mathcal{F}_n$) the category of affine (vector) bundles over objects of $\mathcal{F}_n$ and affine (vector) bundle homomorphisms which cover morphisms of $\mathcal{F}_n$. The projection functor from $\mathcal{AF}_n(\mathcal{VF}_n)$ to $\mathcal{F}_n$ will be denoted by p.

For $A \in \mathcal{AF}_n, LA$ will denote the vector bundle corresponding to A. If $y \in p(A)$ then LA_y is the vector space of translations in A_y and we have the maps

$$A_y \times LA_y \ni (a, v) \to (a + v) \in A_y$$
$$A_y \times A_y \ni (a, b) \to (b - a) \in LA_y\ .$$

For $f \in \mathrm{Hom}_{\mathcal{AF}_n}(A, B)$, $Lf \in \mathrm{Hom}_{\mathcal{VF}_n}(LA, LB)$ will denote the linear part of f, i.e. $f(a + v) = f(a) + Lf(v)$. Then L is a covariant functor from $\mathcal{AF}_n$ to $\mathcal{VF}_n$. We will regard $\mathcal{VF}_n$ as a subcategory of $\mathcal{AF}_n$ so L restricted to $\mathcal{VF}_n$ is the identity functor.

1.5. Definition. Let $F \in \mathcal{NF}_n$. A natural bundle $G : \mathcal{M}_n \to \mathcal{AF}_n (G : \mathcal{M}_n \to \mathcal{VF}_n)$ will be called a natural affine (vector) bundle over F if and only if $p \circ G = F$.

The category $\mathcal{NAF}$ of natural affine bundles over a functor F will be defined so that $f = \{f(M)\}_{M \in \mathcal{M}_n} \in \mathrm{Mor}(\mathcal{NAF})$ if and only if $f \in \mathrm{Mor}(\mathcal{NF}_n)$, $\forall M \in \mathcal{M}_n$, $f(M) \in \mathrm{Mor}(\mathcal{AF}_n)$ and $p(f(M)) = \iota_{F(M)}$. In a similar way one can define the category $\mathcal{NVF}$ of natural vector bundles over F. In the case $F = \iota_{\mathcal{M}_n} \mathcal{NAF}(\mathcal{NVF})$ will be denoted by $\mathcal{NA}$ $(\mathcal{NV})$ and called the category of natural affine (vector) bundles.

The functor $L : \mathcal{AF}_n \to \mathcal{VF}_n$ induces a covariant functor from $\mathcal{NAF}$ to $\mathcal{NVF}$ which will be also denoted by L.

1.6. Examples. Fix $F \in \mathcal{NF}_n$.

1. Let T denote the tangent bundle functor. Then $T \in \mathcal{NV}$ and $TF \in \mathcal{NVF}$.

2. Let $K \in \mathcal{NA}$. We define $K \times F \in \mathcal{NAF}$ in the following way: $K \times F(M) = K(M) \times_M F(M)$, $K \times F(\phi) = K(\phi) \times F(\phi)|_{K \times F(M)}$.

3. Let $K \in \mathcal{NV}, G \in \mathcal{NVF}$. We define $K \otimes G \in \mathcal{NVF}$ so that $K \otimes G(M) = K \times F(M) \otimes G(M)$, $K \otimes F(\phi) \otimes G(\phi)$.

4. $T^* \otimes TF \in \mathcal{NVF}$. Let $j \in \mathrm{Hom}_{\mathcal{NF}_n}(J^1 F, T^* \otimes TF)$ be such that

$$j(M) : J^1 F(M) \ni j^1_x \sigma \to d_x \sigma \in T^* \times TF(M) \ . \tag{1.5}$$

Since $j(M)$ is an immersion we can identify $J^1 F(M)$ with $\mathrm{im}\, j(M) \subset T^* \otimes TF(M)$. Then

$$J^1 F(M) = (\iota_{T^*} \otimes d\pi_F)^{-1} (\iota_{TM}) \ . \tag{1.6}$$

Therefore $J^1 F \in \mathcal{NAF}$ and $LJ^1 F(M) = (\iota_{T^*} \otimes d\pi_F)^{-1}\ (0_{TM}) = T^* \otimes VF(M)$, where $VF(M)$ denotes the vertical bundle of $F(M)$, $VF \in \mathcal{NVF}$, so $LJ^1 F = T^* \otimes VF$.

1.7. Remark. If $W \in \mathcal{NVF}$ and ord $(W) = k$, then W_0 is an L^k_n-vector bundle over F_0, i.e. L^k_n acts on W_0 so that for $a \in L^k_n$, $\rho_a : W_0 \to W_0$ is a vector bundle homomorphism and the bundle projection $p : W_0 \to F_0$ is L^k_n-equivariant. If $f \in \mathrm{Hom}_{\mathcal{NVF}}(G, H)$ then $f_0 : G_0 \to H_0$ is an L^k_n-equivariant homomorphism of vector bundles over ι_{F_0}. Similarly, as in remark (1.2), this can be expressed as equivalence of suitable categories. An analogous statement is true for $\mathcal{NAF}$.

1.8. Definition. Let $G, H, K \ \in \mathcal{NAF}, \alpha \in \mathrm{Hom}(G, H)$, $\beta \in \mathrm{Hom}(H, K)$.
1. The sequence $G \xrightarrow{\alpha} H \xrightarrow{\beta} K$ is exact if and only if $\forall M \in \mathcal{M}_n$ the sequence $LG(M) \xrightarrow{L\alpha(M)} LH(M) \xrightarrow{L\beta(M)} LK(M)$ is exact.
2. We say that α is a monomorphism (an epimorphism) if and only if $\forall M \in \mathcal{M}_n$,

$\alpha(M)$ is a monomorphism (an epimorphism). In the language of exact sequences we will express this fact by saying that the sequence $0 \to G \xrightarrow{\alpha} H$ ($G \xrightarrow{\alpha} H \to 0$) is exact.
3. We say that the morphism α has constant rank if and only if $\exists K \in \mathbf{N}$, $\forall M \in \mathcal{M}_n$, rank $(L\alpha(M)) = k$.

1.9. Proposition. *Let $F \in \mathcal{NF}_n$, ord $(F) = k$. If F is homogeneous, that is , L_n^k acts transitively on F_0, then rank$(\alpha) = const$ for every $\alpha \in$ Mor$(\mathcal{NAF})$.*

Proof. Let $a \in F(M)$, $b \in F(N)$. F is homogeneous, so there exists $\phi \in \mathrm{Hom}(M,N)$ such that $F(\phi)a = b$. Let $\alpha \in \mathrm{Hom}_{\mathcal{NAF}}(G, H$. Then (1.1) implies that $H(\phi) \circ \alpha(M)_a = \alpha(N)_b \circ G(\phi)$. Since $H(\phi)$, $G(\phi)$ are isomorphisms on fibres, rank $(\alpha(M)_a) =$ rank $(\beta(N)_b)$. □

Morphisms of constant rank can be used to define new natural affine bundles. Let $G, H \in \mathcal{NAF}$, $\alpha \in \mathrm{Hom}(G,H)$ and rank(α) =const. Then we can define $\mathrm{im}\,\alpha \in \mathcal{NAF}$, $(\mathrm{im}\,\alpha)(M)$ =$\mathrm{im}\,\alpha(M)$, $(\mathrm{im}\,\alpha)(\phi) = H(\phi)|\mathrm{im}\,\alpha(M)$. We also have coker $\alpha \in \mathcal{NVF}$ where (coker $\alpha)(M) = H(M)/\mathrm{im}\alpha(M)$ and coker $\alpha(\phi)$ is defined by the commutative diagram

$$\begin{array}{ccc} H(M) & \xrightarrow{\quad H(\phi) \quad} & H(N) \\ \downarrow{\scriptstyle q(M)} & & \downarrow{\scriptstyle q(N)} \\ \text{coker } \alpha(M) & \xrightarrow{\text{coker } \alpha(\phi)} & \text{coker } \alpha(N), \end{array}$$

($q(M)$ and q(N) are the canonical projections). If $\alpha : G \to H$ is such that $\alpha(M)$ is an inclusion for every $M \in \mathcal{M}_n$, then G will be called a natural affine (vector, in the case $G, H \in \mathcal{NVF}$) subbundle of H and coker α will be denoted by H/G. If $H \in \mathcal{NVF}$ then one can define ker $\alpha \in \mathcal{NAF}$ in the obvious way.

1.10. Remark. It is easy to see that $L(\mathrm{im}\alpha) = \mathrm{im}L(\alpha)$, $L($ ker $\alpha) =$ ker $L(\alpha)$, $L($ coker $\alpha) =$ coker $L(\alpha)$ and q, from the diagram above, is in Mor$(\mathcal{NAF})$.

Natural vector bundles

Let $W \in \mathcal{NV}$. Then $J^kW(M)$ has the structure of a vector bundle:$aj_x^1\sigma + bj_x^1\eta = j_x^1(a\sigma + b\eta)$, $a, b \in \mathbf{R}$. Therefore $J^kW \in \mathcal{NV}$. Note that J^1W can be regarded as an element of $\mathcal{NAW}$ or of $\mathcal{NV}$. This will not, however, lead to confusion because we will consider the category $\mathcal{NAF}$ only for fixed $F \in \mathcal{NF}_n$.

For $W \in \mathcal{NV}$ we define $\varepsilon_W^k \in \mathrm{Hom}_{\mathcal{NV}}(S^kT^* \otimes W, J^kW)$ linearly extending the formula

$$\varepsilon_W^k(M)(\omega^1 \odot \cdots \odot \omega^k \otimes s) = j_x^k(f^1 \cdot \ldots \cdot f^k\eta) \ , \tag{1.7}$$

where $f^i \in C^\infty(M)$, $f^i(x) = 0, d_xf^i = \omega^i$ for $i = 1, \ldots, k$; $\eta \in \underline{W}(M), \eta(x) = s$. It is not difficult to check in coordinates that $\varepsilon_W^k(M)$ is well defined and the following sequence of natural vector bundles is exact:

$$0 \to S^kT^* \otimes W \xrightarrow{\varepsilon^k_W} J^kW \xrightarrow{\rho^k_{k-1}} J^{k-1}W \to 0 \ , \tag{1.8}$$

where for $F \in \mathcal{NF}_n$ and $r \geq k$ $\rho^r_k \in \mathrm{Hom}(J^rF, J^kF)$ denotes the canonical projection, $\rho^r_k(M) : J^rF(M) \ni j^r_x\sigma \to j^k_x\sigma \in J^kF(M)$. $T, J^kT \in \mathcal{NV}$ and $\rho^k_0 : J^kT \to T$ has constant rank. We define $(J^kT)^0 = \ker \rho^k_0 \in \mathcal{NV}$. Then we have the following exact sequence:

$$0 \to (J^kT)^0 \to J^kT \xrightarrow{\rho^k_0} T \to 0 \ . \tag{1.9}$$

2. The action of $S^{k+1}T^* \times T(M)$ on $J^1F(M)$

In order to study this action we will introduce, following [12], the notion of a connection of order k. Let $F \in \mathcal{NF}_n$, $M \in \mathcal{M}_n$, $X \in \underline{T}(M)$. $F(X) \in \underline{TF}(M)$ will denote the infinitesimal lifting of X. Locally, $F(X)$ is generated by $(F(\varphi_t))$ where (φ_t) is the local 1-parameter group of diffeomorphisms of X. If ord $(F) = k$ then for every $y \in F(M)$, $F(X)_y$ depends only on j^k_xX and we have a map

$$f(M) : K^kT \times F(M) \ni (j^k_xX, y) \to F(X)_y \in TF(M) \ .$$

Then fM is a homomorphism of vector bundles and covers $\iota_F(M)$ (see lemma 2.4 in [12]). It is also possible to prove that $f = \{f(M)\}_{M\in\mathcal{M}_n}$ is a natural transformation of the functor $J^kT \times F$ to TF (proof of prop. 2.9 in [12]). Using the terminology introduced in section 1, one can say that $f \in \mathrm{Hom}_{\mathcal{NVF}}(J^kT \times F, TF)$.

If W is a natural bundle then $W(\mathbf{R}^n) = W_0 \times \mathbf{R}^n$. This implies that $(TF)_0 = T(F_0 \times \mathbf{R}^n)_0 = TF_0 \times \mathbf{R}^n$. we can identify $(J^kT \times F)_0$ with $l^k_n\mathbf{R}^n \times F_0$ where l^k_n denotes the Lie algebra of L^k_n. Then from the formulae in [12] p.24 one can get an explicit formula for f_0:

$$f_0 : l^k_n \times \mathbf{R}^n \times F_0 \ni (\xi, v, y) \to (\rho'_y(\xi), v) \in TF_0 \times \mathbf{R}^n \tag{2.1}$$

where $\rho'_y : l^k_n \to T_yf_0$ is the differential of the orbital projection $\rho_y : L^k_n \ni a \to \rho(a, y) \in F_0$ in the neutral element of L^k_n. If F is homogeneous, then ρ_y is a submersion, so ρ'_y is an epimorphism. Therefore we have the following

2.1. Proposition. $f \in \mathrm{Hom}_{\mathcal{NVF}}(J^kT \times F, TF)$. *If F is homogeneous, then F is an epimorphism.*

Bundle of connections

Let $k \geq 1$, $M \in \mathcal{M}_n$. Then

$$C^k(M) = (\iota_{T^*} \otimes \rho^k_0)^{-1}(\iota_{TM}) \subset T^* \otimes J^kT(M) \tag{2.2}$$

is a bundle of connections of order k. C^k is an affine subbundle of $T^* \otimes J^kT$, $C^k \in \mathcal{NA}, LC^k = T^* \otimes (J^kT)^0$ and ord $(C^k) = k + 1$.

Covariant derivative

If λ is a global section of $C^k(M)$ then it is called a connection on M. For $X \in \underline{T}(M), \sigma \in \underline{F}(M)$ (ord $(F) = k$) the covariant derivative of σ is defined in the following way:

$$(\nabla_X \sigma)(x) = d_x\sigma(X) - F_{\sigma(x)}(\lambda(X)) \in VF(M) \tag{2.3}$$

$C^k \times F$ is an affine subbundle of $T^* \otimes J^kT \times F$. we will prove that $\iota^{T^*} \otimes f(C^k \times F(M)) \subset J^1F(M)$ $(k = \text{ord } (F))$.

Let $\lambda \in T^* \otimes J^kT(M)_x$ $y \in F(M)_x$. From (2.2) and (1.6) we have:

$$\begin{aligned} &(\lambda, y) \in C^k \times F(M) \Longleftrightarrow \forall X \in T_xM \quad \rho_0^k(\lambda(X)) = X \\ &\iota_{T^*} \otimes f(\lambda, y) \in J^1F(M) \Longleftrightarrow \forall X \in T_xM \quad d\pi_F(f(\lambda(X), y)) = X \ . \end{aligned} \tag{2.4}$$

Since $d\pi_F \circ f = \rho_0^k, d\pi_F(f(\lambda(X), y)) = \rho_0^k(\lambda(X))$, and (2.4) implies

$$(\lambda, y) \in C^k \times F(M) \Leftrightarrow \iota_{T^*} \otimes f(\lambda, y) \in J^1F(M) \ . \tag{2.5}$$

Therefore one can define $h : C^k \times F \to J^1F$ so, that the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & C^k \times F & \longrightarrow & T^* \otimes J^kT \times F \\ & & \Big\downarrow h & & \Big\downarrow \iota_{T^*\otimes f} \\ 0 & \longrightarrow & J^1F & \longrightarrow & T^* \otimes TF \end{array}$$

commutes. Then, as a consequence of proposition 2.1, we get the following

2.2. Proposition. *$h \in \mathrm{Hom}_{\mathcal{NAF}}(C^k \times F, J^1F)$. If F is homogeneous then h is an epimorphism.*

2.3. Remark. Let λ be a connection on $M, \rho \in \underline{F}(M)$. Then from (1.5) and (2.3) we have:

$$\nabla\rho(x) = j_x^1\sigma - h(\lambda_x, \sigma_x) \in T^* \otimes VF(M) \ . \tag{2.6}$$

Therefore, λ is a σ-connection (i.e. $\nabla\sigma = 0$) if and only if $h \circ (\lambda, \sigma) = j^1\sigma$.

2.4. Definition. Let $B^k \in \mathcal{NF}_n$ be such, that for $M, N \in \mathcal{M}_n, \phi \in \hom(M, N)$

$$\begin{aligned} B^k(M) &= \{j_x^k\psi : \psi \in \mathrm{Diff}(M; x), j_x^{k-1}\iota_M, x \in M\} \\ b^k(\phi)(j_x^k\psi) &= j_x^k(\phi \circ \psi \circ \phi^{-1}) \ . \end{aligned}$$

B_0^k is a subgroup of L_n^k and it can be identified with $S^k(\mathbf{R}^n)^* \otimes \mathbf{R}^n$:

$$I : B_0^k \ni j_0^k\phi \to (t_{j_1 \ldots j_k}) = (\frac{\partial^k}{\partial x_{j_1} \ldots \partial x_{j_k}}\phi^i) \in S^k(\mathbf{R}^n)^* \otimes \mathbf{R}^n$$

is an isomorphism of groups. L^k_n acts on B^k_0 by the transformation $L^k_n \times B^k_0 \ni (a,b) \to aba^{-1} \in B^k_0$ and $I : B^k_0 \to S^k(\mathbf{R}^n)^* \otimes \mathbf{R}^n$ is L^k_n-equivariant. Therefore natural bundles B^k and $S^k t^*$ are isomorphic and we will identify them.

Let $\Delta^{1,k}$ denote the canonical monomorphism $S^{k+1}T^* \otimes T \to T^* \otimes S^k T^* \otimes T$. We define $\epsilon^{1,k} = (\iota^{T^*} \otimes \epsilon^k_t) \circ \Delta^{1,k}$. Then $\epsilon^{1,k} \in \mathrm{Hom}_{\mathcal{NV}}(S^{k+1}T^* \otimes T, t^* \otimes (J^kT)^0)$ is a monomorphism. By using the formula [12]p. 42, which describes the rule of transformation of a connection under the action of a diffeomorphism, we obtain the formula for the action of $S^{k+1}T^* \otimes T(M)$ on $C^k(M)$:

2.5. Lemma. *If $\lambda \in C^k(M_x), j^{k+1}_x\phi \in S^{k+1T^* \otimes T(M)}$ then*

$$C^k(\phi)\lambda = \lambda + \epsilon^{1,k}(j^{k+1}_x\phi)\ . \tag{2.7}$$

Now we will describe the action of $S^{k+1}T^* \otimes T(M)$ on $J^1F(M)$.

2.6. Theorem. *Let $F \in \mathcal{NF}_n$, ord $(F) = k, z \in J^1F(M)_x$ and $j^{k+1}_x\phi \in S^{k+1}T^* \otimes T(M)$. Then*

$$J^1F(\phi)z = z + \chi(j^{k+1}_x\phi, y) \tag{2.8}$$

where $\chi = Lh \circ \epsilon^{1,k} \in \mathrm{Hom}_{\mathcal{NV}F}(S^{k+1}T^* \otimes T, T^* \otimes VF), y - \rho^1_0(z)$.

Proof. Let $\lambda \in C^k(M)_x$. Since $z - h(\lambda, y) \in T^* \otimes VF(M)$ is an element of a k-th order bundle,

$$\begin{aligned} z - h(\lambda,y) &= T^* \otimes VF(\phi)(z - h(\lambda,y)) = \\ &= J^1F(\phi)z - J^1F(\phi)h(\lambda,y) = \\ &= J^1F(\phi)z - h(C^k(\phi)\lambda, y) = \\ &= J^1F(\phi)z - h(\lambda + \epsilon^{1,k}(j^{k+1}_x\phi), y) = \\ &= J^1F(\phi)z - h(\lambda,y) - Lh \circ \epsilon^{1,k}(j^{k+1}_x\phi, y)\ .\square \end{aligned}$$

3. The structure tensor

We will assume that $F \in \mathcal{NF}_n$, ord $(F) = 1$ and F is homogeneous. $T^* \otimes T$ can be identified with $\mathrm{im}\,\epsilon^1_T = (J^1T)^0$ (see 1.8, 1.9). Then $LC^1 = T^* \otimes (J^1T)^0 = T^1_2$. Let $t : C^1 \to \wedge^2 T^* \otimes T$ denote the torsion transformation: for $M \in \mathcal{M}_n, \lambda \in C^1(M)$ $t(M)(\lambda) \in \wedge^2T^* \otimes T(M)$ is a torsion tensor of the connection λ.

3.1. Lemma. *$Lt \in \mathrm{Hom}^{\mathcal{NV}}(t^1_2, \wedge^2T^* \otimes T)$ is equal to $-\mathcal{A}$ where $\mathcal{A}(N)$ is the skew symmetrization: for $X, Y \in T(M)_x$, $C \in T^1_2(M)$, $\mathcal{A}(M)(C)(X,Y) = C(X,Y) - C(Y,X)$.*

Proof. Let $\phi : U \subset M \to \mathbf{R}^n$ be a coordinate system. Then $C^1(\phi) : C^1(U) \to C^1(\mathbf{R}^n) = \mathbf{R}^3 \times \mathbf{R}^n$ is a coordinate system on $C^1(U)$. It is proved in [12] that

$C^1(\phi)(\lambda^i_{jk} = -\Gamma^i_{jk})$ where Γ^i_{jk} are the Christoffel symbols of λ, and if one calculates in coordinates, one can easily see that $Lt = -\mathcal{A}$. □

SC^1 will denote the natural bundle of torsion-free connections:$SC^1 = \ker\, t$. Then we have the following exact sequence of natural affine bundles:

$$0 \to SC^1 \to C^1 \to \wedge T^* \otimes T \to 0 \tag{3.1}$$

and $L(SC^1) = L(\ker\, t) = \ker\,(-\mathcal{A}) = S^2T^* \otimes T$. We will prove the following:

3.2. Theorem. *The following diagram is exact and commutative:*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & S^2T^*\otimes T\times F \arrow[r, equal] \arrow[d] & S^2T^*\otimes T\times F \arrow[r] \arrow[d, "\chi"] & 0 \\
0 \arrow[r] & \ker Lh \arrow[r] \arrow[d, "-\partial"] & T^1_2\times F \arrow[r, "Lh"] \arrow[d, "-\mathcal{A}"] & T^*\otimes VF \arrow[r] \arrow[d, "LS^1"] & 0 \\
0 \arrow[r] & im\,\partial \arrow[r] \arrow[d] & \wedge^2T^*\otimes T\times F \arrow[r, "q"] \arrow[d] & HF \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.2}$$

where ∂ denotes $\mathcal{A}$ restricted to ker $Lh, HF = \wedge^2T^*\otimes T\times F/im\,\partial, q$ denotes the canonical projection and $S^1 \in \mathrm{Hom}_{\mathcal{NAF}}(J^1F, HF)$ is defined so that $S^1\circ h = q\circ t$:

$$\begin{tikzcd}
C^1\times F \arrow[r, "h"] \arrow[d, "t"] & J^1F \arrow[d, "S^1"] \\
\wedge^2T^*\otimes T\times F \arrow[r, "q"] & HF
\end{tikzcd} \tag{3.3}$$

Proof. It is clear that the diagram (3.2) without the arrow LS^1 is exact and commutative. Therefore ker $Lh \subset$ ker $(q\circ(-\mathcal{A}))$. Since h in (3.3) is an epimorphism, there exists a unique S^1 such that (3.3) is commutative. Now we will prove that the third column in (3.2) is exact. We will use (3.2) with the LS^1-arrow missing. Let $p \in T^*\otimes VF(M)$, $LS^1(p) = 0$. There exists $l \in T^1_2 \times F(M)$ such that $Lh(l) = p$. Then $q\circ(-\mathcal{A})(l) = 0$ and $-\mathcal{A}(l)$ $\in$ker $q = \mathrm{im}\,\partial$. Let $b \in$ ker $Lh(M), -\partial(b) = -\mathcal{A}(l)$. Then $l - b \in$ ker $(-\mathcal{A}) = S^2T^*\otimes T\times F(M)$, and $p = Lh(l-b) = \xi(l-b)$ $\in$im$\,\xi(M)$. Therefore ker $LS^1(M)$ $\subset$im$\,\xi(M)$. It is easy to see that $LS^1\circ\xi = 0$ and consequently ker LS =im$\,\xi$. □

3.3. Definition. $\underline{S} = \underline{S}^1 \circ j^1 : \underline{F} \to \underline{HF}$ will be called the structure tensor operator. For $\sigma \in \underline{F}(M) S_\sigma = \underline{S}(M)(\sigma) \in \underline{HF(M)}$ will be called the structure tensor of σ.

3.4. Remark. HF is a natural bundle of order ≤ 1 and $\underline{S}$ is a natural differential operator of order ≤ 1.

Now we will examine how $\underline{S}$ is related to the structure tensor of a G-structure. Since ord $(F) = 1$, $F(M) = (F^1(M) \times F_0)/L^1_n$, where the action of L^1_n is given by formula (1.2). Let Φ denote the canonical projection $F^1(M) \times F_0 \to F(M)$. For a global section $\sigma \in \underline{F}(M)$, $\bar{\sigma} : F^1(M) \to f_0$ will denote the tensorial 0-form of σ : if $x \in M$, $h \in F^1_\xi(M)$ then $\bar{\sigma} h = \Phi_h^{-1}(\sigma(x))$. Let $y \in F_0$ and G_y be the isotropy group of y. If $\sigma \in \underline{F}(M)$ is a global section, then $Q_y(\sigma) = \bar{\sigma}^{-1}(y) \subset F^1(M)$ is a G_y-structure (see [12], prop. 2.20). Replacing every bundle W by its standard fibre W_0 and every morphism f by f_0 we obtain from the diagram (3.2) an exact and commutative diagram of L^1_n-vector bundles over F_0 and L^1_n-equivariant vector bundle homomorphisms over ι_{F_0} (see remark 1.7). Then taking fibres over fixed $y \in F_0$ we get an exact and commutative diagram of G_y-vector spaces and G_y-equivariant linear maps ($V = \mathbf{R}^n$ and g_y is the Lie algebra of G_y):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V^* \otimes g_y & \longrightarrow & V^* \otimes V^* \otimes V & \xrightarrow{\iota_{V^*} \otimes \rho'_y} & V^* \otimes T_y F_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\partial_y} & & \downarrow{\scriptstyle -\mathcal{A}} & & \downarrow{\scriptstyle LS^1_y} & & \\
0 & \longrightarrow & \operatorname{im} \partial_y & \longrightarrow & \wedge^2 V^* \otimes V & \xrightarrow{q_y} & (HF_0)_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Therefore $(HF_0)_y = \wedge^2 V^* \otimes V \ /\mathrm{im}\, \partial_y = H^{0,2}(g_y)$ is the Spencer cohomology group. Let $\sigma \in \underline{F}(M)$ be a global section, $\lambda \in \underline{C}^1(M)$ be a σ–connection. We will denote by $c_y : Q_y(\sigma) \to H^{0,2}(g_y)$ the structure tensor of the G_y–structure $Q_y(\sigma)$. Then $c_y = q_y \oplus \bar{T}$, where $T = t(\lambda)$, $\bar{T}$ is the tensorial 0-form of T. From remark 2.3 and diagram (3.3) we get:

$$S_\sigma(x) = S^1(j^1_x \sigma) = S^1(h(\lambda_x, \sigma_x)) = q(t(\lambda_x), \sigma_x) = q(T_x, \sigma_x).$$

If $l \in Q_y(\sigma)$, then $\bar{\sigma}(l) = y$, and

$$\bar{S}_\sigma(l) = q_0(\bar{T}(l), \bar{\sigma}(l)) = q_0(\bar{T}(l), y) = q_y(\bar{T}(l)) = c_y(l).$$

Therefore we have the following

3.5. Proposition. *If $\sigma \in \underline{F}(M)$ is a global section then $\bar{S}_\sigma|_{Q_y(\sigma)}$ is the structure tensor of $Q_y(\sigma)$.*

Now we will give an interpretation of the functor HF. In $J^1F(M)$ we have the following relation

$$z_1 \equiv z_2 \leftrightarrow \rho_0^1(z_1) = \rho_0^1(z_2) \text{ and } z_1 - z_2 \in \operatorname{im} \chi(M).$$

The theorem 2.6 implies that $J^1F(M)_y/\operatorname{im}\chi(M)_y/\equiv$ is the set of orbits of $S^2T^* \otimes T(M)_x (x = \pi(y))$ in $J^1F(M)_y$. Since $\ker LS^1 = \operatorname{im}\chi, S^1$ induces a bijection $J^1F(M)/\operatorname{im}\chi(M) \to HF(M)$:

$$\begin{tikzcd} J^1F(M) \arrow[rr, "S^1(M)"] \arrow[dr] & & HF(M) \\ & J^1F(M)/\operatorname{im}\chi(M) \arrow[ur] & \end{tikzcd}$$

where $J^1F(M) \to J^1F(M)/\operatorname{im}\chi(M)$ is the canonical projection. Since $L(\ker S^1) = \ker L(S^1) = \operatorname{im}\chi$, from (3.3) we get the exact commutative diagram ($h_0 = h|_{SC^1 \times F}$)

$$\begin{tikzcd} 0 \arrow[r] & SC^1 \times F \arrow[r] \arrow[d, "h_0"] & C^1 \times F \arrow[r, "t"] \arrow[d, "h"] & \wedge^2T^* \otimes T \times F \arrow[r] \arrow[d, "q"] & 0 \\ 0 \arrow[r] & \ker S^1 \arrow[r] \arrow[d] & J^1F \arrow[r, "S^1"] \arrow[d] & HF \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Let $Fl(M)$ denote the set of first jets of 1-flat sections of $F(M)$, i.e. $j^1_x\sigma \in Fl(M)$ if and only if there exists a map $\phi : U \to \mathbf{R}^n$ such that $j^1_0\phi_*\sigma = j^1_0\sigma_0$ where $\sigma_0 \in \underline{F}(\mathbf{R}^n)$ is a constant section. Then (2.8) implies that $\operatorname{im}\chi = L\ker S^1 = S^2T^* \otimes T \subset L(Fl(M))$. But since 1-flat sections locally admit torsion-free connections, $Fl(M) \subset \operatorname{im} h_0 = \ker S^1$, so $Fl(M) = \ker S^1(M)$. Therefore one can introduce a vector bundle structure on the affine bundle $J^1F(M)/\operatorname{im}\xi(M)$ so that $Fl(M)$ is the 0 of $J^1F(M)/\operatorname{im}\xi(M)$, and then the functors $J^1F/\operatorname{im}\xi$ and HF are isomorphic (in the category $\mathcal{NV}f$). Since the theorem 2.6 was proved for arbitrary $k \in \mathbf{N}$, one can define $HF = J^1F/\operatorname{im}\xi$ and the structure tensor operator $\underline{S} : \underline{F} \to \underline{H}F$ also when $k > 1$. Then HF may have only an affine structure.

4. First order natural differential operators

4.1. Theorem. *Let $F, G \in \mathcal{NF}_n$ and let F be homogeneous, ord $(F) = 1$, ord $(G) \le 1$. If $D \in \operatorname{Hom}_{\mathcal{NF}_n}(J^1F, G)$, then there exists exactly one morphism $K \in$*

$\mathrm{Hom}_{\mathcal{NF}_n} HF, G$ *such that* $D = K \circ S^1$*:*

$$\begin{tikzcd} J^1F \arrow[rr, "D"] \arrow[dr, "S^1"'] & & G \\ & HF \arrow[ur, "K"'] & \end{tikzcd}$$

Proof. Formula (1.1) implies that $D(M)$ is constant on orbits of $S^2T^* \otimes T(M)$ in $J^1F(M)$. Therefore, from the remarks after proposition 3.5, it follows that there exists a map $K(M) : HF(M) \to G(M)$ such that $D(M) = K(M) \circ S^1(M)$. S^1 is a epimorphism so $K = \{K(M)\}_{M \in \mathcal{M}_n}$ is unique and $K \in \mathrm{Mor}(\mathcal{NF}_n)$. □

4.2. Corollary. *Let* F, G *be as in theorem 4.1,* $\underline{D} : \underline{F} \to \underline{G}$ *be a natural differential operator, ord* $(\underline{D}) = 1$*. Then there exists exactly one* $K \in \mathrm{Hom}(HF, G)$ *such that* $\underline{D} = \underline{K} \circ \underline{S}$*.*

4.3. Remark. If we assume in 4.2 that $\underline{D}$ is a $*$-n.d.operator, then $\underline{D}$ can be uniquely factorized by $\underline{S}, \underline{D} = \underline{K} \circ \underline{S}$, where $K : HF \to G$ satisfies the condition (1.1) but it is possible that $K(M)$ is not smooth.

We have assumed in sections 32 and 4 that F is homogeneous. If this is not the case, then F_0 decomposes under action of L^k_n into a set of orbits: $F_0 = \{F^\alpha_0\}_{\alpha \in A}$. F^α_0 are submanifolds (not necessarily regular) with an action of L^k_n induced from F_0. This gives a family of homogeneous natural functors $\{F^\alpha\}_{\alpha \in A}$. Then $\underline{F}^\alpha(M) \subset \underline{F}(M)$.

4.4 Remark. If F^α_0 is a regular submanifold of $F_0, \sigma \in \underline{F}(M)$, then $\sigma \in \underline{F}^\alpha(M)$ if and only if $\sigma(M) \subset F^\alpha(M)$. In particular, if G is a real algebraic subgroup of L^1_n (for example when F is a tensor bundle), then F^α_0 is a regular submanifold of F_0 (see the remark after prop. II 3.1 in [3]).

N.d. operators on $\underline{F}$ give n.d. operators on $\underline{F}^\alpha$, which can be described with use of corollary 4.2. This could help to determine n.d. operators on $\underline{F}$, but one should keep in mind the fact that, in general, there are some operators on $\underline{F}^\alpha$ which do not come from operators on $\underline{F}$.

Examples of structure tensor operators

4.5. Lemma. *If* $K \in \mathcal{NV}$*, then* VK *is isomorphic to* $K \times K$*.*

Proof. The isomorphism is given by the following formula:

$$VK(M) \in [t \to \gamma_t] \to (\frac{d}{dt}\gamma_t(0), \gamma_0) \in K \quad \times K(M)\ .\square$$

Let F be a tensor bundle functor. We will describe $Lh : T^1_2 \times F \to T^* \otimes VF$. $Lh = \iota_{T^*} \otimes (f \circ \epsilon^1_T)$. It is easy to calculate f using (2.1). Here we will use the formula for the Lie derivative from [12], def. 2.7: for $X \in \underline{T}(M), \sigma \in \underline{F}(M)$

$$L_X \sigma = d\sigma(X) - f \circ (j^1_x X, \sigma) \ . \tag{4.1}$$

If $A = d_x f \otimes Z_x, f \in C^\infty(M), f(x) = 0, Z \ \in \underline{T}(M)$, then

$$f \circ \epsilon^1_T(A, \sigma_x) = f(j^1_x(fZ), \sigma_x) = -(L_{fZ}\sigma)_x \ . \tag{4.2}$$

1. Let $F = T$. Then $T^* \otimes VF = T^1_1 \times T$. Using (4.2) we get: $(\sigma = X \in \underline{T}(M))$

$$f \circ \epsilon^1_T(A, X_x) = -(L_{fZ}X)_x = [X, fZ]_x = d_x f(X)Z = A(X_x)$$

and consequently

$$Lh(M) : T^1_2 \times T(M) \ni (C, X) \to (C(;X), X) \in T^1_1 \times T(M) \ . \tag{4.3}$$

2. $F = T^*, T^* \otimes VF = T_2 \times T^*$. In (4.2) we take $\sigma = \omega \in \underline{T}^*(M)$ and we get:

$$Lh(M) : T^1_2 \times T^*(M) \ni (C, \omega) \to (-\omega \circ C, \omega) \in T_2 \times T^*(M) \ . \tag{4.4}$$

3. Using the formula for the Lie derivative of a tensor product, one can obtain the formula for Lh in any tensor bundle. In particular if $F = T^1_1$ then $T^* \otimes VF = T^1_2 \times T^1_1$ and we have:

$$Lh(M) : T^1_2 \times T^1_1(M) \ni (C, J) \to (C \circ (;J) - J \circ C, J) \in T^1_2 \times T^1_1(M) \,, \tag{4.5}$$

where $C \circ (;J)(X, Y) = C(X, J(Y))$ for $X, Y \in T(M)$.

Almost complex and almost product structures

Let $F = T^1_1$ and let F^α_0 denote the orbit of some $y_0 \in F_0 = (\mathbf{R}^n)^* \otimes \mathbf{R}^n$. Then F^α_0 is a regular submanifold of F_0 (remark 4.4). Let $P = \wedge^2 T^* \otimes T$. we define a natural differential operator $\underline{N} : \underline{F} \to \underline{P} \times F$ by the formula $\underline{N}(M) : T^1_1(M) \ni J \to (N_J, J) \in P = \times T^1_1(M)$, where N_J is the Nijenhuis tensor of J: for $X, Y \in \underline{T}(M)$,

$$\frac{1}{2} N_J(X, Y) = [JX, JY] + J^2[X, Y] - J[X, JY] - J[JX, Y] \ . \tag{4.6}$$

Then ord $(\underline{N}) = 1$. Let $N^1 : J^2 1F \to P \times F$ satisfy $\underline{N} = \underline{N}^1 \circ j^1$. We apply theorem 4.1 to N^1 restricted to $J^1 F^\alpha$ and we get $K : HF^\alpha \to P \times F^\alpha$ which makes the following diagram commute:

$$\begin{tikzcd} J^1 F^\alpha \arrow[d, "S^1"] \arrow[dr, "N^1"] & \\ HF^\alpha \arrow[r, "K"] & P \times F^\alpha \end{tikzcd} \tag{4.7}$$

From the definition of the torsion tensor, we have the following:

4.6. Lemma. *If $J \in \underline{F}(M), \lambda \in \underline{C}^1(M)$ is a J-connection, $T = t(\lambda)$, then*

$$\frac{1}{2}N_J(X,Y) = -T(JX,JY) - J^2T(X,Y) + JT(X,JY) + JT(JX,Y)\ . \tag{4.8}$$

Let $p \in \mathrm{Hom}_{\mathcal{NV}F}(\wedge^2T^*\otimes T\times F, P\times F)$ be such that, when restricted to the fibre over $J \in F_x$, it is given by the following formula ($C \in \wedge^2T^*\otimes T, X, Y \in T(M)_x$):

$$p(C,J)(X,Y) = 2(-C(JX,JY) - J^2C(X,Y) + JC(X,JY) + JC(JX,Y))\ . \tag{4.9}$$

Then lemma 4.6 implies that $p(T_x, J_x) = \underline{N}(J)_x$. But $\underline{N}(J)_x = N^1(j^1_xJ) = N^1(h(\lambda_x, J_x))$ so $N^1\circ h = p\circ t$ and from (3.3), (4.7) we obtain the following commutative diagram:

$$\begin{tikzcd}
C^1\times F^\alpha \arrow[r, "h"] \arrow[d, "t"] & J^1F^\alpha \arrow[d, "S^1"] \arrow[dr, "N^1"] & \\
\wedge^2T^*\otimes T\times F^\alpha \arrow[r, "q"] \arrow[drr, "\rho"'] & HF^\alpha \arrow[dr, "K"] & \\
 & & P\times F^\alpha
\end{tikzcd} \tag{4.10}$$

Hence $K \in \ Mor(\mathcal{NV}F), N^1 \in \mathrm{Mor}(\mathcal{NV}F)$.

4.7. Proposition. *If there exist $k \in \mathbf{R}\backslash\{0\}$ such that $y_0^2 = ki_{\mathbf{R}^n}$ then ker $p =$ im ∂.*

Proof. It follows from (4.10) that $K\circ q = p$ so im $\partial =$ ker q is a subbundle of ker p. Let $(C,J) \in$ ker $p(M)$. Then for $X, Y \in T(M)$ we have

$$C(JX,JY) + kC(X,Y) - JC(X,JY) - JC(JX,Y) = 0\ . \tag{4.11}$$

We define $A \in T^1_2(M)_x$ by the following formula:

$$A(X,Y) = \frac{1}{2}C(X,Y) + \frac{1}{4k}J(C(X,JY) + C(Y,JX))\ . \tag{4.12}$$

Then

$$\begin{aligned} A(X,JY) &= \frac{1}{2}C(X,JY) + \frac{1}{4}JC(X,Y) - \frac{1}{4k}JC(JX,JY) = \\ &= J(\frac{1}{2k}JC(X,JY) + \frac{1}{4}C(X,Y) - \frac{1}{4k}C(JX,JY))\ . \end{aligned}$$

Computing $C(JX,JY)$ from (4.11) we obtain from the formula above:

$$\begin{aligned} A(X,JY) &= J(\frac{1}{2k}JC(X,JY) + \frac{1}{2}C(X,Y) - \frac{1}{4k}JC(X,JY) - \frac{1}{4k}JC(JX,Y)) = \\ &= JA(X,Y)\ . \end{aligned}$$

Then (4.5) implies that $A \in$ ker $Lh(M_x)$. But $\partial(A) = C$ so $C \in$ im $\partial(M)$. $\square$

Since ker $p =$ im $\partial =$ ker q, we have (under the assumption of prop. 4.7) ker $p =$ ker q. Consequently K, in diagram (4.10) is a monomorphism and we can identify HF^α with im K, S^1 with N^1 and q with p. In particular we have the following

4.8. Corollary. *If $J \in \underline{T}_1^1(M)$ is an almost complex or almost product structure $(J^2 = -\iota_{TM}$ or $J^2 = \iota_{TM})$ then its structure tensor S_J is equal to $(N_J, J) \in \underline{\wedge}^2 T^* \otimes T \times T_1^1(M)$.*

4.9. Remark. If we identify im K =im p with HF^α, we can get from (4.9) HF^α in the explicit form:

$$HF^\alpha(M) = \{(C, J) \in \wedge^2 T^* \otimes T \times F^\alpha(M) : \forall X, Y \in \underline{T}(M) C(X, JY) = \\ = -JC(X, Y)\}\,.$$

4.10. Definition. For $W \in \mathcal{NV}\ F, k \in \mathbf{N}$ we define $\prod^k W \in \mathcal{NV}F$:

$$\prod^k W(M) = \underbrace{W(M) \times_M \cdots \times_M W(M)}_{k \text{ times}}\ .$$

II Systems of linearly independent vector fields

Let $F = \prod^k T, k \le n$. Then $F_0 = (\mathbf{R}^n)^k$. Let $(e_1, \dots, e_n)$ denote the canonical basis of $(\mathbf{R}^n)$ and let F_0^α be the orbit of $(e_1, \dots, e_k) \in F_0$. Sections of $F^\alpha(M)$ are systems of k linearly independent vector fields. Taking suitable $P \in \mathcal{NV}$, a natural differential operator $\underline{N} : \underline{F} \to \underline{P} \times F$ and $p \in \mathrm{Hom}_{\mathcal{N}}\, \mathcal{V}\ F(\wedge^2 T^* \otimes T \times F, P \times F)$, one can see that the diagram (4.10) is commutative in this case. We take $P = \prod^m T, m = \frac{k(k-1)}{2}$. For $(X_1, \dots, X_k) \in \underline{F}(M)$ we put:

$$\underline{N}(M)(X_1, \dots, X_k) = (([X_i, X_j])_{1 \le i < j \le k} X_1, \dots, X_k)\,. \tag{4.13}$$

For $C \in \wedge^2 T^* \otimes T \times F(M), X_1, \dots, X_k \in T(M)_x$ we define

$$p(M)(C, X_1, \dots, X_k) = ((-C(X_i, X_j)_{1 \le i < j \le k} \quad X_1, \dots, X_k))\,. \tag{4.14}$$

If $\sigma = (X_1, \dots, X_k) \in \underline{F}^\alpha(M), \lambda \in \underline{C}(M)$ is a σ-connection, $T = t(\lambda)$ is the torsion tensor of σ, then $[X_i, X_j] = -T(X_i, X_j)$ and, just as in example I, we obtain the diagram (4.10) (N^1 and K are defined in the same way as in I). As before, we will prove that ker p = im ∂. In this case $T^* \otimes VV = \prod^k T_1^1 \times \prod^k T$. Then (4.3) implies that $Lh(M) : T_2^1 \times \prod^k T(M) \to \prod^k T(M)$ is given by

$$Lh(M)(C, X_1, \dots, X_k) = (C(\cdot, X_1), \dots, C(\cdot, X_k), X_1, \dots, X_k)\,. \tag{4.15}$$

If $(C, X_1, \dots, X_k) \in$ ker $p(M_x)$ then (4.14) gives

$$C(X_i, X_j) = 0 \text{ for } 1 \le i < j \le k\,. \tag{4.16}$$

Let $X_{k+1}, \dots, X_n \in T(M_x)$ be such that $X_1, \dots, X_n$ are linearly independent. We define $A \in T_2^1(M)$ by the formula

$$A(X_i, X_j) = \begin{cases} C(X_i, X_j), & \text{for } i \le k, j > k; \\ 0, & \text{for } \ j \le k; \\ \frac{1}{2} C(X_i, X_j), & \text{for } \ i, j > k\,. \end{cases}$$

Then (4.15) implies that $A \in$ ker $Lh(M)$ and (4.16) gives $C = \partial(A)$.

4.11. Corollary. *If $\sigma = (X_1, \dots, X_k) \in \underline{\prod^k T}(M)$ is a system of k linearly independent vector fields then its structure tensor S_σ can be identified with*

$$([X_i, X_j]_{1\le i<j\le k},\, X_1, \dots, X_k) \in \underline{\prod}^{k+m} T(M).$$

III Systems of linearly independent 1-forms

Let $F = \prod^k T^*$, $k \le n$. Then $F_0 = ((\mathbf{R}^n)^*)^k$. we will denote by $F_0 0^\alpha$ the orbit of $(e^1, \dots, e^k)$. Sections of $F^\alpha(M)$ are systems of k linearly independent 1-forms. In this case we take $P = \prod^k \wedge^2 T^*$ and we define a natural differential operator $\underline{N}(M) : \underline{F}(M) \ni (\omega^1, \dots, \omega^k) \to (d\omega^1, \dots, d\omega^k, \omega^1, \dots, \omega^k) \in \underline{P \times F}(M)$. Let $p \in \mathrm{Hom}_{\mathcal{NV}F}(\wedge^2 T^* \otimes T \times F, P \times F)$ be defined so that

$$p(M)(C, \omega^1, \dots, \omega^k) = (\frac{1}{2}\omega^1 \circ C, \dots, \frac{1}{2}\omega^k \circ C, \omega^1, \dots, \omega^k)\,. \tag{4.17}$$

If $\sigma = (\omega^1, \dots, \omega^k) \in \underline{F}(M)$ and $\lambda \in \underline{C}^1(M)$ is a σ-connection, then $d\omega^i(X,Y) = \frac{1}{2}\omega^i \circ T(X,Y)$, $i = 1, \dots, k$. This implies that diagram (4.10) is commutative in this case. $T^* \otimes VF = \prod^k T_2 \times \prod^k T^*$ and (4.4) implies that $Lh(M) : T^1_2 \times \prod^k T^*(M) \to \prod^k T_2 \times \prod^k T^*(M)$ is given by

$$Lh(M)(C, \omega^1, \dots, \omega^k) = (-\omega^1 \circ C, \dots, -\omega^k \circ C, \omega^1, \dots, \omega^k)\,. \tag{4.18}$$

It follows immediately from (4.17) that ker p = im ∂ and we get the following

4.12. Corollary. *If $\sigma = (\omega^1, \dots, \omega^k) \in \underline{\prod^k T^*}(M)$ is a system of linearly independent 1-forms, then its structure tensor S_σ is equal to $(d\omega^1, \dots, d\omega^k, \omega^1, \dots, \omega^k) \in \underline{\prod^k \wedge^2 T^* \times \prod^k T^*}(M)$.*

Let $F, G \in \mathcal{NF}_n$, ord $(F) = 1$, ord $(G) \le 1$. owing to corollary 4.2, the classification of first order natural differential operators $\underline{D} : \underline{F} \to \underline{G}$, for homogeneous F, can be reduced to the classification of L^1_n-equivariant maps from HF_0 to G_0.

Example. We will find all $*$-n.d. operators $\underline{D} : \underline{T \times T} \to \underline{T}$ of order 1 for $n \ge 3$. Let $W \subset (\mathbf{R}^n)^3$ be an orbit of $e = (e_1, e_2, e_3)$ and $K : W \to R^n$ be L^1_n-equivariant. Then $G_e \subset G_{K(e)}$ (G_x denotes the isotropy group of x). Therefore $K(e) \in$ span $\{e_1, e_2, e_3\}$, so $K(e) = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, for some $\lambda_i \in \mathbf{R}$. If $y = (y_1, y_2, y_3) \in W$ then $y = \rho(a, e)$ for some $a \in L^1_n$, and $K(y) = K(\rho(a,e)) = \rho(a, K(e)) = \rho(a, \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$. Then remark 4.3 and corollary 4.11 imply that, if $\underline{D} : \underline{T \times T} \to \underline{T}$ is a $*$-n.d. operator, ord $(\underline{D}) = 1$ then there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}$ such that if $X, Y \in \underline{T}(M)$, $x \in$ dom $X \cap$ dom Y, $X_x, Y_x, [X,Y]_x$ are linearly independent , then $\underline{D}(X,Y)(x) = \lambda_1 X_x + \lambda_2 Y_x + \lambda_3 [X,Y]_x$. If $X_x, Y_x [X,Y]_x$ are not linearly independent, we use the following

4.13. Lemma. *There exist $\hat{X}, \hat{Y} \in \underline{T}(M)$ defined on some neighbourhood U of x such that $j^1_x\hat{X} = j^1_x X, j^1_x\hat{Y} = j^1_x Y$ and there exists a sequence $\{a_m\}_{m\in\mathcal{N}}$, $a_m \in U$ such that $\lim_{m\to\infty} a_m = x$ and $\hat{X}_{a_m}, \hat{Y}_{a_m}, [\hat{X}, \hat{Y}]_{a_m}$ are linearly independent for every $m \in N$.*

Since ord $(\underline{D}) = 1$, we have:

$$\underline{D}(X,Y)(x) = \underline{D}(\hat{X},\hat{Y})(x) = \lim_{m\to\infty} \underline{D}(\hat{X},\hat{Y})(a_m) =$$
$$= \lim_{m\to\infty} (\lambda_1 \hat{X}_{a_m} + \lambda_2 \hat{Y}_{a_m} + \lambda_3[\hat{X}\hat{Y}]_{a_m}) = \lambda_1 X_x + \lambda_2 Y_2 + \lambda_3 [X,Y]_x$$

and we get the following

4.14. Theorem. *Let $\underline{D} : \underline{T} \times T \to \underline{T}$ be a $*$-n.d. operator, ord $\underline{D} = 1, n \geq 3$. Then exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbf{R}$ such that for every $X, Y \in \underline{T}(M)$*

$$\underline{D}(X,Y) = \lambda_1 X + \lambda_2 Y + \lambda_3 [X,Y]\,.$$

4.15. Corollary. *If $n \geq 3$ then the Lie bracket is the only bilinear first order $*$-n.d. operator from $\underline{T \times T}$ to $\underline{T}$, up to a constant factor.*

In fact the assumption of bilinearity is very strong, and more general result can be obtained (see [7], [5]).

Proof. of lemma 4.13. It is sufficient to take $M = \mathbf{R}^n$, $x = 0$. we define

$$\hat{X} = X + \textstyle\sum_{i,j=1}^n (x^j \frac{\partial}{\partial x_j} X^i|_{x=0}) e_i + (x^1)^2 e_2,$$
$$\hat{Y} = Y + \textstyle\sum_{i,j=1}^n (x^j \frac{\partial}{\partial x_j} Y^i|_{x=0}) e_i + (x^1)^2 e_1 + x^1 x^2 e_3\,.$$

Let $y = (y_1, \dots, y_n) \in \mathbf{R}^n$. we will denote by $_y$ a matrix 3×3 such that $(M_y)^i_1 = \hat{X}^i_y$, $(M_y)^i_2 = \hat{Y}^i_y$, $(M_y)^i_3[\hat{X},\hat{Y}]^i_y$, $i = 1\dots3$. Then det $M_y = -(y_1)^7 + \{\textit{terms of degree } \leq 6\}$. Therefore, it is possible to find a sequence $\{a_m\}$ converging to x with det $M_{a_m} \neq 0$ for $m \in N$, and then $X_{a_m}, Y_{a_m}, [X,Y]_{a_m}$ are linearly independent. □

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