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A COMMUTATIVE NEUTRIX CONVOLUTION OF DISTRIBUTIONS AND THE EXCHANGE FORMULA

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Abstract. The neutrix convolution product $f \boxed{*} g$ of two distributions f and g in $\mathcal{D}'$ is defined to be the neutrix limit of the sequence $\{(f\tau_n) * (g\tau_n)\}$, provided the limit exists, where $\{\tau_n\}$ is a certain sequence of functions τ_n in $\mathcal{D}$ converging to 1. The neutrix product $(Ff)\Box(Fg)$ in $\mathcal{Z}'$, where F denotes the Fourier transform, is defined to be the neutrix limit of the sequence $\{F(f\tau_n).F(g\tau_n)\}$, where

$$F(f\tau_n) = F(f) * \delta_n, \quad F(g\tau_n) = F(g) * \delta_n, \quad \delta_n = F(\tau_n)$$

and $\{\delta_n\}$ is a sequence of functions in $\mathcal{Z}$ converging to the Dirac delta function. It is proved that the exchange formula

$$F(f \boxed{*} g) = F(f)\Box F(g)$$

then holds. Some examples are given.

In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The convolution product of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see for example Gel'fand and Shilov [4].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.
Then the *convolution product* $f * g$ is defined by the equation

$$\langle (f*g)(x), \phi(x)\rangle = \langle g(y), \langle f(x), \phi(x+y)\rangle\rangle \tag{1}$$

for arbitrary test function ϕ in $\mathcal{D}$.

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It follows that if the convolution product $f * g$ exists by Definition 1 then the following equations hold:

$$f * g = g * f, \tag{2}$$

$$(f * g)' = f * g' = f' * g. \tag{3}$$

Definition 1 is rather restrictive and in order to define further convolution products of distributions, Jones in [5] gave the following definition.

Definition 2. Let f and g be distributions in $\mathcal{D}'$ and let τ be an infinitely differentiable function satisfying the following conditions:
(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1, |x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0, |x| \ge 1$.
Let

$$f_n(x) = f(x)\tau(x/n), \quad g_n(x) = g(x)\tau(x/n)$$

for $n = 1, 2, \ldots$. Then the *convolution product* $f * g$ is defined as the limit of the sequence $\{f_n * g_n\}$, providing the limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$.

Note that in this definition the convolution product $f_n * g_n$ exists by Definition 1 since f_n and g_n both have bounded supports. It is clear that if the convolution product $f * g$ exists by this definition, then equation (2) holds. However, equations (3) need not necessarily hold since Jones proved that

$$1 * \operatorname{sgn} x = x = \operatorname{sgn} x * 1$$

and

$$(1 * \operatorname{sgn} x)' = 1, \quad 1' * \operatorname{sgn} x = 0, \quad 1 * (\operatorname{sgn} x)' = 2.$$

Many convolution products could still not be defined by Definition 2 and the following modification of Definition 2 was given in [3]:

Definition 3. Let f and g be distributions in $\mathcal{D}'$, let

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

where τ is as in Definition 2 and let $f_n = f\tau_n$, $g_n = g\tau_n$. Then the *neutrix convolution product* $f \boxed{*} g$ is defined to be the neutrix limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \langle f_n * g_n, \phi\rangle = \langle h, \phi\rangle$$

for all ϕ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range the real numbers with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all functions which converge to zero as n tends to infinity.

The convolution product $f_n * g_n$ in this definition is again in the sense of Definition 1, the supports of f_n and g_n being bounded. The neutrix convolution product $f \boxed{*} g$ clearly satisfies equation (2) if it exists, although it does not necessarily satisfy equations (3). A non-commutative neutrix convolution product, denoted by $f \circledast g$ was defined in [2].

It was proved in [3] that if the convolution product $f * g$ exists by Definition 1, then the neutrix convolution product $f \boxed{*} g$ exists and

$$f * g = f \boxed{*} g.$$

As in [4], we define the Fourier transform of a function ϕ in $\mathcal{D}$ by

$$F(\phi)(\sigma) = \tilde{\phi}(\sigma) = \int_{-\infty}^{\infty} \phi(x) e^{ix\sigma}\, dx.$$

Here $\sigma = \sigma_1 + i\sigma_2$ is a complex variable and it is well known that $\tilde{\phi}(\sigma)$ is an entire analytic function with the property

$$|\sigma|^q |\tilde{\phi}(\sigma)| \le C_q e^{a|\sigma_2|} \tag{4}$$

for some constants C_q and a depending on $\tilde{\phi}$. The set of all analytic functions $\mathcal{Z}$ with property (4) is in fact the space

$$F(\mathcal{D}) = \{\psi : \exists \phi \in \mathcal{D}, F(\phi) = \psi\}.$$

The Fourier transform $\tilde{f}$ of a distribution f in $\mathcal{D}'$ is an ultradistribution in $\mathcal{Z}'$, i.e. a continuous linear functional on $\mathcal{Z}$. It is defined by Parseval's equation

$$\langle \tilde{f}, \tilde{\phi} \rangle = 2\pi \langle f, \phi \rangle.$$

The *exchange formula* is the equality

$$F(f * g) = F(f).F(g). \tag{5}$$

It is well known that the exchange formula holds for all convolution products of distributions f and g satisfying Definition 1, provided f and g both have compact support, see for example Treves [6].

We now consider the problem of defining multiplication in $\mathcal{Z}'$. To do this we need the Fourier transform $F(\tau_n)$ of τ_n and write

$$\delta_n(\sigma) = \frac{1}{2\pi} F(\tau_n),$$

which is a function in $\mathcal{Z}$. Putting $\psi = \tilde{\phi}$, we have from Parseval's equation

$$\langle \tau_n, \phi \rangle = \frac{1}{2\pi} \langle F(\tau_n), F(\phi) \rangle = \langle \delta_n, \psi \rangle.$$

Since

$$\lim_{n\to\infty} \langle \tau_n, \phi \rangle = \lim_{n\to\infty} \int_{-\infty}^{\infty} \tau_n(x)\phi(x)\, dx = \int_{-\infty}^{\infty} \phi(x)\, dx = \langle 1, \phi \rangle$$

for all ϕ in $\mathcal{D}$ and since $F(1) = 2\pi\delta$, we obtain

$$\lim_{n\to\infty} \langle \delta_n, \psi \rangle = \langle \delta, \psi \rangle$$

for all ψ in $\mathcal{Z}$. Thus $\{\delta_n\}$ is a sequence in $\mathcal{Z}$ converging to the Dirac delta function δ.

If f is an arbitrary distribution in $\mathcal{D}'$, then since δ_n is a function in $\mathcal{Z}$, the convolution product $\tilde{f} * \delta_n$ is defined by

$$\langle (\tilde{f} * \delta_n)(\sigma), \psi(\sigma) \rangle = \langle \tilde{f}(\nu), \langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle \rangle \tag{6}$$

for arbitrary ψ in $\mathcal{Z}$. If $\psi = \tilde{\phi}$, we have

$$\psi(\sigma + \nu) = F[e^{ix\nu}\phi(x)]$$

and it follows from Parseval's equation that

$$\begin{aligned} \langle \delta_n(\sigma), \psi(\sigma + \nu) \rangle &= \frac{1}{2\pi} \langle F(\tau_n)(\sigma), F(e^{ix\nu}\phi)(\sigma) \rangle = \langle \tau_n(x), e^{ix\nu}\phi(x) \rangle \\ &= \int_{-\infty}^{\infty} \tau_n(x) e^{ix\nu} \phi(x)\, dx \\ &\to \int_{-\infty}^{\infty} e^{ix\nu}\phi(x)\, dx = \psi(\nu). \end{aligned} \tag{7}$$

Thus

$$\lim_{n\to\infty} \langle \tilde{f} * \delta_n, \psi \rangle = \langle \tilde{f}, \psi \rangle$$

for arbitrary ψ in $\mathcal{Z}$ and it follows that $\{\tilde{f} * \delta_n\}$ is a sequence of infinitely differentiable functions converging to $\tilde{f}$ in $\mathcal{Z}'$.

This leads us to the following definition:

Definition 4. Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathcal{Z}'$ and let $\tilde{f}_n = \tilde{f} * \delta_n$ and $\tilde{g}_n = \tilde{g} * \delta_n$. Then the *neutrix product* $\tilde{f} \Box \tilde{g}$ is defined to be the neutrix limit of the sequence $\{\tilde{f}_n . \tilde{g}_n\}$, provided the limit $\tilde{h}$ exists in the sense that

$$\operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}_n . \tilde{g}_n, \psi \rangle = \langle \tilde{h}, \psi \rangle$$

for all ψ in $\mathcal{Z}$.

In this definition we use $\tilde{f} \Box \tilde{g}$ to denote the neutrix product of $\tilde{f}$ and $\tilde{g}$ to distinguish it from the usual definition of the product $\tilde{f}_n . \tilde{g}_n$ of two infinitely differentiable functions $\tilde{f}_n$ and $\tilde{g}_n$. If

$$\lim_{n \to \infty} \langle \tilde{f}_n . \tilde{g}_n, \psi \rangle = \langle \tilde{h}, \psi \rangle$$

for all ψ in $\mathcal{Z}$, we simply say that the *product* $\tilde{f}.\tilde{g}$ exists and equals $\tilde{h}$. We then of course have

$$\tilde{f} \Box \tilde{g} = \tilde{f}.\tilde{g}$$

.

It is immediately obvious that if the neutrix product $\tilde{f} \Box \tilde{g}$ exists then the neutrix product is commutative.

The product of ultradistributions in $\mathcal{Z}'$ also has the following property:

Theorem 1. *Let $\tilde{f}$ and $\tilde{g}$ be ultradistributions in $\mathcal{Z}'$ and suppose that the neutrix products $\tilde{f} \Box \tilde{g}$ and $\tilde{f} \Box \tilde{g}'$ (or $\tilde{f}' \Box \tilde{g}$) exist. Then the neutrix product $\tilde{f}' \Box \tilde{g}$ (or $\tilde{f} \Box \tilde{g}'$) exists and*

$$(\tilde{f} \Box \tilde{g})' = \tilde{f}' \Box \tilde{g} + \tilde{f} \Box \tilde{g}'. \tag{8}$$

Proof. Let ψ be an arbitrary function in $\mathcal{Z}$. Then

$$\langle \tilde{f} \Box \tilde{g}, \psi \rangle = \operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}_n . \tilde{g}_n, \psi \rangle, \quad \langle \tilde{f} \Box \tilde{g}', \psi \rangle = \operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}_n . \tilde{g}'_n, \psi \rangle.$$

Further,

$$\begin{aligned} \langle (\tilde{f} \Box \tilde{g})', \psi \rangle &= -\langle \tilde{f} \Box \tilde{g}, \psi' \rangle = -\operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}_n . \tilde{g}_n, \psi' \rangle \\ &= -\operatorname*{N-lim}_{n \to \infty} \langle \tilde{g}_n, (\tilde{f}_n . \psi)' - \tilde{f}'_n . \psi \rangle \\ &= \operatorname*{N-lim}_{n \to \infty} \langle \tilde{g}'_n, \tilde{f}_n . \psi \rangle + \operatorname*{N-lim}_{n \to \infty} \langle \tilde{g}_n, \tilde{f}'_n . \psi \rangle \end{aligned}$$

and so

$$\operatorname*{N-lim}_{n \to \infty} \langle \tilde{f}'_n . \tilde{g}_n, \psi \rangle = \langle (\tilde{f} \Box \tilde{g})', \psi \rangle - \langle \tilde{f} \Box \tilde{g}', \psi \rangle.$$

Hence the neutrix product $\tilde{f}'.\tilde{g}$ exists and equation (8) follows.

It follows similarly that if $\tilde{f}' \Box \tilde{g}$ exists then $\tilde{f} \Box \tilde{g}'$ exists.

We can now prove the exchange formula.

Theorem 2. *Let f and g be distributions in $\mathcal{D}'$ having Fourier transforms $\tilde{f}$ and $\tilde{g}$ respectively in $\mathcal{Z}'$. Then the neutrix convolution product $f \boxed{*} g$ exists in $\mathcal{D}'$, if and only if the neutrix product $\tilde{f}\Box\tilde{g}$ exists in $\mathcal{Z}'$ and the exchange formula*

$$F(f \boxed{*} g) = \tilde{f}\Box\tilde{g}$$

is then satisfied.

Proof. We have from equation (7) that

$$\langle \delta_n(\sigma), \psi(\sigma + \nu)\rangle = F(\tau_n \phi)$$

and then from equation (6) that

$$\begin{aligned} \langle \tilde{f}_n, \psi\rangle &= \langle \tilde{f} * \delta_n, \psi\rangle = \langle \tilde{f}, F(\tau_n\phi)\rangle = 2\pi\langle f, \tau_n\phi\rangle \\ &= 2\pi\langle f_n, \phi\rangle = \langle F(f_n), \psi\rangle \end{aligned}$$

on using Parseval's equation twice. It follows that $F(f_n) = \tilde{f}_n$. Similarly, we have $F(g_n) = \tilde{g}_n$. Now since the convolution product $f_n * g_n$ exists by Definition 1 and f_n and g_n both have compact support

$$F(f_n * g_n) = F(f_n).F(g_n) = \tilde{f}_n.\tilde{g}_n$$

and so on using Parseval's equation again

$$2\pi\langle f_n * g_n, \phi\rangle = \langle F(f_n * g_n), \psi\rangle = \langle \tilde{f}_n.\tilde{g}_n, \psi\rangle.$$

Suppose the neutrix convolution product $f \boxed{*} g$ exists. Then

$$\begin{aligned} 2\pi\langle f \boxed{*} g, \phi\rangle &= \mathrm{N-}\lim_{n\to\infty} 2\pi\langle f_n * g_n, \phi\rangle = \mathrm{N-}\lim_{n\to\infty}\langle F(f_n * g_n), \psi\rangle \\ &= \mathrm{N-}\lim_{n\to\infty}\langle \tilde{f}_n.\tilde{g}_n, \psi\rangle = \langle \tilde{f}\Box\tilde{g}, \psi\rangle \end{aligned}$$

for arbitrary ϕ in $\mathcal{D}$ and $F\phi$ in $\mathcal{Z}$, proving the existence of the neutrix product $\tilde{f}\Box\tilde{g}$ and the exchange formula.

Conversely, if the neutrix product $\tilde{f}\Box\tilde{g}$ exists then the argument can be reversed to prove the existence of the neutrix convolution product $f \boxed{*} g$ and the exchange formula. This completes the proof of the theorem.

Theorem 3. *The products $(\sigma + i0)^\lambda.(\sigma + i0)^\mu$ and $(\sigma - i0)^\lambda.(\sigma - i0)^\mu$ exist and*

$$(\sigma + i0)^\lambda.(\sigma + i0)^\mu = (\sigma + i0)^{\lambda+\mu} \tag{9}$$

$$(\sigma - i0)^\lambda.(\sigma - i0)^\mu = (\sigma - i0)^{\lambda+\mu} \tag{10}$$

for all λ and μ .

Proof. It is well known that

$$x_+^\lambda * x_+^\mu = B(\lambda + 1, \mu + 1)x_+^{\lambda+\mu+1} \tag{11}$$

for $\lambda, \mu, \lambda + \mu + 1 \neq -1, -2, \dots$, where B denotes the Beta function.

Further, see Gel'fand and Shilov [4],

$$F(x_+^\lambda) = ie^{i\lambda\pi/2}\Gamma(\lambda + 1)(\sigma + i0)^{-\lambda-1} \tag{12}$$

for $\lambda \neq -1, -2, \dots$. On using the exchange formula, it follows from equations (11) and (12) that

$$\begin{aligned} -e^{i(\lambda+\mu)\pi/2}\Gamma(\lambda+1)\Gamma(\mu+1)(\sigma+i0)^{-\lambda-1}.(\sigma+i0)^{-\mu-1} &= \\ = B(\lambda+1, \mu+1)ie^{i(\lambda+\mu+1)\pi/2}\Gamma(\lambda+\mu+2)(\sigma+i0)^{-\lambda-\mu-2} \end{aligned}$$

for $\lambda, \mu, \lambda+\mu+1 \neq -1, -2, \dots$, the product $(\sigma+i0)^{-\lambda-1}.(\sigma+i0)^{-\mu-1}$ existing since the convolution product $x_+^\lambda * x_+^\mu$ exists. Equation (9) now follows for $\lambda, \mu, \lambda + \mu \neq 0, 1, 2, \dots$.

Now suppose that $\lambda, \mu, \lambda + \mu > -1$ and put

$$(\sigma + i0)_n^\lambda = (\sigma + i0)^\lambda * \delta_n(\sigma).$$

Then since

$$(\sigma + i0)^\lambda = \sigma_+^\lambda + e^{i\lambda\pi}\sigma_-^\lambda,$$

see [4], it follows that $\{(\sigma + i0)_n^\lambda.(\sigma + i0)_n^\mu\}$ is a sequence of locally summable functions which converges to the locally summable function $(\sigma+i0)^{\lambda+\mu}$. Equation (9) follows for $\lambda, \mu, \lambda + \mu > -1$.

Now suppose that equation (9) holds when $-k - 1 < \lambda < -k$, for some positive integer k, and $\lambda + \mu = 0, \pm 1, \pm 2, \dots$. This is certainly true when $k = 0$. Then

$$\lim_{n\to\infty}(\sigma + i0)_n^\lambda.(\sigma + i0)_n^\mu = (\sigma + i0)^{\lambda+\mu},$$

by our assumption when $-k - 1 < \lambda < -k$. It follows that

$$\begin{aligned} \lim_{n\to\infty}[(\sigma + i0)_n^\lambda.(\sigma + i0)_n^\mu]' &= \\ &= \lim_{n\to\infty}[\lambda(\sigma + i0)_n^{\lambda-1}.(\sigma + i0)_n^\mu + \mu(\sigma + i0)_n^\lambda.(\sigma + i0)_n^{\mu-1}] \\ &= (\lambda + \mu)(\sigma + i0)^{\lambda+\mu-1} \end{aligned}$$

and so

$$\lim_{n\to\infty}(\sigma + i0)_n^{\lambda-1}.(\sigma + i0)_n^\mu = (\sigma + i0)^{\lambda+\mu-1}.$$

Equation (9) follows by induction for $\lambda \neq -1, -2, \dots$ and $\lambda + \mu = 0, \pm 1, \pm 2, \dots$.

We are finally left to prove equation (9) for the case $\lambda = r = -1, -2, \dots$ and $\mu = s = 0, 1, 2, \dots$. Since

$$\ln(\sigma + i0) = \ln|\sigma| + i\pi H(-\sigma)$$

and

$$(\sigma + i0)^s = \sigma^s$$

for $s = 0, 1, 2, \dots$, see [4], are locally summable functions, it follows as above that if

$$\ln(\sigma + i0)_n = \ln(\sigma + i0) * \delta_n(\sigma),$$

then the sequence $\{\ln(\sigma + i0)_n.(\sigma + i0)_n^s\}$ converges to the locally summable function $(\sigma + i0)^s \ln(\sigma + i0)$. Thus

$$\begin{aligned}
\lim_{n\to\infty} [\ln(\sigma + i0)_n.(\sigma + i0)_n^s]' &= \\
&= \lim_{n\to\infty} [(\sigma + i0)_n^{-1}.(\sigma + i0)_n^s + s\ln(\sigma + i0)(\sigma + i0)_n^{s-1}] \\
&= [(\sigma + i0)^s \ln(\sigma + i0)]' \\
&= s(\sigma + i0)^{s-1} \ln(\sigma + i0) + (\sigma + i0)^{s-1},
\end{aligned}$$

see [4], and so

$$\lim_{n\to\infty} (\sigma + i0)_n^{-1}.(\sigma + i0)_n^s = (\sigma + i0)^{s-1}.$$

Equation (9) follows for $\lambda = -1$ and $\mu = 0, 1, 2, \dots$. Another induction argument shows that equation (9) holds for $\lambda = -1, -2, \dots$ and $\mu = 0, 1, 2, \dots$. This completes the proof of the theorem.

Corollary 1.

$$\sigma^{-r}.\sigma^s = \sigma^{s-r} \tag{13}$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$ and

$$\delta^{(r-1)}(\sigma).\sigma^s = \begin{cases} 0, & s \geq r, \\ \frac{(-1)^s (r-1)!}{(r-s-1)!} \delta^{(r-s-1)}(\sigma), & r > s \end{cases} \tag{14}$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots$.

Proof. Since

$$(\sigma + i0)^s = \sigma^s$$

for $s = 0, 1, 2, \dots$ and

$$(\sigma + i0)^{-r} = \sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(\sigma)$$

for $r = 1, 2, \dots$, see [4], it follows from equation (9) that

$$\begin{aligned}
(\sigma + i0)^{-r}.\sigma^s &= \begin{cases} \sigma^{s-r}, & s \geq r, \\ \sigma^{s-r} + \frac{i\pi(-1)^{r+s}}{(r-s-1)!} \delta^{(r-s-1)}(\sigma), & r > s \end{cases} \\
&= \sigma^{-r}.\sigma^s + \frac{i\pi(-1)^r}{(r-1)!} \delta^{(r-1)}(\sigma).\sigma^s,
\end{aligned}$$

the product clearly being distributive with respect to addition. Equating real and imaginary parts, equations (13) and (14) follow.

Corollary 2.

$$\sigma_+^{-r-1/2}.\sigma_-^{-r-1/2} = \frac{(-1)^r\pi}{2(2r)!}\delta^{(2r)}(\sigma) \tag{15}$$

for $r = 0, 1, 2, \dots$.

Proof. It follows from equation (9) that

$$\begin{aligned}(\sigma + i0)^{-r-1/2}.(\sigma + i0)^{-r-1/2} &= (\sigma + i0)^{-2r-1} \\ &= \left[\sigma_+^{-r-1/2} - i(-1)^r\sigma_-^{-r-1/2}\right].\left[\sigma_+^{-r-1/2} - i(-1)^r\sigma_-^{-r-1/2}\right] \\ &= \sigma^{-2r-1} - \frac{i\pi}{(2r)!}\delta^{(2r)}(\sigma)\end{aligned}$$

for $r = 0, 1, 2, \dots$. Expanding and equating the imaginary parts gives equation (15).

Corollary 3.

$$\sigma^{-r}.\delta^{(r-1)}(\sigma) = \frac{(-1)^r(r-1)!}{2(2r-1)!}\delta^{(2r-1)}(\sigma) \tag{16}$$

for $r = 1, 2, \dots$.

Proof. It follows from equation (9) that

$$\begin{aligned}(\sigma + i0)^{-r} \cdot (\sigma + i0)^{-r} &= (\sigma + i0)^{-2r} \\ &= \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right] \cdot \left[\sigma^{-r} + \frac{i\pi(-1)^r}{(r-1)!}\delta^{(r-1)}(\sigma)\right] \\ &= \sigma^{-2r} + \frac{i\pi}{(2r-1)!}\delta^{(2r-1)}(\sigma)\end{aligned}$$

for $r = 1, 2, \dots$. Expanding and equating imaginary parts gives equation (16).

Theorem 4. *The neutrix product* $\sigma_+^\lambda \Box \delta^{(s)}(\sigma)$ *exists and*

$$\sigma_+^\lambda \Box \delta^{(s)}(\sigma) = 0 \tag{17}$$

for real $\lambda \neq 0, \pm1, \pm2, \dots$ *and* $s = 0, 1, 2, \dots$.

Proof. It was proved in [3] that

$$x_+^\lambda \boxed{*}\, x^s = 0, \quad x_-^\lambda \boxed{*}\, x^s = 0$$

for real $\lambda \neq 0, \pm1, \pm2, \dots$ and $s = 0, 1, 2, \dots$. Thus

$$(x - i0)^\lambda \boxed{*}\, x^s = (x_+^\lambda + e^{-i\lambda\pi}x_-^\lambda)\boxed{*}\, x^s = 0$$

for real $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $s = 0, 1, 2, \dots$. On applying the exchange formula to this equation we get

$$\sigma_+^{-\lambda-1} \Box \delta^{(s)}(\sigma) = 0$$

for real $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $s = 0, 1, 2, \dots$, since

$$F[(x - i0)^\lambda] = \frac{2\pi e^{-i\lambda\pi/2}}{\Gamma(-\lambda)} \sigma_+^{-\lambda-1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and

$$F(x^s) = 2(-i)^s \pi \delta^{(s)}(\sigma)$$

for $s = 0, 1, 2, \dots$, see Gel'fand and Shilov [4]. Equation (17) follows immediately.

Corollary 1. *The neutrix product* $\sigma_-^\lambda \Box \delta^{(s)}(\sigma)$ *exists and*

$$\sigma_-^\lambda \Box \delta^{(s)}(\sigma) = 0$$

for real $\lambda \neq 0, \pm 1, \pm 2, \dots$ *and* $s = 0, 1, 2, \dots$.

Proof. The result follows immediately from equation (17) on replacing x by $-x$ in equation (17).

Theorem 5. *The neutrix product* $(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu$ *exists and*

$$(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu = \sigma_+^{\lambda+\mu} + e^{i(\mu-\lambda)\pi} \sigma_-^{\lambda+\mu} \tag{18}$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$.

Proof. It was proved in [3] that

$$x_+^{-\lambda-1} \boxed{*} x_-^{-\mu-1} = B(\lambda + \mu + 1, -\mu) x_-^{-\lambda-\mu-1} + B(\lambda + \mu + 1, -\lambda) x_+^{-\lambda-\mu-1}$$

for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$. Applying the exchange formula to this equation and using equation (12) and the equation

$$F(x_-^\lambda) = -ie^{-i\lambda\pi/2} \Gamma(\lambda + 1)(\sigma - i0)^{-\lambda-1}$$

we get

$$\begin{aligned} e^{i(\lambda-\mu)\pi/2} \Gamma(-\lambda)\Gamma(-\mu)(\sigma - i0)^\lambda \Box (\sigma + i0)^\mu &= \\ &= e^{i(\lambda+\mu)\pi/2} B(\lambda + \mu + 1, -\mu)\Gamma(-\lambda - \mu)(\sigma - i0)^{\lambda+\mu} + \\ &+ e^{-i(\lambda+\mu)\pi/2} B(\lambda + \mu + 1, -\lambda)\Gamma(-\lambda - \mu)(\sigma + i0)^{\lambda+\mu} \end{aligned}$$

and so

$$\begin{aligned} (\sigma - i0)^\lambda \Box (\sigma + i0)^\mu &= e^{i\mu\pi} \sin(\lambda\pi) \operatorname{cosec}[(\lambda + \mu)\pi](\sigma - i0)^{\lambda+\mu} + \\ &+ e^{-i\lambda\pi} \sin(\mu\pi) \operatorname{cosec}[(\lambda + \mu)\pi](\sigma + i0)^{\lambda+\mu} \\ &= \sigma_+^{\lambda+\mu} + e^{i(\mu-\lambda)\pi} \sigma_-^{\lambda+\mu}, \end{aligned}$$

proving equation (18) for real $\lambda, \mu \neq 0, \pm 1, \pm 2, \dots$.

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