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METRICALLY REGULAR SQUARE OF METRICALLY REGULAR BIGRAPHS II

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. Metrically regular bigraphs the square of which are metrically regular graphs are investigated in the case of graphs with 6 distinct eigenvalues (these eigenvalues can have variuos multiplicities).

1. Terminology and notation

We use the same terminology as in [2]. Our graphs are finite and undirected and have no loops or multiple edges. The second power or equivalently the square of a graph G is the graph G^2 with the same vertex set as G and different vertices are adjacent if and only if there is at least one path of length 2 or 1 in G between them.

The present paper deals with the metrically regular graphs (MRG) with 6 distinct eigenvalues and having metrically regular squares and it extends the results obtained in [2].

2. Metrically regular bipartite graphs with 6 distinct eigenvalues

Let $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$ are the eigenvalues of MRG G with respective multiplicities $m_1, m_2, m_3, m_4, m_5, m_6$. As G is a bipartite graph we obtain from [2] Theorem 1.10

$$\begin{aligned} \lambda_1 &= -\lambda_6, \quad m_1 = m_6 = 1 \\ \lambda_2 &= -\lambda_5, \quad m_2 = m_5 \\ \lambda_3 &= -\lambda_4, \quad m_3 = m_4 \end{aligned} \tag{1}$$

and it holds for the structural constants of G

$$p_{ij}^k = 0 \quad \text{for} \quad i,j,k \in \{0,1,2,3,4,5\}, \tag{2}$$
$$i+j+k \equiv 0 \,(\mathrm{mod}\ 2) \text{ and also for } \quad i+j<k \quad \text{and } |i-j|>k \quad .$$

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According to [2] Theorem 1.3 $\lambda_i (i = 1, 2, 3, 4, 5, 6)$ is the solution of the equation $|\lambda I - P_1| = 0$ and we get

$$\begin{aligned}
\lambda^6 - \lambda^4[\lambda_1 + p^1_{12}p^2_{11} + p^2_{13}p^3_{12} + p^3_{14}p^4_{13} + p^4_{15}p^5_{14}] \\
+\lambda^2[p^1_{12}p^2_{11}p^3_{14}p^4_{13} + p^1_{12}p^2_{11}p^4_{15}p^5_{14} + p^2_{13}p^3_{12}p^4_{15}p^5_{14}+ \\
+ \lambda_1(p^2_{13}p^3_{12} + p^3_{14}p^4_{13} + p^4_{15}p^5_{14})]- \\
- \lambda_1 p^2_{13}p^3_{12}p^4_{15}p^5_{14} = 0
\end{aligned} \tag{3}$$

The condition for G to have the square G^2 metrically regular gives the following relations for the structural constants ${}^2p^k_{ij}$ of G^2

$$\begin{aligned}
&(4) & {}^2p^1_{11} &= 2p^1_{12} &&= p^2_{11} + p^2_{22} \\
&(5) & {}^2p^1_{12} &= p^1_{23} &&= p^2_{13} + p^2_{24} \\
&(6) & {}^2p^1_{22} &= 2p^1_{34} &&= p^2_{33} + p^2_{44} \\
&(7) & {}^2p^1_{23} &= p^1_{45} &&= p^2_{35} \\
&(8) & {}^2p^1_{33} &= p^1_{55} &&= p^2_{55} = 0 \\
&(9) & {}^2p^2_{11} &= 2p^3_{12} &&= p^4_{22} \\
&(10) & {}^2p^2_{12} &= p^3_{14} + p^3_{23} &&= p^4_{13} + p^4_{24} \\
&(11) & {}^2p^2_{13} &= p^3_{25} &&= p^4_{15} \\
&(12) & {}^2p^2_{22} &= 2p^3_{34} &&= p^4_{33} + p^4_{44} \\
&(13) & {}^2p^2_{23} &= p^3_{45} &&= p^4_{35} \\
&(14) & {}^2p^2_{33} &= p^3_{55} &&= p^4_{55} = 0 \\
&(15) & {}^2p^3_{12} &= p^5_{23} + p^5_{14} && \\
&(16) & {}^2p^3_{13} &= p^5_{25} && \\
&(17) & {}^2p^3_{22} &= 2p^5_{34} && \\
&(18) & {}^2p^3_{23} &= p^5_{45} && \\
&(19) & {}^2p^3_{33} &= p^5_{55} &&= 0
\end{aligned}$$

If A denotes the adjacency matrix of G and A_2 is the adjacency matrix of G^2 it is easy to see

$$A_2 = \frac{1}{p^2_{11}}A^2 + \frac{p^2_{11} - p^1_{11}}{p^2_{11}}A - \frac{\lambda_1}{p^2_{11}}I \quad .$$

The eigenvalues of G^2 are in regard of (2) in the form

$$\mu_i = \frac{\lambda_i^2 + p^2_{11}\lambda_i - \lambda_1}{p^2_{11}} \quad , \qquad i \in \{1, 2, \ldots, 6\} \quad . \tag{20}$$

As G^2 is a metrically regular graph with diameter 3 it must have just 4 distinct numbers as its eigenvalues. So it must hold $\mu_i = \mu_j = \mu_k$ or $\mu_i = \mu_j$ and $\mu_k = \mu_l$ (for distinct numbers i, j, k, l; $i, j, k, l \neq 1$ because G^2 is connected and therefore its index μ_1 has the multiplicity 1).

A. $\mu_i = \mu_j = \mu_k$
According to (20) we obtain

$$\lambda_i + \lambda_j = \lambda_i + \lambda_k = \lambda_j + \lambda_k = -p^2_{11}$$

and we get the contradiction with $\lambda_i \neq \lambda_j \neq \lambda_k \neq \lambda_i$.

B. $\mu_i = \mu_j, \mu_k = \mu_l$.
As $\lambda_2 > 0$, $p^2_{11} > 0$ and $\lambda_2 \geq |\lambda_t|$, $\quad t \in \{3, 4, 5\}$, and $\mu_3 \neq \mu_4$, there are only the following cases:

a) $\mu_2 = \mu_6$, $\mu_3 = \mu_5$.
From (20) we obtain $\lambda_2 + \lambda_6 = -p^2_{11} = \lambda_3 + \lambda_5$
and from (1) we get $\lambda_2 = \lambda_1 - p^2_{11}$,
As

$$\lambda_3 > 0 \text{ it holds } \quad \lambda_1 > 2p^2_{11} \quad . \tag{21}$$

b) $\mu_2 = \mu_6, \mu_4 = \mu_5$.
By the same way as in a) it follows from (1) and (20)

$$\lambda_2 = \lambda_1 - p^2_{11}, \lambda_3 = p^2_{11} - \lambda_2 = 2p^2_{11} - \lambda_1 .$$

As $\lambda_2 > \lambda_3 > 0$ it follows

$$2p^2_{11} > \lambda_1 > \frac{3}{2} p^2_{11} \tag{22}$$

c) $\mu_3 = \mu_5, \mu_4 = \mu_6$.
From (1) and (20) we obtain $\quad \lambda_3 = p^2_{11} - \lambda_1$.
As $\lambda_1 > p^2_{11} (v_1 = \lambda_1 = \sum_j p^k_{1j}$, see [1]) we get the contradiction with $\lambda_3 > 0$.

d) $\mu_3 = \mu_6, \mu_4 = \mu_5$.
In this case it follows from (1) and (20)
$\lambda_2 = 2p^2_{11} - \lambda_1$, $\lambda_3 = \lambda_1 - p^2_{11}$. As $\lambda_2 > \lambda_3 > 0$ we get

$$\frac{3}{2} p^2_{11} > \lambda_1 > p^2_{11} \tag{23}$$

In the next part it will be shown that the conditions b) and d) cannot occur. We use the well-known relations for the structure constants of association schemes (see [1])

$$v_i = p^0_{ii} = \sum_j p^k_{ij} \quad , \tag{24}$$

$$v_i p^i_{kj} = v_k p^k_{ij} \quad . \tag{25}$$

(Let x is an arbitrary vertex, v_i denotes the number of vertices y for which $d(x,y) = i$ where $d(x,y)$ is the distance from the vertex x to the vertex y.)

From (24) ($i=1;\ k=1,2,5$) it follows

$$p^1_{12} = \lambda_1 - 1, \quad p^2_{13} = \lambda_1 - p^2_{11}, \quad p^5_{14} = \lambda_1 \tag{26}$$

With respect to (8), (14) and (25) we get

$$v_5 p^5_{25} = v_2 p^2_{55} = 0; \quad p^5_{25} = 0 \tag{27}$$

$$v_5 p^5_{45} = v_4 p^4_{55} = 0; \quad p^5_{45} = 0 \tag{28}$$

So, the relations (8), (24), (27), (28) give

$$v_5 = \sum_{k=0}^{5} p^5_{k5} = 1 \tag{29}$$

$$v_5 = \sum_{k=0}^{5} p^1_{k5} = p^1_{45} = 1 \tag{30}$$

$$\textit{and} \qquad v_5 = \sum_{k=0}^{5} p^2_{k5} = p^2_{35} = 1 \tag{31}$$

From (25) we get $v_5 p^5_{23} = v_3 p^3_{25}$ and because $p^5_{23} \neq 0$ we obtain from (24) ($i = 5,\ k=3$) - note that p^3_{25}, p^3_{45} are non-negative integers and $p^3_{25} > 0$ -

$$p^3_{25} = 1, \quad p^3_{45} = 0. \tag{32}$$

As $p^5_{14} \neq 0$ we get from (25) $v_5 p^5_{14} = v_4 p^4_{15}$ and $p^4_{15} \neq 0$. So, from (24) ($i=5,\ k=4$) we obtain

$$p^4_{15} = 1, \quad p^4_{35} = 0. \tag{33}$$

The relations (25), (26), (33) give

$$v_5 p^5_{14} = v_4 p^4_{15} \quad \text{and} \quad v_4 = \lambda_1. \tag{34}$$

With regard of (24)($i=4,\ k=1$) it follows

$$p^1_{34} = \lambda_1 - 1 \tag{35}$$

and from (25) we get

$$v_1 p^1_{34} = v_4 p^4_{13}, \quad \text{so} \quad p^4_{13} = \lambda_1 - 1 \tag{36}$$

With respect to (24) ($i=2,4;\ k=5$), (26), (28) it follows

$$p^5_{23} = v_2, \ p^5_{34} = 0 \tag{37}$$

From (25) we get $v_3 p^3_{25} = v_5 p^5_{23}$, so

$$v_2 = v_3 \tag{38}$$

and the relation (25) gives

$$p^2_{13} = p^3_{12}, \tag{39}$$
$$p^2_{33} = p^3_{23}. \tag{40}$$

With regard of (1) we obtain from (3)

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 = \lambda_1 p^2_{13} p^3_{12} p^4_{15} p^5_{14} \tag{41}$$

In the case d) we obtain from (26), (33), (39), (41)

$$\lambda_1^2 (2p^2_{11} - \lambda_1)^2 (\lambda_1 - p^2_{11})^2 = \lambda_1^2 p^2_{13} p^2_{13}$$
$$\text{so} \qquad |2p^2_{11} - \lambda_1| = 1. \tag{42}$$

As $p^2_{11} \in N$ we get the contradiction with (23).

By the same way we obtain the relation (42) in the cases a) and b).

In the case b) we obtain with regard of (22)

$$\lambda_1 = 2p^2_{11} - 1, \quad p^2_{11} > 2 \tag{43}$$

and from (25), (26), (43) it follows

$$v_1 p^1_{12} = v_2 p^2_{11} \quad \text{and} \quad v_2 = \frac{2\lambda_1(\lambda_1 - 1)}{\lambda_1 + 1} = 2\lambda_1 - 4 + \frac{4}{\lambda_1 + 1}$$

As $v_2 \in N$ we obtain $\lambda_1 \leq 3$, so $p^2_{11} \leq 2$ and we get the contadiction with (43).

The relations (21) and (42) imply for the case a)

$$\lambda_1 = 2p^2_{11} + 1 \quad , \tag{44}$$
$$p^2_{11} = \frac{\lambda_1 - 1}{2} \quad . \tag{45}$$

From (25) with respect to (26), (38) we get

$$v_1 p^1_{12} = v_2 p^2_{11} \quad \text{and} \quad v_2 = v_3 = 2\lambda_1 \tag{46}$$

The relations (24) ($i = 1,\ k = 2$) and (45) imply

$$p^2_{13} = \frac{\lambda_1 + 1}{2} \tag{47}$$

and from (9), (26), (39), (47) we get

$$p^4_{22} = \lambda_1 + 1 \tag{48}$$

With respect to (24) ($i = 2,\ k = 1$), (2), (46) it follows

$$p^1_{23} = \lambda_1 + 1 \tag{49}$$

From (5), (47), (49) we obtain

$$p^2_{24} = \frac{\lambda_1 + 1}{2} \tag{50}$$

and the relations (4), (26) and (45) imply

$$p^2_{22} = \frac{3(\lambda_1 - 1)}{2} \quad . \tag{51}$$

The relation (24) ($i = 4,\ k = 2$) implies with respect to (34), (50)

$$p^2_{44} = \frac{\lambda_1 - 1}{2} \tag{52}$$

From (6), (35), (38) and (52) we get

$$p^2_{33} = p^3_{23} = \frac{3}{2}(\lambda_1 - 1) \tag{53}$$

The relations (25), (34), (36) and (46) imply

$$v_3 p^3_{14} = v_4 p^4_{13} \quad \text{and} \quad p^3_{14} = \frac{\lambda_1 - 1}{2} \tag{54}$$

and from (2), (24) ($i = 4,\ k = 3$), (32), (54) we obtain

$$p^3_{34} = \frac{\lambda_1 + 1}{2} \tag{55}$$

From the relations (25), (34), (46), (52) and (55) we get

$$v_4 p^4_{33} = v_3 p^3_{34} \quad \text{and} \quad p^4_{33} = \lambda_1 + 1 \tag{56}$$

$$\text{and} \qquad v_4 p^4_{24} = v_2 p^2_{44} \quad \text{so} \quad p^4_{24} = \lambda_1 - 1. \tag{57}$$

(12), (55) and (57) imply

$$p^4_{44} = 0 \tag{58}$$

Because of $p_{ij}^k \in N$ and $\lambda_1 > 1$ the relations (45), (47), (50), (51), (52), (53), (54) and (55) imply

$$\lambda_1 = 2k+1, \quad k \in N \tag{59}$$

and the nonzero structural constants of the considered metrically regular bigraphs are

$$\begin{aligned}
&p_{i0}^i = p_{45}^1 = p_{35}^2 = p_{25}^3 = p_{15}^4 = 1,\\
&p_{11}^2 = p_{44}^2 = p_{14}^3 = k,\\
&p_{13}^2 = p_{24}^2 = p_{12}^3 = p_{34}^3 = k+1,\\
&p_{12}^1 = p_{34}^1 = p_{13}^4 = p_{24}^4 = 2k,\\
&p_{14}^5 = 2k+1,\\
&p_{23}^1 = p_{22}^4 = p_{33}^4 = 2k+2,\\
&p_{22}^2 = p_{33}^2 = p_{23}^3 = 3k,\\
&p_{23}^5 = 2(2k+1),\\
&v_0 = v_5 = 1, \qquad \lambda_1 = 2k+1 = -\lambda_6,\\
&v_1 = v_4 = 2k+1, \qquad \lambda_2 = k+1 = -\lambda_5,\\
&v_2 = v_3 = 2(2k+1), \quad \lambda_3 = 1 = -\lambda_4.
\end{aligned} \tag{60}$$

For the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_6$ and their corresponding multiplicities $m_1, m_2, \ldots, m_6$ of the considered graphs it holds
$m_1 = 1$, since the graph is connected,
$\sum_{i=1}^{6} m_i = \sum_{j=0}^{5} v_j = n$, the number of vertices,
$\sum_{i=1}^{6} m_i\lambda_i = 0$, since the graph has no loops,
$\sum_{i=1}^{6} m_i\lambda_i^2 = n\lambda_1$, since the graph is regular.
So, with respect to (1), (2) and (60) we obtain

$$\begin{aligned}
&m_1 = 1\\
&2m_1 + 2m_2 + 2m_3 = 4(3k+2),\\
&2m_1(2k+1)^2 + 2m_2(k+1)^2 + 2m_3 = 4(3k+2)(2k+1)\ .
\end{aligned}$$

These equations imply

$$m_2 = 8 - \frac{12}{k+2}\ ,$$
$$m_3 = 6k - 5 + \frac{12}{k+2}\ .$$

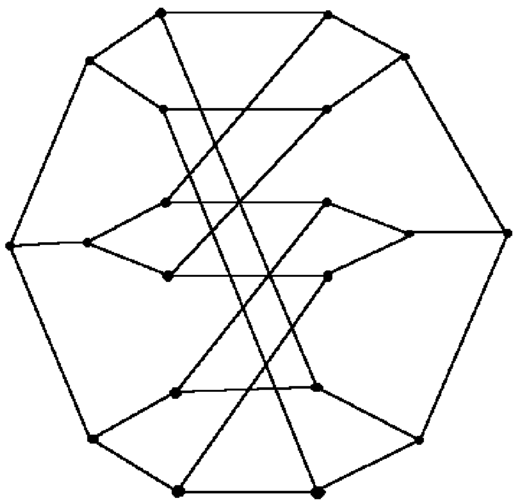
As $m_2, m_3 \in N$ it must hold

$$k \in \{1, 2, 4, 10\}.$$

So we have proved the following theorem:

Theorem. *There are only four tables of the parameters of association schemes of the type (60) for $k \in \{1, 2, 4, 10\}$ so that the corresponding metrically regular bipartite graphs with 6 distinct eigenvalues have the metrically regular square.*

The realization of the table (60) for $k = 1$ is shown in the figure below. In the case $k = 2$ it is the 5-dimensional unit cube.



References

[1] Bauer L., *Association Schemes I*, Arch. Math. (Brno) **17** (1981), 173-184.

[2] Vetchý V., *Metrically regular square of metrically regular bigraphs I*, Arch. Math. (Brno), Tomus 27b (1992), 183-197.

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