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A CONTINUOUS DEPENDENCE OF FIXED POINTS OF ϕ–CONTRACTIVE MAPPINGS IN UNIFORM SPACES

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Abstract. The main purpose of the present paper is to established conditions for a continuous dependence of fixed points of ϕ-contractive mappings in uniform spaces. An application to nonlinear functional differential equations of neutral type have been made.

The main purpose of the present paper is to establish when the convergence of a sequence of ϕ-contractive mappings in a uniform space implies a convergence of the sequence of their fixed points. The notion a ϕ-contractive mapping in a uniform space has been introduced in [1]. In veiw of the applications given in [1] the problem of a continuous dependence of fixed points can be formulated as a continuous dependence of the solutions of a nonlinear functional differential equation on its right-hand side. As a particular case we obtain an extension of the results from [2] and [3] in metric spaces.

Since [4] contains the most general version of fixed point for ϕ-contractive mappings in uniform spaces we shall recall some basic definitions and results from [4].

Let $(X, \mathcal{A})$ be a complete Hausdorff uniform space with a uniformity generated by saturated family of pseudometrics $\mathcal{A} = \{d_\alpha(x,y) : \alpha \in A\}$, A being an index set (cf [5]) Let $j : A \to A$ be a mapping and let $j^k(\alpha) = j(j^{k-1}(\alpha)), j^0(\alpha) = \alpha, (k = 1,2,3,\dots)$. Since j is not image of the element $\alpha_0 \in A$, that is $j^{-1}(\alpha_0) = \{\alpha \in A : j(\alpha) = \alpha_0\}$. In general we define

$$j^{-n}(\alpha_0) = \{\alpha \in A : j^n(\alpha) = \alpha_0\} \quad (n = 2,3,\dots) \quad .$$

The space X is called j-bounded if for every $x, j \in X$ and $\alpha \in A$ there exists a constant $Q = Q(\alpha, x, y) > 0$ such that

$$d_{j^{-n}(\alpha)}(x,y) \leqq Q \quad (\alpha, x, y) < \infty \quad (n = 0,1,2,\dots) \quad .$$

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The last inequality is in the sense defined in [4]. Further on we shall assume that X is a j-bounded space.

Let (ϕ) be a family of contractive functions $\Phi_\alpha(t)R^1_+ \to R^1_+,\ R^1_+ = [0,\infty), \alpha \in$ $\in A$ with the properties:

$$\begin{aligned} &\Phi_\alpha(t) \quad \text{is strictly increasing, continuous from the right,}\\ &0 < \Phi_\alpha(t) < t \quad \text{and} \quad \Phi_{j(\alpha)}(t) \leqq \Phi_\alpha(t) \quad \text{for} \quad t \geqq 0 \quad \text{and}\\ (\Phi 1) \qquad &\Phi_\alpha(t_1+t_2) \leqq \Phi_\alpha(t_1) + \Phi_\alpha(t_2) \quad \text{for every} \quad t_1, t_2 > 0 \quad . \end{aligned}$$

$$(\Phi 2) \qquad \lim_{n\to\infty} \Phi_\alpha(\Phi_{j^{-1}(\alpha)}(\dots\Phi_{j^{-n}(\alpha)}(t)\dots)) = 0$$

which ought to be understand in the following sense: for every sequence $\alpha, \alpha_1, \dots$ $\dots, \alpha_n, \dots (\alpha_n \in j^{-n}(\alpha)) \lim_{n\to\infty} \Phi_\alpha(\Phi_{\alpha_1}(\dots\Phi_{\alpha_n}(t)\dots)) = 0.$

The mapping T is called: 1) ϕ-contractive if $d_{j(\alpha)}(Tx, Ty \leqq \Phi_\alpha(d_\alpha(x,y))$ for every $x, y \in X$ and $\alpha \in A$; 2) contractive if $d_{j(\alpha)}(Tx,Ty) < d_\alpha(x,y)$ for every $x, y \in X$ and $\alpha \in A$; 3) j-regular when if $\{T^n x\}_{n=0}^\infty$ is not d_α-Cauchy sequence, then it is not $d_{j(\alpha)}$-Cauchy sequence for every $x \in$ (or equivalently, if $\{T^n x\}_{n=0}^\infty)$ is $d_{j(\alpha)}$-Cauchy sequence, then it is d_α-Cauchy sequence).

Theorem A. [4] *Every ϕ-contractive j-regular mapping $T : X \to X$ has a unique fixed point $\bar{x} \in X$ and $\bar{x} = \lim_{n\to\infty} T^n x$ for arbitrary $x \in X$.*

Main results

Let $\{T_k\}_{k=1}^\infty$ be a sequence of operators $T_k : X \to X$. Every T_k has at last one fixed point $y_k (k = 1, 2, 3, \dots)$. Let $T_0 : X \to X$ be a ϕ-contractive j-regular mapping with fixed point y_0. We say that the sequence $\{T_k\}_{k=1}^\infty$ tends uniformly to T_0 if for every $\varepsilon > 0$ there exists $\nu = \nu(\varepsilon)$ such that $d_\alpha(T_k x, T_0 x) < \varepsilon$ for every $k > \nu, x \in X, \alpha \in A$

Theorem 1. *If the sequence $\{T_k\}_{k=1}^\infty$ converges uniformly to T_0, then the sequence $\{y_k\}_{k=1}^\infty$ converges to y_0.*

Proof. In view of the uniform convergence of the sequence $\{T_k\}_{k=1}^\infty$ to T_0 for every $\varepsilon > 0$, $\alpha \in A$ there is ν_1 such that $d_\alpha(T_k y, T_0 y) < \varepsilon/2$ for every $y \in X$. So that we have for $k > \nu_1$

$$d_\alpha(y_k, y_0) = d_\alpha(T_k y_k, T_0 y_0) \leqq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(d_{j^{-1}(\alpha)}(y_k, y_0)) \quad .$$

For $\varepsilon/2^2$ we find ν_2 such that when $k > \nu_2$ we have

$$d_{j^{-1}(\alpha)}(y_k, y_0) \leqq \frac{\varepsilon}{2^2} + \Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0)) \quad .$$

Therefore for $k > \max\{\nu_1, \nu_2\}$ the following inequalities are fulfilled:

$$
\begin{aligned}
d_\alpha(y_k, y_0) &\leqq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(\frac{\varepsilon}{2^2} + \Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0))) \leqq \\
&\leqq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(\frac{\varepsilon}{2^2}) + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0))) \leqq \\
&\leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2}) + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0))) \quad .
\end{aligned}
$$

We can proceed in an analogous way and then obtain for $k > N_n = \max\{\nu_1, \nu_2, \ldots$ $\ldots, \nu_n\}$

$$
\begin{aligned}
d_\alpha(y_k, y_0) \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots + \frac{\varepsilon}{2^n} + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\ldots \\
\ldots \Phi_{j^{-n-1}(\alpha)}(d_{j^{-n-1}(\alpha)}(y_k, y_0)) \ldots) \leqq \\
\leqq \varepsilon + \Phi_{j^{-1}(\alpha)}\Phi_{j^{-2}(\alpha)}(\ldots \Phi j^{-n-1}(\alpha)(d_{j^{-n-1}(\alpha)}(y_k, y_0)) \ldots)) \quad .
\end{aligned}
$$

Let us fix n sufficiently large such that

$$
\Phi_{j^{-1}(\alpha)}\Phi_{j^{-2}(\alpha)}(\ldots \Phi j^{-n-1}(\alpha)(d_{j^{-n-1}(\alpha)}(y_k, y_0)) \ldots) < \varepsilon \quad .
$$

Then for $k > N_n$ we have $d_\alpha(y_k, y_0) < 2\varepsilon$. Theorem 1 is thus proved. □

Remark 1. If we replace the definition of ϕ-contractive mapping by the following one : $d_\alpha(T_x, T_y) \leqq \Phi_\alpha(d_{j(\alpha)}(x, y))$ then the assertion of Theorem 1 is also valid. We must only modify conditions (Φ1) and (Φ2), namely, $\Phi_\alpha(t) \leqq \Phi_{j(\alpha)}(t)$ and

$$
\lim_{n\to\infty} \Phi_\alpha(\Phi_{j(\alpha)}(\ldots \Phi_{j^n(\alpha)}(t)) \ldots) = 0 \quad .
$$

The definition of a j-bounded and j-regular mapping can be modified in an obvious way.

Remark 2. Let X be a quasicomplete uniform space. This means that every closed bounded subset of X is complete in the induced topology. Consequently if T : : $M \to M$ is a ϕ-contractive mapping of a bounded closed set $M \subset X$ into itself, then T has a unique fixed point in M (cf. [1], Theorem 1).

Further on we shall assume that X is a locally compact space. Let us recall the relations between locally compact spaces and uniformizable spaces (cf. [6]). Every completely regular T_1-space is said to be a Tikhonoff's one. It is known (cf.[6]) that every locally compact Hausdorff space is a Tikhonoff's space. On the other hand X is uniformizable if and only if X is a completely regular space. So that we shall assume that X is a locally compact quasicomplete Hausdorff space. We shall denote again by $\mathcal{A}$ its uniformity, that is, $\mathcal{A} = \{d_\alpha(x, y) : \alpha \in A\}$ (cf.[5], [6]).

Theorem 2. *Let $(X, \mathcal{A})$ be a locally compact Hausdorff quasicomplete j-bounded uniform space. Let $T_k : X \to X$ be a ϕ-contractive mapping with fixed point y_k for any $k = 0, 1, 2, \ldots$,i.e. $d_\alpha(T_k x, T_k y) \leqq \Phi_\alpha(d_{j(\alpha)}(x, y))$. If $\{T_k\}_{k=1}^{\infty}$ converges pointwise to y_0, then the sequence $\{y_k\}_{k=1}^{\infty}$ converges to y_0.*

Proof. Let us choose $\varepsilon > 0$ and $\alpha_1, \ldots, \alpha_p \in A$ such that the neighbourhood $N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0) = \{x \in X : d_{\alpha_i}(y_0, x) \leqq \varepsilon\}$ of y_0 is a compact subset of X The sequence $\{T_k\}_{k=1}^{\infty}$ is equicontinuous and converges pointwise to T_0. But $N\varepsilon(\alpha_1, \ldots$ $\ldots, \alpha_p)(y_0)$ is compact and in view of the results of Ch. VII [6], $\{T_k\}_{k=1}^{\infty}$ converges uniformly on $N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$ to 0. Then let for $k > \nu_s$ we have $d_{\alpha_i}(T_k y, T_0 y) <$ $< \dfrac{\varepsilon}{2^{s+1}}$ $(i = 1, 2, \ldots, p)$ for every $y \in N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$ and for $k > N_n =$ $= \max\{\nu_1, \ldots, \nu_n\}$ $(s = 1, \ldots, p)$ we have

$$
\begin{aligned}
d_{\alpha_i}(T_k y, y_0) &= d\alpha_i(T_k y, T_0 y_0) \leqq d_{\alpha_i}(T_k y, T_0 y) + d_(\alpha_i)(T_0 y, T_0 y_0) \leqq \\
&\leqq \frac{\varepsilon}{2^2} + \Phi_{\alpha_0}(d_{j(\alpha)}(y, y0)) \leqq \frac{\varepsilon}{2^2} + \Phi_{\alpha_0}(\frac{\varepsilon}{2^3} + \Phi_{j(\alpha)}(d_{j-2(\alpha)}(y, y_0))) \leqq \\
&\leqq \frac{\varepsilon}{2^2} + \Phi_{\alpha_0}(\frac{\varepsilon}{2^3}) + \Phi_{\alpha_0}(\Phi_{j(\alpha)0}(d_{j^2(\alpha)}(y, y_0))) \leqq \ldots \\
&\leqq \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^n} + \Phi_{\alpha_0}(\Phi_{j(\alpha)0}(\ldots \Phi_{j^n(\alpha)0}(d_{j^{n+1}(\alpha)}(y, y_0)) \ldots)) \leqq \\
&\leqq \frac{\varepsilon}{2} + \Phi_{\alpha_0}(\ldots \Phi j^n \alpha_0(Q) \ldots) \leqq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad .
\end{aligned}
$$

We obtained that T_k maps $N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$ into itself.

Denote by T_k/N_ε the restriction of T_k to $N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$. But X is quasicomplete and j-bounded. The same properties has and $N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$. Then T_k/N_ε possesses a fixed point $\bar{y}_k$ for $k > N_n$ in $N\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$. On the other hand T_k has only one fixed point y_k. Consequently $\bar{y}_k = y_k \in N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)$ for $k > N_n$. So we have $\lim\limits_{k \to \infty} y_k = y_0$ which completes proof of Theorem 2.

Let $\mathcal{A}_1 = \{d_{\alpha_1}(x, y) : \alpha_1 \in A1\}$ be two families of pseudometrics for the same set X. They will be called equivalent if and only if the identity mapping from $(X, \mathcal{A}_1)$ to $(X, \mathcal{A}_2)$ is a homomorphism. Further on, we shall assume that card $\mathcal{A}_1 = \text{card}\mathcal{A}_2$, so that we shall not differ the index sets of equivalent families of pseudometrics.

A sequence of families of pseudometrics $\{\mathcal{A}_n\}_{n=1}^{\infty}$ tends uniformly to a family $\mathcal{A}_0$ if for every $\varepsilon > 0$ there is N such that for every $n > N, x, y \in X$ and $\alpha \in A$

$$|d_\alpha^{(n)}(x, y) - d_\alpha^{(0)}(x, y)| < \varepsilon \quad .$$

Proposition 1. *Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of families of pseudometrics on X which tends uniformly to the family $\mathcal{A}_0$ such that each $\mathcal{A}_n$ is equivalent to $\mathcal{A}_0$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of ϕ-contractive mappings converging pointwise on X to a mapping T_0. Then $\{T_n\}_{n=1}^{\infty}$ converges $\mathcal{A}_0$-uniformly for every compact set $K \in X$ to T_0 $(T_n$ is ϕ-contractive with respect to the family $\mathcal{A}_n)$.*

Proof. For an arbitrary $\varepsilon > 0$ we choose $\eta = \frac{\varepsilon}{3}$. Let for $n > N$, $\alpha \in A$ and $x, y \in X$, $\bar{\alpha} \in j^{-1}(\alpha)$ the inequality holds

$$|d_{\bar{\alpha}}^{(n)}(x,y) - d_{\bar{\alpha}}^{(0)}(x,y)| < \eta \quad .$$

If $n > N$ and $x, y \in X$ for which $d_{\alpha}^{(0)}(x,y) < \eta$ then

$$d_{\alpha}^{(0)}(T_n x, T_n y) < \eta + d_{\alpha}^{(n)}(T_n x, T_n y) < \eta + d_{\bar{\alpha}}^{(n)}(x,y) < \eta + \eta + d_{\bar{\alpha}}^{(0)}(x,y) < 3\eta = \varepsilon \quad .$$

We obtained: for every $x, y \in X$, $\alpha \in A$, and $\bar{\alpha} \in j^{-1}(\alpha)$ the inequality $d_{\bar{\alpha}}^{(0)}(x,y) < < \eta$ implies $d_{\alpha}^{(0)}(T_n x, T_n y) < \varepsilon$ for $n > N$. On the other hand every $T_k (k = = 1, \ldots, N)$ is uniformly continuous on the compact set K with respect to the family $\mathcal{A}_0$. Consequently the sequence $\{T_n\}_{n=1}^{\infty}$ is equicontinuous on K with respect to $\mathcal{A}_0$. But K is compact and then pointwise convergence of $\{T_n\}_{n=1}^{\infty}$ to T_0 implies a uniform convergence to T_0 with respect to $\mathcal{A}_0$, which completes the proof of Proposition 1. $\square$

We shall introduce the notion j-locally compact space. The uniform space X is said to be j-locally compact if for every point y_0 and for every finite collection $\alpha_1, \ldots, \alpha_p \in A$ there exists $\varepsilon = \varepsilon(\alpha_1, \ldots, \alpha_p) > 0$ such that the set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) = \{x \in X : d_{\alpha_i}(x, y0) \leqq \varepsilon(\alpha_1, \ldots, \alpha_p)\}$ is compact and $\varepsilon(\alpha_1, \ldots, \alpha_p) \leqq \varepsilon(j(\alpha_1, \ldots, j(\alpha_p))$ and $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon(\alpha_1, \ldots, \alpha_p)) \subset \subset K(\bar{\alpha}_1, \ldots, \bar{\alpha}_p)(y_0, \varepsilon(\bar{\alpha}_1, \ldots, \bar{\alpha}_p))$ for $\bar{\alpha}_i \in j^{-1}(\alpha_i) \quad (i = 1, 2, \ldots, p) \quad .$

Theorem 3. *Let $(X, \mathcal{A})$ be a j-locally compact j-bounded quasicomplect uniform space. The sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are as in Proposition 1. If T_0 is ϕ-contractive with respect to $\mathcal{A}_0$ and T_n has a fixed point y_n $(n = 0, 1, \ldots)$, then the sequence $\{y_n\}_{n=1}^{\infty}$ tends to y_0.*

Proof. For every finite collection $\alpha_1, \ldots, \alpha_p \in A$ we find $\varepsilon = \varepsilon(\alpha_1, \ldots, \alpha_p) > 0$ such that the set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ is a compact. By Proposition 1 the sequence $\{T_n\}_{n=1}^{\infty}$ tends uniformly to T_0 on $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$.

Let $\bar{\alpha}_i \in j^{-1}(\alpha_i) \quad (i = 1, \ldots, p)$. Then for a collection $\bar{\alpha}_1, \ldots, \bar{\alpha}_p$ there exists $\varepsilon = \varepsilon(\bar{\alpha}_1, \ldots, \bar{\alpha}_p) > 0$ such that $K(\bar{\alpha}_1, \ldots, \bar{\alpha}_p)(y_0, \varepsilon(\bar{\alpha}_1, \ldots, \bar{\alpha}_p))$ is a compact set. Consider the continuous function $f_n^i(z) = d_{\alpha_i}^0(y_0, T_n z)$ on the compact set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon(\alpha_1, \ldots, \alpha_p))$. Having in mind the definition of j-locally compactness we have

$$f_n^i(z) = d_{\alpha_i}^0(y_0, T_n z) \leqq \Phi_{\bar{\alpha}_i}(d_{\bar{\alpha}_i}^0(y_0, z)) \leqq$$
$$\Phi_{\bar{\alpha}_i}(\varepsilon(\alpha_1, \ldots, \alpha_p)) \leqq \Phi_{\bar{\alpha}_i}(\varepsilon(j(\bar{\alpha}_1), \ldots, j(\bar{\alpha}_p)) < \varepsilon(\alpha_1, \ldots, \alpha_p)$$

for every $\bar{\alpha}_i \in j^{-1}(\alpha_i)$. Then

$$\bar{f}^{-i} = \sup\{f_n'(z) : z \in K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)\} < \varepsilon \quad (i = 1, 2, \ldots, p)$$

and $\eta = \max\left\{f^{-i} : i = 1, 2, \ldots, p\right\} < \varepsilon$.

In view of uniform convergence of $\{T_n\}_{n=1}^{\infty}$ to T_0 for $\mu = \varepsilon - \eta > 0$ we find N such that for $n \geqq N$ and $x \in K(\alpha_i, \ldots, \alpha_p)(y_0, \varepsilon)$ we have $d^0_{\alpha_i}(T_n x, T_0 x) < \mu$ Therefore we obtain

$$d^0_{\alpha_i}(T_n x, y_0) \leqq d^0_{\alpha_i}(T_n x, T_0 x) + d_{\alpha_i}(T_0 x, T_0 y_0) \leqq \mu + \varepsilon - \mu = \varepsilon \quad ,$$

that is, for $n \geqq N$ the operator T_n maps $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ into itself.

Denote by $T_n|_K$ the restriction of T_n to $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ for each $n \geqq N$ But $T_n|_K$ is ϕ-contractive with respect to $\mathcal{A}_n$ which maps $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ into itself. On the other hand $\mathcal{A}_0$ and $\mathcal{A}_n$ are equivalent and $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ is a compact with respect to $\mathcal{A}_n$. Consequently $T_n|_K$ has a fixed point y_n and since T_n has only one fixed point then $y_n \in K(\alpha_1, m \ldots, \alpha_p)(y_0, \varepsilon)$ for $n \geqq N$ It follows that $y_n \to y_0$.

Theorem 3 is thus proved. □

Applications

Here we shall apply the results obtained to some initial value problems consider in [1].

Let us consider the initial value problems

$$(1_k) \qquad \begin{gathered} \varphi'(t) = F_k(t, \varphi(\Delta, (t)), \ldots, \varphi(\Delta_m(t)), \varphi'(\tau_1(t)), \ldots, \varphi'(\tau_n(t))), \\ t > 0 \varphi(t) = \psi(t) \ , \quad \varphi'(t) = \psi'(t) \ , \quad t \leqq 0 \ , \end{gathered}$$

where $\varphi(t)$ is the unknown function. The deviations $\Delta_i(t) = \tau_l(t) \quad (i - 1, \ldots, m;$ $l = 1, \ldots, n)$ are of mixed type and in general case unbounded. After usual transformations, assuming $\psi(0) = 0$ problem (1_k) can be reduced to the following one $(x(t) = \varphi'(t))$ for $t > 0$ and $\theta(t) = \psi'(t)$ for $t \leqq 0$):

$$(2_k) \qquad \begin{gathered} x(t) = F_k(t, \int_0^{\Delta_1(t)} x(s)\, ds, \ldots, \int_0^{\Delta_m(t)} x(s)\, ds, x(\tau_1(t)), \ldots, x(\tau_n(t))), \\ t > 0 \ , \quad x(t) = \theta(t) \ , \quad t \leqq 0 \quad . \end{gathered}$$

Let $C(R^1)$ be the linear topological space consisting of all continuous function $f(t) : R^1 \to R^1$ with a topology generated by a saturated family of seminorms $\mathcal{A} = \{\| \cdot \|_K\}$, $\quad \|f\|_K = \sup \{|f(t)| : \ t \in K \ \}$ where $K \subset R^1$ runs over all compact subsets of R^1. In view of Theorem 2 we shall look for a solution of (2_k) in a locally compact set of functions. Namely, let us consider the set $c_L = \{f \in C(R^1) :$ $|f(t) - f(\bar{t})| \leqq L|t - \bar{t}|$ for every $t, \bar{t} \in R^1\}$, where the Lipschitz constant L does not depend on K. It easy to verify that C_L is closed convex and every point has a neighbourhood with a compact closure by Arzela-Ascoli theorem. We shall find a solution of $(2_k$ in the set $C^0_L = \{f \in C_L : |f(t)| \leqq r_0(t)\}$ where $r_0(t) : R^1 \to$ $\to R^1_+, \quad r_0(t)$ is continuous positive function on R^1.

We shall make the following assumption (cf. [1]):

$$\text{(C1)}\qquad \begin{aligned}&\Delta_i(t),\tau(t): R^1_+ \to R^1 \quad (R^1_+ = [0,\infty)) \text{ are continuous}\\ &\Delta_i(0) \leqq 0 \quad , \ \tau_l(0) \leqq 0 \text{ and } |\Delta_i(t) - \Delta_i(\bar t)| \leq P_i|t-\bar t| \ ,\\ &|\tau_l(t) - \tau_l(\bar t)| \leqq Q_l|t - \bar t|.\end{aligned}$$

The map $j : A \to A$ is defined as in [1], where the index set A consists all compact subsets of R^1:

(C2) For every $k = 0,1,2,\dots$ the functions $F_k(t,u_1,\dots,\ u_m,v_1,\dots,v_n)$: $R^1_+ \times R^{mn} \to R^1$ are continuous and satisfy the conditions:

$$|F_k(t,u_1,\dots,u_m,v_1,\dots,v_n)| \leqq \omega(t)\left[1+\sum_{i=1}^m |u_i| + \sum_{l=1}^n |v_l|\right]$$

$$|F_k(t,u_1,\dots,u_m,v_1,\dots,v_n) - F_k(t,\bar u_1,\dots,\bar u_m,\bar v_1,\dots,\bar v_n)| \leqq$$
$$\leqq \Omega[|u_1-\bar u_1| + \dots + |u_m - \bar u_m| + |v_1 - \bar v_1| + \dots + |v_n - \bar v_n|]$$

where Ω is a positive constant:

$$|F_k(t,u_1,\dots,u_m,v_1,\dots,v_n) - F_k(\bar t,u_1,\dots,u_m,v_1,\dots,v_n) \leqq L_0|t-\bar t|$$

where L_0 is a positive constant and

$$L_0 + \Omega\left[r_0(t)\sum_{i=1}^m P_i + L\sum_{l=1}^n Q_l\right] \leqq L \quad ;$$

$$\omega(t)\left[1+\sum_{i=1}^m |\Delta_i(t)| r_0(t) + n r_0(t)\right] \leqq r_0(t); \quad \Omega(m\bar\Delta_K + n) < 1$$

for every compact $K \subset R^1$ where $\bar\Delta_K = \sup\{|\Delta(t)| : t \in K\}$.

Conditions (C3) and (C4) are the same as in [1], assuming that the initial functions have Lipschitz constants.

Theorem 4. *Let the assumptions (C1) – (C4) be fulfilled. If the sequence of functions $\{F_K\}_{K=1}^\infty$ tends pointwise to F_0, then the sequence of solutions of (2_k) tends to the solution of (2_0).*

Proof. We form by the right hand side of (2_k) the sequence of operators $\{T_K\}_{K=1}^\infty$. It is easy to see that T_K maps the set $C_L^0 = \{f \in C_L(R^1) : |f(t)| \leqq r_0(t),\ t \geqq 0\}$ into itself. We shall verify only that $(T_K f)(t)$ has a Lipschitz constant equals to L, because another details of the proof are as in [1]. For $t, \bar t > 0$ we have

$$\begin{aligned}|(T_K f)(t - (T_K f))(\bar t)| &\leqq L_0|t-\bar t| + \Omega\left[r_0(t)\sum_{i=1}^m |\Delta_i(t) - \Delta_i(\bar t)| + \right.\\ &\left. + \sum_{l=1}^n L|\tau_l(t) - \tau_l(\bar t)|\right] \leqq L_0|t-\bar t| + \\ &+ \Omega\left[r_0(t)\sum_{i=1}^m P_i + L\sum_{l=1}^n Q_l\right]|t-\bar t| \leqq L|t-\bar t|.\end{aligned}$$

Now we can apple Theorem 2 in order to conclude that the solution of (2_k) tends to the solution of (2_0) . This is possible because T_K is an equicontinuous family of operators and then pointwise convergence on compact sets implies a uniform convergence. □

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