ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 11 – 16

BOUNDS OF LENGTHS OF OPEN HAMILTONIAN WALKS

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Dedicated to Professor F. Šik on the occasion of his seventieth birthday

Abstract. If G is a graph, an open Hamiltonian walk is any open sequence of edges of minimal length which includes every vertex of G. In this paper bounds of lengths of open Hamiltonian walks are studied.

Since Hamilton defined the useful concept of Hamiltonian cycle or path in graphs, a lot of related notions have been studied; in particular, Hamiltonian connectedness, pancyclic graphs, hypohamiltonian graphs and many others. Almost of them are stronger than the existence of a Hamiltonian cycle or a Hamiltonian path while a few are weaker; namely Hamiltonian walk defined in [1], [2], [3]. Here we propose some results on open Hamiltonian walks.

In this paper the graph means a finite, connected, undirected graph without loops and multiple adges. If G is a graph, $V(G)$ and $E(G)$ denote the sets of vertices and edges of G respectively. An open sequence of edges passing through each vertex of a graph G is called an open walk in the graph G. Any open walk of minimal length is called an open Hamiltonian walk (see [4]). Throughout the paper we shall denote l_G the length of open Hamiltonian walk in the graph G.

Let G be a graph on n vertices, $n \geq 3$, then $n-1 \leq l_G \leq 2n-4$ [4]. Let S be a spanning tree of the graph G and l_S the length of open Hamiltonian walk in the graph S. Then obviously $l_G \leq l_S$ and $l_S = 2(n-1) - \operatorname{diam} S$, where $\operatorname{diam} S$ is the diameter of S (see [4]).

Consider the set of all spanning trees of the graph G. Let $S_{\max}$ be a spanning tree, which has the maximal diameter and denote diam $S_{\max}$ the diameter of it. Then $l_G \leq 2(n-1)-$ diam $S_{\max}$.

Let G be a graph on n vertices and H a path in G whose length $h = \operatorname{diam} G$. The path H can be completed to a spanning tree of the graph G. Then obviously $l_G \leq 2(n-1) - \operatorname{diam} G$.

1991 *Mathematics Subject Classification*: 05C45.
Key words and phrases: graph, tree, Hamiltonian path, open Hamiltonian walk, diameter of graph.
Received March 15, 1989.

Let k be a minimal number of edges which we have to add to the graph G to obtain a graph containing a Hamiltonian path. Obviously, $0 \le k \le n-3$ and $k=0$ iff G has a Hamiltonian path and $k = n-3$ iff G is a star graph $K_{1,n-1}$.

Theorem 1. *Let S be a tree on n vertices, $n \ge 4$, and k be a minimal number of edges which we have to add to the graph S to obtain a graph G containing a Hamiltonian path ($k \le n-3$). Then*

$$l_S \le 2(n-1) - \frac{n+k-1}{k+1}$$

Proof. If $k=0$, the Theorem 1 clearly holds.

Let $k \ge 1$. Let H be a Hamiltonian path in G which we obtained from S after adding k edges. H has to contain each of these k edges. If we omit these k edges, the rest of H consists of $(k+1)$ disjoint paths coverning $V(G)$. We denote $\alpha_0, \alpha_1, \dots, \alpha_k$ the lengths of these paths, where $0 \le \alpha_0 \le \alpha_1 \le \dots \le \alpha_k$. Obviously $\sum_{i=0}^{k} \alpha_i = n-1-k$. Let $\beta_{s,t}$ denote the distance of path of lengths α_s and α_t in S. Obviously, $\beta_{s,t} \ge 1$ and there exists a path of length

$$\max_{s,t} \left(\frac{\alpha_s+\alpha_t}{2} + \beta_{s,t}\right)$$

in S and so

$$\operatorname{diam} S \ge \max_{s,t} \left(\frac{\alpha_s+\alpha_t}{2} + \beta_{s,t}\right).$$

Since $0 \le \alpha_0 \le \alpha_1 \le \dots \le \alpha_k$,

$$\operatorname{diam} S \ge \frac{\alpha_k+\alpha_{k-1}}{2} + \beta_{k,k-1}.$$

Let $k=1$. Then $\alpha_0+\alpha_1 = n-2$ and $\operatorname{diam} S \ \ge \dfrac{n-2}{2}+1 = \dfrac{n}{2}$ and so the Theorem 1 holds again.

Let now $k>1$. We shall study the numbers $\beta_{k,k-1}, \beta_{k,k-2}, \beta_{k-1,k-2}$. At least one of these numbers has to be at least 2, otherwise S is not a tree. Therefore, the lower bound of diam S, the diameter of the tree S, is the number M, where

$$M = \max\left\{\frac{\alpha_k+\alpha_{k-1}}{2} + \beta_{k,k-1}\,;\ \frac{\alpha_k+\alpha_{k-2}}{2} + \beta_{k,k-2}\,;\ \frac{\alpha_{k-1}+\alpha_{k-2}}{2} + \beta_{k-1,k-2}\right\}$$

We distinguish 3 cases :

1. $\alpha_{k-2} = \alpha_k$
2. $\alpha_{k-2} = \alpha_k - 1$
3. $\alpha_{k-2} = \alpha_k - 2$

1. If $\alpha_{k-2} = \alpha_k$, then $\alpha_{k-1} = \alpha_{k-2} = \alpha_k$ and so

$$\alpha_k + \alpha_{k-1} = \alpha_k + \alpha_{k-2} = \alpha_{k-1} + \alpha_{k-2}.$$

Since $\alpha_k + \alpha_{k-1} \ \geq 2\dfrac{n-1-k}{k+1}$,

$$\operatorname{diam} S \geq \frac{n-1-k}{k+1} + 2 = \frac{n+k+1}{k+1}.$$

2. If $\alpha_{k-2} = \alpha_k - 1$ we distinguish 2 cases a) and b).

a) Let $\alpha_{k-2} = \alpha_{k-1} = \alpha_k - 1$. Then

$$n-1-k = \alpha_0 + \alpha_1 + \cdots + \alpha_k \leq k\alpha_{k-2} + (\alpha_{k-2}+1) = (k+1)\alpha_{k-2} + 1$$

and so $\alpha_{k-2} \geq \dfrac{n-k-2}{k+1}$. Therefore $\alpha_{k-1} + \alpha_{k-2} \geq 2\dfrac{n-k-2}{k+1}$ and we get

$$\operatorname{diam} S \geq \frac{n-k-2}{k+1} + 2 = \frac{n+k}{k+1}.$$

b) Let $\alpha_{k-2} = \alpha_{k-1} - 1 = \alpha_k - 1$. Then

$n-1-k \leq (k+1)\alpha_{k-2} + 2$ and $\alpha_{k-2} \geq \dfrac{n-k-3}{k+1}$. Therefore

$\alpha_{k-1} + \alpha_{k-2} \geq 2\dfrac{n-k-3}{k+1} + 1 = \dfrac{2n-k-5}{k+1}$ and so

$$\operatorname{diam} S \geq \frac{2n-k-5}{2(k+1)} + 2 = \frac{n+\frac{3}{2}k-\frac{1}{2}}{k+1}$$

.

3. If $\alpha_{k-2} \leq \alpha_k - 2$, then $\alpha_k - \alpha_{k-2} \geq 2$.

a) If $\alpha_{k-2} > \dfrac{n-k-1}{k+1} - \dfrac{2}{k+1}$, then

$\alpha_k \geq \dfrac{n-k-1}{k+1} - \dfrac{2}{k+1} + 2, \quad \alpha_{k-1} \geq \alpha_{k-2}$ and so

$$\alpha_k + \alpha_{k-1} \geq 2\left(\frac{n-k-1}{k+1} - \frac{2}{k+1}\right) + 2 = 2\frac{n-2}{k+1}.$$

b) If $\alpha_{k-2} \leq \dfrac{n-k-1}{k+1} - \dfrac{2}{k+1}$, then

$\alpha_0 + \alpha_1 + \ldots + \alpha_{k-2} \leq (k-1)\left(\dfrac{n-k-1}{k+1} - \dfrac{2}{k+1}\right)$ and so

$$\alpha_k + \alpha_{k-1} = n-1-k-(\alpha_0 + \alpha_1 + \cdots + \alpha_{k-2}) \geq$$
$$\geq n-1-k-(k-1)\left(\frac{n-k-1}{k+1} - \frac{2}{k+1}\right) = 2\frac{n-2}{k+1}.$$

In both cases we get

$$\alpha_k + \alpha_{k-1} \geq 2\frac{n-2}{k+1}$$

and, therefore

$$\operatorname{diam} S \geq \frac{n-2}{k+1} + 1 = \frac{n+k-1}{k+1}.$$

We find out that, if $k > 1$, the bound $\operatorname{diam} S \geq \dfrac{n+k-1}{k+1}$ holds in all 3 cases 1 – 3. This proves the Theorem 1. $\square$

Remark 1. The bound of the diameter of the tree from the Theorem 1 is the best possible. In the figure 1 there is an example of a tree on 11 vertices. The tree can be completed to a graph with a Hamiltonian path by 2 edges. The diameter of this tree is $4 = \dfrac{n+k-1}{k+1}$.

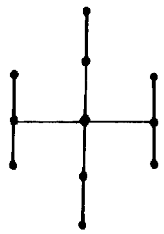

Fig. 1

Now we show that the conclusion of the Theorem 1 remains to hold even if G is not a tree.

Theorem 2. *Let G_1 be a graph on n vertices, $n \geq 4$, and k be the minimal number of edges which we have to add to the graph G_1 to obtain a graph G containing a Hamiltonian path ($k \leq n-3$). Then*

$$l_{G_1} \leq 2(n-1) - \frac{n+k-1}{k+1}.$$

Proof. Let H be a Hamiltonian path in G, which we obtained from G_1 after adding k edges. H has to contain each of these k edges. If we omit these k edges, the rest of H consists of $(k+1)$ disjoint paths covering $V(G)$. This system of paths can be completed by edges of G_1 to a spanning tree S of the graph G_1. Spanning tree S can be completed by the same edges as G_1 to a graph with a Hamiltonian path. According to the Theorem 1, $l_S \leq 2(n-1) - \dfrac{n+k-1}{k+1}$ and because $l_{G_1} \leq l_S$, is the Theorem 2 proved. $\square$

Theorem 3. *Let G be a graph which we obtain from a graph G_1 by omitting a unique edge. Then*

$$l_{G_1} \geq \frac{2l_G + 1}{3}.$$

Proof. Let $\{[x,y]\} = E(G_1) - E(G)$ and L_{G_1} be an open Hamiltonian walk in G_1 of length l_{G_1}. If $[x,y] \notin E(L_{G_1})$ then $l_{G_1} = l_G$ and the Theorem 3 holds.

Let $[x,y] \in E(L_{G_1})$. The edge $[x,y]$ is contained at most twice in L_{G_1}.

Suppose that the edge $[x,y]$ is obtained twice in L_{G_1}. If at the first occurence the edge $[x,y]$ is walked in the direction from x to y, then at the second occurence it is walked in the direction from y to x. Let

$$L_{G_1} = \{x_0, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{i+k}, x_{i+k+1}, \ldots, x_{l_{G_1}}\}$$

and $x_i = x_{i+k+1} = x, x_{i+1} = x_{i+k} = y$. Obviously, for any index s, $i+2 \leq s \leq i+k-1$ there does not exist an index t, $t \leq i-1$ or $t \geq i+k+2$ so that $x_s = x_t$, otherwise there exists an open walk of length less than l_{G_1} in G_1 (not containing the edge $[x,y]$). Since G is a connected graph indexes r, s ($i+2 \leq r \leq i+k-1$ and $s \leq i-1$ or $s \geq i+k+2$) have to exist so that $[x_r, x_s]$ is an edge in G.

Suppose that $s \leq i-1$. Then the sequence of edges $\{x_0, x_1, \ldots, x_s, x_r, x_{r-1}, \ldots, x_{i+1}, x_{i+k-1}, x_{i+k-2}, \ldots, x_r, x_s, x_{s+1}, \ldots, x_i, x_{i+k+2}, x_{i+k+3}, \ldots, x_{l_{G_1}}\}$ is an open walk in G having length l_{G_1} and not containing the edge $[x,y]$.

If $s \geq i+k+2$, then the sequence of edges $\{x_0, x_1, \ldots, x_i, x_{i+k+2}, x_{i+k+3}, \ldots, x_s, x_r, x_{r-1}, \ldots, x_{i+1}, x_{i+k-1}, x_{i+k-2}, \ldots, x_r, x_s, x_{s+1}, \ldots, x_{l_{G_1}}\}$ is an open walk in G having length l_{G_1} and not containing the edge $[x,y]$.

In both cases we have $l_G = l_{G_1}$. It means that the edge $[x,y]$ occurs in L_{G_1} only once.

The open Hamiltonian walk L_{G_1} is divided by the edge $[x,y]$ into two sequences of edges P and Q having lengths p and q. Obviously, $l_{G_1} = p + 1 + q$. Since G is a connected graph, vertices u, v ($u \in P$, $v \in Q$, $u \neq x$, $v \neq y$) have to exist so that $[u,v]$ is an edge in G.

In G, there exists an open walk (see Fig. 2) of length

$$\begin{aligned} l_G &\leq p + \left\lfloor \frac{p}{2} \right\rfloor + 1 + \left\lfloor \frac{q}{2} \right\rfloor + q = l_{G_1} + \left\lfloor \frac{p}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor \leq \\ &\leq l_{G_1} + \frac{p+q}{2} = l_{G_1} + \frac{l_{G_1} - 1}{2} = \frac{3l_{G_1} - 1}{2} \end{aligned}$$

and so $l_{G_1} \geq \dfrac{2l_G + 1}{3}$. □

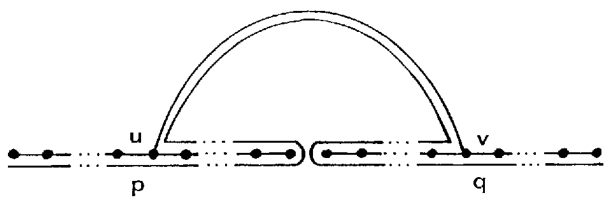

Fig. 2

Remark 2. The bound of the Theorem 3 is the best possible as seen in the figure 3. There is a graph G which can be completed by unique edge to the graph G_1 with a Hamiltonian path. Clearly $l_G = 13$, $l_{G_1} = 9$ and

$$l_{G_1} = \frac{2l_G + 1}{3}.$$

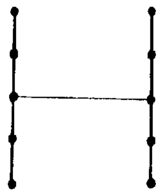

Fig. 3

Corollary 1. *Let G be a graph on n vertices which we obtain from a graph G_1 by omitting a unique edge. Let $l_G \geq \frac{1}{2}(3n-1)$. Then $l_{G_1} > n-1$ i.e. G_1 does not contain a Hamiltonian path.*

Proof. It is sufficient to find when $\dfrac{2l_G+1}{3} \geq n$ considering $l_{G_1} \geq \dfrac{2l_G+1}{3}$. This in equality holds for $l_G \geq \dfrac{1}{2}(3n-1)$. □

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