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EDGE SHIFT DISTANCE BETWEEN TREES

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Dedicated to Professor F. Šik on the occasion of his seventieth birthday

Abstract. Edge shift distance between isomorphism classes of graphs, introduced by M. Johnson, is investigated in the case of trees and compared with other distances.

Various distances between isomorphism classes of graphs were studied by various authors. The author of this paper has introduced the distance based on common subgraphs [4] and on common subtrees [5]. The edge distance was introduced by V. Baláž, J. Koča, V. Kvasnička and M. Sekanina [1] and has found applications in the organic chemistry. The edge rotation distance was defined by G. Chartrand, F. Saba and H.-B. Zou [2]. The edge shift distance was introduced by M. Johnson [3] and is closely related to the edge rotation distance.

We consider finite undirected graphs without loops and multiple edges.

Let G be a finite undirected graph. Let u, v, w be tree pairwise distinct vertices of G such that u is adjacent to v and is not adjacent to w. To perform the rotation of the edge uv into the position uw means to delete the edge uv from G and to add the edge uw to G. Let $\Gamma(n, m)$ denote the class of all undirected graphs with n vertices and m edges. In [2] it is proved that if G_1, G_2 are two graphs from $\Gamma(n, m)$, then G_1 can be transformed into a graph isomorphic to G_2 by a finite number of edge rotations. The minimum number of edge rotations necessary for doing this called the edge rotation distance between the graphs G_1, G_2 and denoted by $d_{er}(G_1, G_2)$. We speak about the distance between graphs, but more precisely we should have to speak about the distance between isomorphism classes of graphs. This is a metric on the set of all isomorphism classes of graphs from $\Gamma(n, m)$.

A special kind of the edge rotation is the edge shift. The shift of the edge uv into the position uw is the rotation of uv into the position uw in the case when the vertices v, w are adjacent in G. Let $\Gamma_c(n, m)$ denote the class of all connected undirected graphs with n vertices and m edges. In [3] it is proved that

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if G_1, G_2 are two graphs from $\Gamma_c(n, m)$, then G_1 can be transformed into a graph isomorphic to G_2 by a finite number of edge shifts. The minimum number of edge shifts necessary for doing this is called the edge shift distance between the graphs G_1, G_2 and denoted by $d_{es}(G_1, G_2)$. This is a metric on the set of all isomorphism classes of graphs from $\Gamma_c(n, m)$.

Note that we must consider $\Gamma_c(n, m)$ instead of $\Gamma(n, m)$, because it is not possible to transform a connected graph into a disconnected one by an edge shift, while by an edge rotation it is possible.

We shall study the edge shift distance on the class of the trees with a given number n of vertices; this is the class $\Gamma_c(n, n-1)$.

The first theorem holds for the graphs in general.

Theorem 1. *Let G_1, G_2 be two graphs from $\Gamma_c(n, m)$, let Δ denote the maximum degree of a vertex in a graph. Then*

$$d_{es}(G_1, G_2) \geqq |\Delta(G_1) - \Delta(G_2)| .$$

Proof. If in the graph G we perform the edge shift of uv into the position uw, then the degree of v decreases by one and the degree of w increases by one, while the degrees of all other vertices remain unchanged. thus the maximum degree of a vertex of G can increase at most by one. Without loss of generality let $\Delta(G_1) \leqq \leqq \Delta(G_2)$. Then for transforming G_1 into a graph isomorphic to G_2 it is necessary to perform at least $\Delta(G_2) - \Delta(G_1)$ edge shifts, which implies the assertion. $\square$

Theorem 2. *Let T_1, T_2 be two trees from $\Gamma_c(n, n-1)$, let d denote the diameter of a graph. Then*

$$d_{es}(T_1, T_2) \geq |d(T_1) - d(T_2)| .$$

Proof. Without loss of generality let $d(T_1) \leqq d(T_2)$. Let u, v, w be tree pairwise distinct vertices of T_1 such that v is adjacent to u and w, while u and w are not adjacent. By deleting the edges uv and vw from T_1 we obtain a graph having three connected components $T_1(u)$, $T_1(v)$, $T_1(w)$ such that u is in $T_1(u)$, v is in $T_1(v)$ and w is in $T_1(w)$. If we perform the shift of the edge uv into the position uw, then the distance between any vertex of $T_1(u)$ and any vertex of $T_1(v)$ increases by one and the distance between any vertex of $T_1(u)$ and any vertex of $T_1(w)$ decreases by one, while the distances between all other pairs of vertices remain unchanged. Hence the diameter increases by at most one. For transforming T_1 into a tree isomorphic to T_2 it is necessary to perform at least $d(T_2) - d(T_1)$ edge shifts, which implies the assertion. $\square$

Theorem 3. *Let S be a star with n vertices, let $T \in \Gamma_c(n, n-1)$. Then*

$$d_{es}(S, T) = n - 1 - \Delta(T).$$

Proof. If T is a star, then $T \cong S$ and $d_{es}(S, T) = n - 1 - \Delta(T) = 0$. Suppose that T is not a star. As $\Delta(S) = n - 1 \geqq \Delta(T)$. Theorem 1 implies $d_{es}(S, T) \geqq$

$\geqq n-1-\Delta(T)$. Let w be a vertex of T of degree $\Delta(T)$. As T is not a star, there exists a vertex v of T which is adjacent to w and to another vertex u. As T is a tree, the vertices u and w are not adjacent. Thus it is possible to perform the shift of the edge uv into the position uw. In the tree thus obtained the degree of w is $\Delta(T)+1$. We shall proceed $n-1-\Delta(T)$ times in this way; then we obtain a tree from $\Gamma_c(n, n-1)$ in which the degree of w is $n-1$ and this tree is isomorphic to S. This implies the assertion. □

Theorem 4. *Let P be a snake with n vertices, let $T \in \Gamma_c(n, n-1)$. Then*

$$d_{es}(P, T) = n-1-d(T) .$$

Remark. A snake is a tree consisting of one path.

Proof. As $d(P) = n-1 \geqq d(T)$, Theorem 2 implies $d_{es}(P, T) \geqq n-1-d(T)$. If T is a snake, then $T \cong P$ and $d_{es}(P, T) = 0 = n-1-d(T)$. If T is not a snake, then let D be a diametral path of T. As T is not a snake, there exists a vertex v of D adjacent to a vertex w not belonging to D. Let u be a vertex of D adjacent to v; as T is a tree, the vertices u and w are not adjacent. Thus is possible to perform the shift of the edge uv into the position uw. In the tree thus obtained there exists a diametral path D' of length $d(T)+1$ obtained from D by substituting the edge uv by a path of length 2 with the inner vertex w. We shall proceed $n-1-d(T)$ times in this way; then we obtain a tree from $\Gamma_c(n, n-1)$ having the diameter $n-1$ and this tree is isomorphic to P. This implies the assertion. □

Corollary 1. *Let S be a star with n vertices, let P be a snake with n vertices. Then*

$$d_{es}(P, S) = n-3 .$$

As every edge shift is an edge rotation, but not conversely, it is easy to see that $d_{es}(G_1, G_2) \geqq d_{es}(G_1, G_2)$ for any two graphs G_1, G_2 from $\Gamma_c(n, m)$. The next theorem will show that the difference between these two distances can be arbitrarily large.

Theorem 5. *Let q be a positive integer. Then there exists a positive integer n and two trees T_1, T_2 from $\Gamma_c(n, n-1)$ such that*

$$d_{es}(T_1, T_2) - d_{er}(T_1, T_2) = q .$$

Proof. Let $n = 3q+4$. Let T_1 be a snake with n vertices, let T_2 be a tree obtained from three snakes with $q+2$ vertices each by choosing one terminal vertex in each of them and identifying these three vertices. Let the center of T_2 be v, let u be a vertex adjacent to v in T_2 and let w be a terminal vertex of T_2 such that u does not lie between v and w. As T_2 is a tree, the vertices u and w are not adjacent. By the rotation of uv into the position uw a snake, i. e. a tree isomorphic to T_1, is obtained and hence $d_{er}(T_1, T_2) = 1$. On the other hand, $d(T_1) = n-1 = 3q+3$,

$d(T_2) = 2q + 2$ and thus $d_{es}(T_1, T_2) \geqq |d(T_1) - d(T_2)| = q + 1$. By q edge shifts along the q edges between v and w the edge uv is transferred into the position uw and a snake is obtained. Hence $d_{es}(T_1, T_2) = q + 1$, which implies the assertion. *square*

Now we return to Theorem 3 and Theorem 4 and look for the bounds for the edge shift distance.

Theorem 6. *Let T_1, T_2 be two trees from $\Gamma_c(n, n-1)$. Then*

$$d_{es}(T_1, T_2) \leqq 2n - 2 - \Delta(T_1) - \Delta(T_2)\ .$$

Proof. This follows from the triangle inequality for T_1, T_2 and the star S with n vertices and from Theorem 3. □

Theorem 7. *Let T_1, T_2 be two trees from $\Gamma_c(n, n-1))$. Then*

$$d_{es}(T_1, T_2) \leqq 2n - 2d(T_1) - d(T_2)\ .$$

Proof. This follows from the triangle inequality for T_1, T_2 and the snake P with n vertices and from Theorem 4. □

Corollary 2. *Let T_1, T_2 be two trees from $\Gamma_c(n, n-1)$ for $n \geqq 4$. Then*

$$d_{es}(T_1, T_2) \leqq 2n - 7\ .$$

Proof. We have $\Delta(T_1) \geqq 2$, $\Delta(T_2) \geqq 2$. If $\Delta(T_1) = \Delta(T_2) = 2$, the both T_1, T_2 are snakes, hence $T_1 \cong T_2$ and $d_{es}(T_1, T_2) = 0$. If $\Delta(T_1) \geqq 3$ or $\Delta(T_2) \geqq 3$, then the inequality follows from Theorem 5. □

Probably this upper bound is not the best possible.

Conjecture. *There exists a constant k such that*

$$d_{es}(T_1, T_2) \leqq n + k$$

for every two trees T_1, T_2 from $\Gamma_c(n, n-1)$ at arbitrary n.

It would be also interesting to compare d_{es} with the distance d_T defined in [5] in such a way that $d_T(T_1, T_2)$ for T_1, T_2 from $\Gamma_c(n, n-1)$ is equal to n minus the maximum number of vertices of a tree which is isomorphic simultaneously to a subtree of T_1 and to a subtree of T_2. In [6] it was proved that the edge rotation distance of two trees from $\Gamma_c(n, n-1)$ is always less than or equal to the distance d_T. Here we shall show that for the edge shift distance an analogous inequality does not hold.

Theorem 8. *For each positive integer q there exists a positive integer n and two trees T_1, T_2 from $\Gamma_c(n, n-1)$ such that*

$$d_T(T_1, T_2) - d_{es}(T_1, T_2) = q .$$

Proof. Let p be an even integer, $p > 6q$. Let $n = p + q + 1$. Let P be a snake with $p+1$ vertices, let c be the center of P, let z be a vertex of P adjacent to a terminal vertex of P. Let S be a star with $q+1$ vertices. Let T_1 (or T_2) be the tree obtained by identifying the center of S with c (or z respectively). Both T_1, T_2 are trees with n vertices. The snake B is a subtree of both T_1 and T_2 with $p+1$ vertices and evidently no tree with more than $p+1$ vertices is isomorphic to subtrees of both T_1 and T_2. Hence $d_T(T_1, T_2) = n - (p+1) = q$. The diameters of both T_1 and T_2 are equal to p and thus by Theorem 7 we have $d_{es}(T_1, T_2) \leqq 2q$. Each of the graphs T_1, T_2 contains exactly one vertex of degree $q+2$. In T_2 such a vertex is a terminal vertex of a path of length $p-1$, while in T_1 the longest paths outgoing from it have the length $p/2$. As $p > 6q$ and one edge shift can change the length of a path or the degree of a vertex at most by one, it is easy to see that $d_{es}(T_1, T_2) = 2q$ which implies the assertion. □

Theorem 9. *For each positive integer q there exists a positive integer n and two trees T_1, T_2 from $\Gamma_c(n, n-1)$ such that*

$$d_{es}(T_1, T_2) - d_T(T_1, T_2) = q .$$

Proof. Let p be an odd integer, $p > 2q$. Let $n = p + q + 1$. Let P be a snake with p vertices, let c be the center of P, let z be a vertex of P adjacent to c. Let S be a star with $q+2$ vertices. Let T_1 (or T_2) be the tree obtained by identifying one terminal vertex of S with c (or z respectively). Similarly as in the proof of the Theorem 8 we have $d_T(T_1, T_2) = q + 1$. On the other hand, if c_0 is the center of S, then by the shift of the edge c_0c into the position c_0z the tree T_1 is transformed into T_2 and hence $d_{es}(T_1, T_2) = 1$, which implies the assertion. □

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