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ON THE OSCILLATORY BEHAVIOR OF CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION

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Dedicated to Professor V. Šeda on the occasion of his sixtieth birthday

Abstract. In this paper we shall study some oscillatory and nonoscillatory properties of solutions of a nonlinear third order differential equation, using the results and methods of the linear differential equation of the third order.

The aim of this paper is to study the oscillatory or nonoscillatory properties of solutions of the nonlinear differential equation

$$u''' + q(t)u' + p(t)\,h(u) = 0 \tag{1}$$

where $q'(t)$ and $p(t)$ are continuous function of $t \in (a, \infty)$, $-\infty < a < \infty$; $h(u)$ is continuous function of $u \in (-\infty, \infty)$ and

(i) $h(u)u > 0$ for $u \neq 0$,

(ii) $\lim\limits_{u\to 0} \dfrac{h(u)}{u} = \Theta, \quad 0 \leq \Theta < \infty.$

In this paper, a solution of equation (1) we will understand a nontrivial solution of (1) defined on the interval $[T, \infty]$, $T > a$. A nontrivial solution of (1) is said to be oscillatory if it has zeros for arbitrarily large values of (the independent variable) t. Otherwise a solution is called nonoscillatory.

The object of generalization are the results of the paper [1] concerning oscillatory or nonoscillatory solutions of equation (1) in the case $q(t) \equiv 0$ on (a, ∞) and results of the paper [2].

In this paper we use the results and the methods of proofs of the theory of a third order linear differential equation, [3] and [5].

1. First of all we list some results for the linear differential equation

$$y''' + 2A(t)y' + [A'(t) + b(t)]\,y = 0 \tag{a}$$

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where $A'(t)$, $b(t)$ are continuous function on (a,∞) and $b(t)$ has the property (v), i.e. $b(t) \geq 0$, $t \in (a,\infty)$ such that $b(t) \not\equiv 0$ on each subinterval.

The adjoint equation to (a) is

$$z''' + 2A(t)\, z' + [A'(t) - b(t)]\; z = 0 \tag{b}$$

Suppose $w(t) > 0$, $t \in (t_0, \infty)$, $a < t_0 < \infty$ is a solutions of equation (b). Then there exists two-parameter family of solutions y of (a) that satisfy the second order equation

$$w\, y'' - w'y' + (w'' + 2Aw)\; y = 0 \tag{c}$$

If $w(t_0) = w'(t_0) = 0$, $w''(t_0) > 0$, $w(t) > 0$ for $t > t_0$ and y are the solutions of (a) such that $y(t_0) = 0$, than each member of this family satisfies equation (c) and is called a band of solutions of the differential equation (a) at the point t_0.

If an equation of the band at the point t_0 is oscillatory to the right of t_0, than each solution of the equation (a) with one zero oscillates to the right of this zero.

If we introduce the substitution $y = \sqrt{w}v$ into (c), then for $t > t_0$ equation (c) becomes

$$v'' + \left[\frac{3}{2}\frac{w''}{w} - \frac{3}{4}\frac{w'^2}{w^2} + 2A\right]\; v = 0. \tag{c$_1$}$$

Beside (a) let us consider the equation

$$Y''' + 2A(t)y' + [A'(t) + b_1(t)]\; Y = 0 \tag{a$_1$}$$

where $b_1(t) \geq 0$ is continuous in (a,∞).

The following theorems [3], [4] will be useful.

Theorem A. [3, Theorem 2.5 and Corollary 2.5]
Let $b(t)$ have the property (v) and let $b(t) \leq b_1(t)$ for $t \in (a,\infty)$.

If the differential equation (a) is oscillatory in (a,∞) (i.e. each solution with one zero oscillates to the right of this zero) then the differential equation (a_1) is oscillatory, too. If (a_1) is nonoscillatory, then also (a) is nonoscillatory in (a,∞).

Theorem B. [3, Theorem 2.1]
Let $b(t)$ have the property (v) for $t \in (a,\infty)$ and

$$\text{let } A(t) \leq 0, \quad t \in (a,\infty) \text{ and } \quad |A(t)| \geq \int_{t_0}^{t} b(\tau)\, d\tau,\; t_0 > a,$$

$t \geq t_0 \in (a,\infty)$. Then the differential equation (a) is disconjugated on (a,∞) (i.e. each solution of (a) has at most two zeros, or one double zero).

Theorem B was originally formulated and proved by G. Sansone [5].

Theorem C. [3, **Theorem 3.1**]
Let $A(t) \geq m > 0$, $A'(t) + b(t) \geq m$, $b(t) - A'(t) \geq 0$ for every $t \in (a, \infty)$. Then every solution of the differential equation (a) is oscillatory in (a, ∞) except one solution y (up to the linear dependence) for which $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} y'(t) = 0$ and y is in the class L^2, i.e.

$$\int_{t_0}^{\infty} y^2(t)\, dt < \infty.$$

Theorem D. [3, **Theorem 2.17 and Corollary 2.3**]
Let $b(t)$ have the property (v) on (a, ∞) and let the differential equation $y'' + \frac{1}{2} A(t)\, y = 0$ be oscillatory in (a, ∞). A necessary and sufficient condition for a nontrivial solution y of the differential equation (a) to be nonoscillatory in (a, ∞) is that

$$y(t)y''(t) - \frac{1}{2}\, y'^2(t) + A(t)\; y^2(t) > 0$$

for every $t \geq t_0 > a$.

2. Let $q'(t)$ and $p(t)$ be continuous functions of $t \in (a, \infty)$ and let (i), (ii) hold. Let u_1 be a nontrivial solution of (1) on $[T, \infty)$, $T > a$. Then u_1 fulfils linear differential equation

$$u''' + q(t)\, u' + p(t)\, H(u_1)\; u = 0, \tag{2}$$

where

$$H(u_1) = \begin{cases} \frac{h(u_1)}{u_1} & \text{for } u_1 \neq 0 \\ \Theta & \text{for } u_1 = 0 \end{cases} \tag{3}$$

The adjoint differential equation to the equation (2) is

$$v''' + q(t)\, v' + [q'(t) - p(t)\, H(u_1)]\; v = 0. \tag{4}$$

If we multiply equation (2) by u and equation (4) by v and integrate from t_0 to t we obtain identities

$$u\, u'' - \frac{1}{2} u'^2 + \frac{1}{2} q u^2 + \int_T^t [p(\tau)\, H(u_1(\tau)) - \frac{1}{2} q'(\tau)]\; u^2(\tau)\, d\tau = const. \tag{U}$$

and

$$v\, v'' - \frac{1}{2} v'^2 + \frac{1}{2} q v^2 - \int_{t_0}^t [p(\tau)\, H(u_1(\tau)) - \frac{1}{2} q'(\tau)]\; v^2(\tau)\, d\tau = const. \tag{V}$$

The following lemma follows immediately from identities (U) and (V):

Lemma 1. *Let $p(t) > 0$, $q'(t) \leq 0$ for $t \in (a, \infty)$. Then every solution u of the differential equation (2) defined on $[T, \infty)$, $T > a$, with the property $u(t_0) = u'(t_0) = 0$, $u''(t_0) > 0$, $t_0 \geq T$ has no zeros to the left of t_0, and every solution v of the differential equation (4) with the property $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$, $t_0 > T$, has no zeros to the right of t_0. Every solution u of (2) and every solution v of (4) has at most one double zero.*

Let $w(t)$ be a solution of the differential equation (4) with the property $w(t) > 0$ for $t \geq t_0$. Then the set of solutions of the linear differential equation of the second order

$$w\, u'' - w'u' + [w'' + q(t)\, w]\; u = 0 \tag{5}$$

fulfils at the same time equation (2) on the interval $[t_0, \infty)$ [3].

Via the substitution $u(t) = \sqrt{w(t)}\, y(t)$ we obtain from equation (5) the equation

$$y'' + \left[\frac{3}{2}\frac{w''}{w} - \frac{3}{4}\frac{w'^2}{w^2} + q(t)\right]\; y = 0. \tag{6}$$

Theorem 1. *Let $p(t) > 0, q'(t) \leq 0$ for $t \in (a, \infty)$ and let (i) (ii) hold. Let $f(t)$ be a given positive function defined on (a, ∞) with continuous third derivative on this interval. Let u_1 be a solution of the differential equation (1) defined on $[t_0, \infty)$ with the property*

$$\begin{aligned} &p(t)\, H(u_1(t)) - \frac{1}{2}q'(t) \leq \\ &\frac{f''(t) + q(t)\, f'(t) + \frac{1}{2}q'(t)\, f(t)}{f(t)} = F(t) \end{aligned} \tag{7}$$

and let the differential equation

$$y'' + \left[\frac{3}{2}\frac{f''(t)}{f(t)} - \frac{3}{4}\frac{f'^2(t)}{f^2(t)} + q(t)\right]\; y = 0 \tag{8}$$

be nonoscillatory on $[t_0, \infty)$. Then the solution u_1 of (1) is nonoscillatory on $[t_0, \infty)$.

Proof. Let the solution u_1 of (1) satisfy (7) on $[t_0, \infty)$. Together with equation (2) let us consider equation

$$v''' + q(t)\, v' + \left[\frac{1}{2}q'(t) + F(t)\right]\; v = 0 \tag{9}$$

obtained by differentiating the second order differential equation

$$f v'' - f'v' + (f'' + q(t)\, f)\; v = 0. \tag{10}$$

As we did earlier, the substitution $v = \sqrt{f}\, y$ transforms equation (10) into (8). Since equation (8) is nonoscillatory, equation (10) is nonoscillatory. This implies that equation (9) is nonoscillatory. By Theorem A (Comparison Theorem) and condition (7) we have that equation (2) is nonoscillatory. Thus the solution u_1 of (1) is nonoscillatory, too. $\square$

Corollary 1. *Let $q(t) \equiv 0$, $p(t) > 0$ for $t \geq t_0 > 1$. Let u_1 be a solution of the equation*

$$u''' + p(t)\,h(u) = 0$$

on $[t_0, \infty)$ with the property

$$H(u_1(t)) \leq \frac{1}{p(t)} \frac{6}{t^3(t^2-1)}\ ,\ t \geq t_0 \tag{11}$$

Then the solution u_1 is nonoscillatory on $[t_0, \infty)$.

Proof. Let $t_0 > 1$ and $f(t) = t - \frac{1}{t}$. Then $f'(t) = 1 + \frac{1}{t^2}$, $f''(t) = -\frac{2}{t^3}$, $f'''(t) = \frac{6}{t^4}$ and

$$\begin{aligned}
&\frac{3}{2}\frac{f''(t)}{f(t)} - \frac{3}{4}\frac{f'^2(t)}{f^2(t)} = -\frac{3}{t^2(t^2-1)}\left[1 + \frac{1}{4}\frac{(t^2+1)^2}{t^2-1}\right] \leq 0,\\
&F(t) = \frac{6}{t^3(t^2-1)} \quad \text{for} \quad t > t_0.
\end{aligned}$$

Therefore the differential equation (8) in this case is nonoscillatory and hence equation (10) is nonoscillatory. This implies that equation (9), where (11) holds is nonoscillatory and Theorem 1 implies the assertion of Corollary 1. $\square$

Theorem 2. *Let $p(t) > 0$ for $t \in (a, \infty)$ and let (i), (ii) hold. Further, let $B(t) \geq 0$ be continuous function of $t \in (a, \infty)$ and $B(t) \not\equiv 0$ on any subinterval of (a, ∞) and let the differential equation*

$$y''' + q(t)\,y' + \left[\frac{1}{2}q'(t) + B(t)\right]\ y = 0 \tag{B}$$

be oscillatory on (a, ∞). If u_1 is a solution of (1) defined on $[t_0, \infty)$ and if $u_1(t_1) = 0$ $t_1 > t_0$ and u_1 fulfils the condition

$$B(t) \leq p(t)\,H(u_1(t)) - \frac{1}{2}q'(t) \ \text{for}\ \ t \in [t_1, \infty) \tag{12}$$

then u_1 is oscillatory in $[t_1, \infty)$.

The proof follows immediately from Theorem A by comparing equation (B) with equation (2) where (12) holds.

Corollary 2. *Let $q'(t) \leq 0$, $p(t) > 0$ for $t \in (a, \infty)$ and $q'(t) \not\equiv 0$ in any subinterval of (a, ∞). Let, further, the differential equation of the second order $y'' + q(t)\,y = 0$ be oscillatory on (a, ∞) and (i), (ii) hold.*

Then every solution u_1 of (1) defined on $[t_0, \infty]$ $t_0 > a$ with one zero is oscillatory.

The proof follows from Theorem 2 in the case if we take $B(t) = -\frac{1}{2}q'(t)$.

Theorem 3. *Let $q'(t) \le 0$, $p(t) > 0$ for $t \in (a, \infty)$ and let (i), (ii) hold. Let $f(t) > 0$ and $F(t) \ge 0$ have the properties as in Theorem 1 and let $F(t) \not\equiv 0$ in any subinterval of (a, ∞). Further let the differential equation (8) be oscillatory on (a, ∞).*
If u_1 is a solution of (1), defined on $[t_0, \infty)$ and moreover $F(t) \le p(t)\, H(u_1(t)) - \frac{1}{2} q'(t)$ for $t \in \langle t_0, \infty)$, then u_1 is oscillatory in $[t_0, \infty)$.

Using Theorem A the method of proof is the same as in the case of Theorem 1.

Corollary 3. *Let $p(t) > 0$, $q(t) \equiv 0$ for $t \in (a, \infty)$ and let (i), (ii) hold and $\Theta > 0$. If u_1 is a solution of (1) defined on $[t_0, \infty)$, $t_0 > 0$, with one zero and*

$$\frac{6}{t^3} \le p(t)\, H(u_1(t)) \tag{13}$$

for $t > t_0$, then u_1 is oscillatory on $\langle t_0, \infty)$.

Proof. Let $f(t) = t^3$. Then there is

$$F(t) = \frac{6}{t^3}, \quad \frac{3}{2} \frac{f''(t)}{f(t)} - \frac{3}{4} \frac{f'^2(t)}{f^2(t)} = \frac{9}{4t^2},$$

that is equation (8) is oscillatory and (13) holds and so Theorem 3 implies the assertion of Corollary 3. □

Lemma 2. *Let $q(t) + 1 \le 0$ and $q(t) + 1 - \frac{1}{2} q'(t) < 0$ for $t \in (a, \infty)$. Then the differential equation*

$$v''' + q(t)\, v' + \left[\frac{1}{2} q'(t) + \frac{1}{2} q'(t) - q(t) - 1\right] v = 0 \tag{14}$$

is disconjugated on (a, ∞) (i.e. each solution of (14) has at most two zeros, or one double zero on (a, ∞)).

Proof. of Lemma 2 see in [3], the proof of Lemma 2.2.

Theorem 4. *Let the assumptions of Lemma 2 be fulfilled and let $p(t) > 0$, $q'(t) \le 0$ for $t \in (a, \infty)$ and (i), (ii) hold. Then every solution u_1 of (1) defined on $[t_0, \infty)$ with the property*

$$p(t)\, H(u_1(t)) \le q'(t) - q(t) - 1 \ \text{for} \ \ t \in [t_0, \infty), \tag{15}$$

is nonoscillatory on $[t_0, \infty)$.

Proof. Let u_1 be a solution of (1), defined on $[t_0, \infty)$ with property (15). It fulfils equation (2) that can be written in the form

$$u''' + q(t)\, u' + \left[\frac{1}{2} q'(t) + p(t)\, H(u_1(t)) - \frac{1}{2} q'(t)\right] u = 0 \tag{16}$$

Compare equation (16) with (14). If we take into consideration (15), i.e.

$$p(t)\, H(u_1(t)) - \frac{1}{2} q'(t) \le \frac{1}{2} q'(t) - q(t) - 1,$$

then Theorem A implies the assertion of Theorem 4. □

With the use of Theorem B we prove

Theorem 5. *Let $p(t) > 0$, $q(t) \le 0$ and $q'(t) \le 0$ for $t \in (a, \infty)$. Let, further, (i), (ii) hold and let $H(u) \le k$, $k > 0$, for every $u \in (-\infty, \infty)$.*

If
$$\frac{1}{2}\,|q(t)| \ \ge \int_{t_0}^{t} \left[k\,p(\tau) - \frac{1}{2} q'(\tau) \right] d\tau, \quad t_0 > a,$$
for $t \ge t_0 \in (a, \infty)$, then every solution u_1 of (1) defined on $[t_0, \infty)$ has at most two zeros, or one double zero in $[t_0, \infty)$.

Proof. Let u_1 be a solution of (1). It fulfils also equation (2) which can be written in the form (16). Equation (16) is of the form (a) and it fulfils the assumptions of Theorem B and therefore every solution of (16) has at most two zeros, or one double zero. Function u_1 is the solution of (1) and of (16). □

Now, we prove that Theorem C implies the assertion of the following theorem.

Theorem 6. *Let (i), (ii) hold and let $H(u) \ge \Theta > 0$ for every $u \in (-\infty, \infty)$. Further let, $q(t) \ge M > 0$, $p(t) > M$ and $\Theta p(t) - q'(t) \ge 0$ for $t \in (a, \infty)$. Then every solution u_1 of (1) defined on the interval $[t_0, \infty), t_0 > a$ is either oscillatory on $[t_0, \infty)$, or it has no zeros on $[t_0, \infty)$ and then $u_1(t) \to 0$, $u_1'(t) \to 0$ for $t \to \infty$ and $u_1(t) \in L^2([t_0, \infty))$.*

Proof. Let u_1 be a solution of (1) defined on $[t_0, \infty)$. It fulfils equation (16). The coeficients of the linear differential equation (16) fulfil the assumptions of Therem C, because $A(t) = \frac{1}{2} q(t) \ge \frac{M}{2} > 0$, $A'(t) + b(t) = p(t)\ H(u_1(t)) \ge \Theta\, M$, $b(t) - A'(t) = p(t)\, H(u_1(t)) - q'(t) \ge \Theta\, p(t) - q'(t) \ge 0$ for $t \in [t_0, \infty)$. If $m = \min\left(\frac{M}{2}, \Theta\, M\right)$, there the assumptions of Theorem C are fulfiled. If u_1 has at least one zero on $[t_0, \infty)$ then it is oscillatory . (It follows from the theory of bands of solutions of linear differential equation of the third order, see [3], and (16) is linear differential equation). If $u_1(t \ne 0)$ for $t \in [t_0, \infty)$, then from Theorem C, it must have the properties:
$u_1 \to 0$, $\ u_1' \to 0$ for $t \to \infty$ and $u_1 \in L^2$ on $\langle t_0, \infty)$. □

At the end we prove that Theorem D implies the assertion of the following theorem.

Theorem 7. *Let (i), (ii) hold. Let $q'(t) \le 0, p(t) > 0$ for $t \in (a, \infty)$ and let the differential equation*
$$u'' + \frac{1}{4} q(t)\ u = 0$$
be oscillatory in (a, ∞). Then a necessary and sufficient condition for a nontrivial solution u_1 of (1) defined on $[t_0, \infty)$ to be nonoscillatory in $[t_0, \infty)$ is that
$$u_1(t)\, u_1''(t) - \frac{1}{2} u_1'^2(t) + \frac{1}{2} q(t)\, u_1^2(t) > 0$$
for all $t \ge t_1 \ge t_0$.

Proof. Let u_1 be a solution of (1) defined on $[t_0, \infty)$ and let the assumptions of Theorem 7 be fulfiled. The solution u_1 fulfils equation (16) which is of the form (a).

It is easy to see that the assumptions of Theorem C on the coeficients of equation (16) are fulfiled and therefore the assertion of Theorem 7 is true. □

Remark. The assertion of Theorem 2 of [4] is a special case of the assertion of Theorem 7.

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