ARCHIVUM MATHEMATICUM (BRNO)
Tomus 28 (1992), 57 – 65

EXISTENCE OF MULTIPLE SOLUTIONS FOR SOME FUNCTIONAL BOUNDARY VALUE PROBLEMS

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. Let $\mathbf{X}$ be the Banach space of C^0-functions on $\langle 0,1\rangle$ with the sup norm and $\alpha,\beta \in \mathbf{X} \to \mathbf{R}$ be continuous increasing functionals, $\alpha(0)=\beta(0)=0$. This paper deals with the functional differential equation (1) $x'''(t) = Q[x,x',x''(t)](t)$, where $Q : \mathbf{X}^2 \times \mathbf{R} \to \mathbf{X}$ is locally bounded continuous operator. Some theorems about the existence of two different solutions of (1) satisfying the functional boundary conditions $\alpha(x)=0=\beta(x')$, $x''(1)-x''(0)=0$ are given. The method of proof makes use of Schauder linearizatin technique and the Schauder fixed point theorem. The results are modified for 2nd order functional differential equations.

1. Introduction

There are many papers devoted to the existence of multiple solutions for ordinary and partial differential equations. We refer, for recent results on ordinary differential equations, to the papers by Chiappinelli, Mawhin and Nugari [2], Ding and Mawhin [4], Fabry, Mawhin and Nkashama [5], Gaete and Manasevich [6], Kiguradze and Půža [9], Kiguradze [10], Nkashama [11], Mawhin [13], Rachůnková [14], Ruf and Srikanth [15], Schmitt [18], Šenkyřík [20] and Vidossich [21].

On the other hand, several authors have recently obtained results on the existence of nonnegative solutions for differential equations. We refer to Castro and Shivaji [1], Danzer and Schmitt [3], Islamov and Shneiberg [7], Kolesov [8], Santanilla [16], Schaaf and Schmitt [17] and Smoller and Wasserman [19].

In the interesting paper [12], Nkashama and Santanilla consider first and second order nonlinear ordinary differential equations when the nonlinearity is a Carathéodory function and there are established criteria for the existence of nonnegative and nonpositive solutions for problems with periodic, Neumann and Dirichlet boundary conditions.

The proofs of results in these papers are mostly based upon a priori estimates, degree theory and the technique of lower and upper solutions.

1991 *Mathematics Subject Classification*: 34K10, 34B15, 34B10.

Key words and phrases: Schauder linearization technique, Schauder differential equation, functional boundary conditions, boundary value problem.

Received October 2, 1991.

Let $\mathbf{X}$ be the Banach space of C^0-functions on $\langle 0,1\rangle$ with the sup norm $\|\cdot\|$. In this paper we consider the 3rd order functional differential equation

$$x'''(t) = Q[x, x', x''(t)](t) \tag{1}$$

in which $Q : \mathbf{X}^2 \times \mathbf{R} \to \mathbf{X}$ is a locally bounded continuous operator. We see that $x''' = Q[\varphi, \varphi', x''](t)$ is an ordinary differential equation for each $(\varphi, \varphi') \in \mathbf{X}^2$. A special case of (1) is the differential equation

$$x''' = q(t, x, x', x''),$$

where $q : \langle 0,1\rangle \times \mathbf{R}^3 \to \mathbf{R}$ is a continuous function.

Let $\alpha, \beta : \mathbf{X} \to \mathbf{R}$ be continuous increasing (i.e. $u, v \in \mathbf{X}$, $u(t) < v(t)$ for $t \in \langle 0,1\rangle \Rightarrow \alpha(u) < \alpha(v)$, $\beta(u) < \beta(v)$) functionals, $\alpha(0) = 0 = \beta(0)$.

The purpose of this paper is to obtain by the Schauder linearization technique and the Schauder fixed point theorem sufficient conditions for the existence

(i) at least one solution x of (1) with $x''(t) \geq 0$ on $\langle 0,1\rangle$,

(ii) at least one solution x of (1) with $x''(t) \leq 0$ on $\langle 0,1\rangle$,

(iii) at least two different solutions x_1, x_2 of (1) with

$$x_1''(t) \leq 0 \leq x_2''(t) \text{ on } \langle 0,1\rangle,$$

satisfying the functional boundary conditions

$$\alpha(x) = 0, \quad \beta(x') = 0, \quad x''(1) - x''(0) = 0. \tag{2}$$

It will be easily seen from the proofs of theorems for problem (1) - (2) that evident modified results hold for the 2nd order functional differential equation

$$x''(t) = P[x, x'(t)](t), \tag{3}$$

where $P : \mathbf{X} \times \mathbf{R} \to \mathbf{X}$ is a locally bounded continuous operator and solution x of (3) satisfies the functional boundary conditions

$$\alpha(x) = 0, \quad x'(1) - x'(0) = 0. \tag{4}$$

2. Notations, lemmas

Convention. If $a \in \mathbf{R}$, then $Q[a, a, a](t)$ and $P[a, a](t)$ denotes $Q[w, w, a](t)$ and $P[w, a](t)$ with $w(t) \equiv a$ on $\langle 0,1\rangle$, respectively.

Let $c_1, c_2 \in \mathbf{R}$, $c_1 < c_2$ and let $A = \max\{|c_1|, |c_2|\}$, $D = \{(x, x'); x' \in \mathbf{X}, \|x\| \leq A, \|x'\| \leq A\}$, $D_1 = \{x; x \in \mathbf{X}, \|x\| \leq A\}$, $H = \{x; x \in \mathbf{X}, c_1 \leq x(t) \leq c_2 \text{ for } t \in \langle 0,1\rangle\}$.

In this paper we shall assume that some of the following assumptions are fulfilled:

(H$_1$) $Q[\varphi,\varphi',c_1](t)\ge 0$, $Q[\varphi,\varphi',c_2](t)\le 0$ for all $(\varphi,\varphi')\in D$, $t\in\langle 0,1\rangle$;

(H$_2$) Either

$$(Q[\varphi,\varphi',u](t)-Q[\varphi,\varphi',v](t))(u-v)<0 \text{ for all } (\varphi,\varphi')\in D,$$
$$u,v\in\langle c_1,c_2\rangle, u\ne v\ , \quad t\in(0,1) \tag{5}$$

or

$$(Q[\varphi,\varphi',u](t)-Q[\varphi,\varphi',v](t))(u-v)\le h_0(t)(u-v)^2 \text{ for all} \tag{6}$$
$$(\varphi,\varphi')\in D,\quad u,v\in\langle c_1,c_2\rangle,\quad t\in\langle 0,1\rangle,$$
$$\text{where } h_0\in C^0(\langle 0,1\rangle),\ \int_0^1 h_0(t)\,dt<0;$$

(H$_3$) $Q[\varphi,\varphi',c_1](t)\le 0,\quad Q[\varphi,\varphi',c_2](t)\ge 0$ for all $(\varphi,\varphi')\in D,\quad t\in\langle 0,1\rangle$;

(H$_4$) Either

$$(Q[\varphi,\varphi',u](t)-Q[\varphi,\varphi',v](t))(u-v)>0 \text{ for all } (\varphi,\varphi')\in D,$$
$$u,v\in\langle c_1,c_2\rangle,\quad u\ne v\ ,\quad t\in(0,1)$$

or

$$(Q[\varphi,\varphi',u](t)-Q[\varphi,\varphi',v](t))(u-v)\ge h_1(t)(u-v)^2 \text{ for all}$$
$$(\varphi,\varphi')\in D,\ u,v\in\langle c_1,c_2\rangle,\ t\in\langle 0,1\rangle,\ \text{where } h_1\in C^0(\langle 0,1\rangle),\ \int_0^1 h_1(t)\,dt>0;$$

(S$_1$) $P[\varphi,c_1](t)\ge 0$, $P[\varphi,c_2](t)\le 0$ for all $\varphi\in D_1$, $t\in\langle 0,1\rangle$;

(S$_2$) Either

$$(P[\varphi,u](t)-P[\varphi,v](t))(u-v)<0 \text{ for all } \varphi\in D_1,\ u,v\in\langle c_1,c_2\rangle,$$
$$u\ne v,\quad t\in(0,1)$$

or

$$(P[\varphi,u](t)-P[\varphi,v](t))(u-v)\le k_0(t)(u-v)^2 \text{ for all } \varphi\in D_1,$$
$$u,v\in\langle c_1,c_2\rangle,\quad t\in\langle 0,1\rangle,\ \text{where } k_0\in C^0(\langle 0,1\rangle),\int_0^1 k_0(t)\,dt<0;$$

(S$_3$) $P[\varphi,c_1](t)\le 0$, $P[\varphi,c_2](t)\ge 0$ for all $\varphi\in D_1$, $t\in\langle 0,1\rangle$;

(S$_4$) Either

$$(P[\varphi,u](t)-P[\varphi,v](t))(u-v)>0 \text{ for all } \varphi\in D_1,\ u,v\in\langle c_1,c_2\rangle,$$
$$u\ne v,\quad t\in(0,1)$$

or

$$(P[\varphi, u](t) - P[\varphi, v](t))(u - v) \geq k_1(t)(u - v)^2 \text{ for all } \varphi \in D_1,$$
$$u, v \in \langle c_1, c_2 \rangle, \quad t \in \langle 0, 1 \rangle, \text{ where } \ k_1 \in C^0(\langle 0, 1 \rangle), \int_0^1 k_1(t)\, dt > 0.$$

Remark 1. Let $Q[x, y, z] = g(z)Q_1[x, y]$ for $[x, y, z] \in \mathbf{X}^2 \times \mathbf{R}$, where $Q_1 : \mathbf{X}^2 \to \mathbf{X}$ is a locally bounded continuous operator, $g : \mathbf{R} \to \mathbf{R}$ is a continuous function and $Q_1[x, y] \geq 0$ for $[x, y] \in \mathbf{X}^2$. Then assumption (H_1) $((\mathrm{H}_2)$; (H_3); $(\mathrm{H}_4))$ is fulfilled for example if $g(c_1) \geq 0$, $g(c_2) \leq 0$ (g is decreasing on $\langle c_1, c_2 \rangle$ and $Q_1[\varphi, \varphi'](t) > 0$ for $(\varphi, \varphi') \in D$, $t \in (0, 1)$; $g(c_1) \leq 0$, $g(c_2) \geq 0$; g is increasing on $\langle c_1, c_2 \rangle$ and $Q_1[\varphi, \varphi'](t) > 0$ for $(\varphi, \varphi') \in D$, $t \in (0, 1)$).

Remark 2. Let $P[x, y] = g(y)P_1[x]$ for $[x, y] \in \mathbf{X} \times \mathbf{R}$, where $P_1 : \mathbf{X} \to \mathbf{X}$ is locally bounded continuous operator, $g : \mathbf{R} \to \mathbf{R}$ is a continuous function and $P_1[x] \geq 0$ for $x \in \mathbf{X}$. Then assumption (S_1) $((\mathrm{S}_2)$; (S_3); $(\mathrm{S}_4))$ is fulfilled for example if $g(c_1) \geq 0$, $g(c_2) \leq 0$ (g is decreasing on $\langle c_1, c_2 \rangle$ and $P_1[\varphi](t) > 0$ for $\varphi \in D_1$, $t \in (0, 1)$; $g(c_1) \leq 0$, $g(c_2) \geq 0$; g is increasing on $\langle c_1, c_2 \rangle$ and $P_1[\varphi] > 0$ for $\varphi \in D_1$, $t \in (0, 1)$).

Lemma 1. *Let $h \in C^0(\langle 0, 1 \rangle)$. If there exists $\beta_j \in C^1(\langle 0, 1 \rangle)$ $(j = 1, 2)$, $\beta_1(t) \leq \beta_2(t)$ for $t \in \langle 0, 1 \rangle$ and a number $\varepsilon \in \{-1, 1\}$ such that*

$$\varepsilon(\beta_1(0) - \beta_1(1)) \leq 0, \quad \varepsilon(\beta_2(0) - \beta_2(1)) \geq 0,$$
$$\varepsilon(h(t, \beta_1(t)) - \beta_1'(t)) \geq 0, \quad \varepsilon(h(t, \beta_2(t)) - \beta_2'(t)) \leq 0 \text{ for } \ t \in \langle 0, 1 \rangle,$$

then the problem

$$u' = h(t, u), \quad u(0) - u(1) = 0$$

has at least one solution $u(t)$ satisfying

$$\beta_1(t) \leq u(t) \leq \beta_2(t) \quad \text{for} \ \ t \in \langle 0, 1 \rangle.$$

Proof. Lemma 1 follows from Corollary 2 in [9] and also from Theorem 4.1 in [10]. □

Lemma 2. *Let either (H_1), (H_2) or (H_3), (H_4) be fulfilled with constants $c_1 < c_2$ and let $(\varphi, \varphi') \in D$. Then the differential equation*

$$u' = Q[\varphi, \varphi', u](t) \tag{7}$$

admits a unique solution u satisfying

$$c_1 \leq u(t) \leq c_2 \quad \text{for} \quad t \in \langle 0, 1 \rangle, \quad u(1) - u(0) = 0. \tag{8}$$

Proof. Let assumption (H_1) ((H_3)) be fulfilled with constants $c_1 < c_2$. Setting $h(t,u) = Q[\varphi, \varphi', u](t)$, $\beta_j(t) = c_j$ for $(t,u) \in \langle 0,1\rangle \times \langle c_1, c_2\rangle$, $j = 1,2$, then equation (7) admits a solution u satisfying (8) by Lemma 1 with $\varepsilon = 1$ $(= -1)$.

Let (H_2) be satisfied and let u_1, u_2 be solutions of (7) satisfying (8) with $u = u_j$ $j = 1,2$, $u_1 \neq u_2$. If (5) is satisfied, then $0 \not\equiv (u_2(t) - u_1(t))' = (Q[\varphi, \varphi', u_2(t)](t) - Q[\varphi, \varphi', u_1(t)](t))\,(u_2(t) - u_1(t)) \leq 0$ for $t \in \langle 0,1\rangle$ and with regard to $u_2(0) - u_1(0) = u_2(1) - u_1(t)$ we have $u_1 = u_2$, a contrary. If (6) is satisfied, then

$$\frac{d}{dt}(u_2(t) - u_1(t))^2 \leq 2h_0(t)(u_2(t) - u_1(t))^2 \text{ for } t \in \langle 0,1\rangle,$$

hence

$$(u_2(t) - u_1(t))^2 \leq (u_2(0) - u_1(0))^2 \exp(2\int_0^t h_0(s)\,ds) \text{ for } t \in \langle 0,1\rangle.$$

In the case $u_2(0) = u_1(0)$ we obtain $u_2 = u_1$, a contrary. In the case $u_2(0) \neq u_1(0)$ we have

$$(u_2(1) - u_1(1))^2 \leq (u_2(0) - u_1(0))^2 \exp(2\int_0^t h_0(s)\,ds) < (u_2(0) - u_1(0))^2,$$

which contradicts $u_2(0) - u_1(0) = u_2(1) - u_1(1)$.

We can similarly prove that assumption (H_4) guarantees the uniqueness of problem (7) - (8).

Lemma 3. *Assume either assumptions (H_1), (H_2) or assumptions (H_3), (H_4) are fulfilled with constants $c_1 < c_2$, and assume $(\varphi, \varphi') \in D$.*
Then the equation

$$x''' = Q[\varphi, \varphi', x''](t) \tag{9}$$

admits a unique solution x satisfying (2) and

$$(x, x') \in D, \quad x'' \in H. \tag{10}$$

Proof. We can rewrite equation (9) in the form (7) with $u = x''$. With respect to Lemma 2 there exists a unique solution u of (7) satisfying (8). Setting $p(b) = \beta(b + \int_0^t u(s)\,ds)$ for $b \in \mathbf{R}$, p is continuous increasing on $\mathbf{R}$, $\lim_{b\to-\infty} p(b) < 0$, $\lim_{b\to\infty} p(b) > 0$, hence $p(b) = 0$ for a unique $b = b_0$. Set $r(c) = \alpha(c + b_0 t + \int_0^t\int_0^s u(\tau)\,d\tau\,ds)$ for $c \in \mathbf{R}$. Then r is continuous increasing on $\mathbf{R}$ and since $\lim_{c\to-\infty} r(c) < 0$, $\lim_{c\to\infty} r(c) > 0$ there exists a unique solution of the equation $r(c) = 0$, say c_0. We see $x(t) = c_0 + b_0 t + \int_0^t\int_0^s u(\tau)\,d\tau\,ds$ is a unique solution of (9) satisfying (2). Next $x(\xi) = 0 = x'(\eta)$ for some $\xi, \eta \in \langle 0,1\rangle$ because on the opposite case $\alpha(x) \neq 0$, $\beta(x') \neq 0$. Using the equalities $x'(t) = \int_\eta^t u(s)\,ds$ and $x(t) = \int_\xi^t x'(s)\,ds$, we get $\|x'\| \leq A$, $\|x\| \leq A$, consequently $(x, x') \in D$. $\square$

3. Multiple solutions for BVP (1) - (2)

Theorem 1. *Assume either assumptions (H_1), (H_2) or assumptions (H_3), (H_4) are fulfilled with constants $c_1 < c_2$. Then there exists a solution x of (1) satisfying (2) and $\|x\| \le A$, $\|x'\| \le A$, $\quad c_1 \le x''(t) \le c_2$ for $t \in \langle 0,1\rangle$.*

Proof. Let $\mathbf{Y}$ be the Banach space of C^2-functions on $\langle 0,1\rangle$ with the norm $\|x\|_2 = \|x\|+\|x'\|+\|x''\|$ for $x \in \mathbf{Y}$. Let $\kappa = \{x; (x,x') \in D, x'' \in H\}$. κ is bounded convex closed subset of $\mathbf{Y}$. According to Lemma 3, to each $\varphi \in \kappa$ there exists a unique solution x of (9) satisfying (2) and $x \in \kappa$. Setting $T(\varphi) = x$ we obtain an operator $T : \kappa \to \kappa$. To prove Theorem 1 it is sufficient to show T has a fixed point.

First we shall prove T is a continuous operator. Let $\{\varphi_n\} \subset \kappa$ be a convergent sequence, $\lim_{n\to\infty} \varphi_n = \varphi$ and let $x_n = T(\varphi_n)$, $\ x = T(\varphi)$. Then

$$x_n'''(t) = Q[\varphi_n, \varphi_n', x_n''(t)](t) \text{ for } \ t \in \langle 0,1\rangle \text{ and } \quad n \in \mathbf{N},$$
$$x'''(t) = Q[\varphi, \varphi', x''(t)](t) \text{ for } \quad t \in \langle 0,1\rangle$$

and

$$\alpha(x_n) = 0 = \beta(x_n'), \quad x_n''(1) - x_n''(0) = 0 \text{ for } n \in \mathbf{N},$$
$$\alpha(x) = 0 = \beta(x'), \ x''(1) - x''(0) = 0.$$

Let $\{\bar{x}\}$ be a subsequence of $\{x_n\}$. Since $\|x_n\| \le A$, $\ \|x_n'\| \le A$, $\ c_1 \le x_n''(t) \le c_2$, $\|x_n'''\| \le L$ for $t \in \langle 0,1\rangle$ and $n \in \mathbf{N}$, where $L = \sup\{\|Q[x,x',c]\|;\ [x,x',c] \in D \times \langle c_1, c_2\rangle\}(<\infty)$, due to the Ascoli-Arzela theorem exists a convergent subsequence $\{\tilde{x}_n\}$ of $\{\bar{x}\}$, $\lim_{n\to\infty} \tilde{x}_n = z$. One can realy verify z is a solution of the differential equation $y''' = Q[\varphi, \varphi', y''](t)$, $z \in \kappa$, $\alpha(z) = 0 = \beta(z')$, $\ z''(1) - z''(0) = 0$. By Lemma 3 the above functional boundary value problem admits a unique solution, due to the definition of T necesarily equal to x. Hence $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = x$.

Next wee see $T(\kappa) \subset \{x;\ x \in C^3(\langle 0,1\rangle), \|x^{(j)}\| \le A \text{ for } j = 0,1,2, \|x'''\| \le L\}$ with the constant L as above, hence $T(\kappa)$ is a precompact subset of $\mathbf{X}$.

This proves T is a completely continuous operator and by Schauder fixed point theorem there exists a fixed point of T in κ. □

Example 1. Consider the functional differential equation

$$x'''(t) = (t^{1/2} + x^2(t^2) - 2x''(t)) \int_0^t \cos^2(x'(s))\, ds. \tag{11}$$

Assumptions (H_1), (H_2) are fulfilled with $c_1 = 0$, $c_2 = 1$, hence by Theorem 1 with $\alpha(y) = y(0)$, $\beta(y) = y(1)$ for $y \in \mathbf{X}$ there exists a solution x of (11) satisfying

$$x(0) = 0,\ x'(1) = 0,\ x''(1) - x''(0) = 0, \|x\| \le 1, \|x'\| \le 1, 0 \le x''(t) \le 1$$
$$\text{for } t \in \langle 0,1\rangle.$$

Theorem 2. *Let $a_1, a_2 \in \mathbf{R}$, $a_1 < 0 < a_2$. Assume assumptions (H_1), (H_2) with $c_1 = 0$, $c_2 = a_2$ and assumptions (H_3), (H_4) with $c_1 = a_1$, $c_2 = 0$ are fulfilled. If $Q[0,0,0](t) \not\equiv 0$ on $\langle 0, 1\rangle$, then there exist at least two different solutions x_1, x_2 of (1) satisfying (2) with $x = x_j$ and*

$$\|x_j\| \le |a_j|,\ \|x_j'\| \le |a_j|,\ a_1 \le x''(t) \le 0 \le x_2''(t) \le a_2 \quad \textit{for } t \in \langle 0,1\rangle \quad (j = 1,2). \tag{12}$$

Proof. By Theorem 1 there exist solutions x_1, x_2 of (1) satisfying (2) and (12). If $Q[0,0,0](t) \not\equiv 0$ then $x = 0$ is not a solution of (1), hence $x_1 \ne x_2$. □

Analogously using Theorem 1 we can prove the following theorem

Theorem 3. *Let $a_1, a_2 \in \mathbf{R}$, $a_1 < 0 < a_2$. Assume assumptions (H_3), (H_4) with $c_1 = 0$, $c_2 = a_2$ and assumptions (H_1), (H_2) with $c_1 = a_1$, $c_2 = 0$ are fulfilled. If $Q[0,0,0] \not\equiv 0$ on $\langle 0, 1\rangle$, then there exist at least two different solutions x_1, x_2 of (1) satisfying (2) with $x = x_j$ and (12).*

Remark 3. If equation (1) satisfies the asssumptions of Theorem 2, then equation $x''' = -Q[x, x', x''(t)](t)$ satisfies assumptions of Theorem 3 and also vice versa.

Example 2. Consider the functional differential equation

$$x'''(t) = \varepsilon \exp\{t x'(t^2) x(\sin t)\} \ln\left(\frac{t+1}{2} + (x''(t))^2\right), \tag{13}$$

where $\varepsilon = \mp 1$. If $\varepsilon = -1$ $(\varepsilon = 1)$, then assumptions of Theorem 2 (Theorem 3) are fulfilled with $a_1 = -2^{1/2}/2$, $a_2 = 2^{1/2}/2$. Since $Q[0,0,0](t) = \varepsilon \ln((t+1)/2) \not\equiv 0$ on $\langle 0,1\rangle$, there exist solutions x_1, x_2 of (13), $x - 1 \ne x_2$ such that $\alpha(x_j) = 0 = \beta(x_j')$, $x_j''(1) - x_j''(0) = 0$, $\|x_j\| \le 2^{1/2}/2$, $\|x_j'\| \le 2^{1/2}/2$, $-2^{1/2}/2 \le x_1''(t) \le 0 \le x_2''(t) \le 2^{1/2}/2$ for $t \in \langle 0,1\rangle$ and $j = 1,2$. If for example $\alpha(x) = \int_0^1 x(s)\,ds = \beta(x)$ for $x \in \mathbf{X}$, then there exist solutions y_1, y_2 of (13), $y_1 \ne y_2$ satisfying $(j = 1,2)$

$$\int_0^1 y_j\,(s)\,ds = 0, \quad y_j(1) - y_j(0) = 0, \quad y_j''(1) - y_j''(0) = 0$$

and

$$\|y_j\| \le 2^{1/2}/2, \quad \|y_j'\| \le 2^{1/2}/2, \quad -2^{1/2}/2 \le y_1''(t) \le 0 \le y_2''(t) \le 2^{1/2}/2 \quad \text{for } t \in \langle 0,1\rangle.$$

From Remark 3, Theorem 2 and Theorem 3 it follows the following

Corollary 1. *Assume equation (1) satisfies assumptions of Theorem 2 and $Q[0,0,0]\,(t) \not\equiv 0$ on $\langle 0,1\rangle$. Then the equation*

$$x'''(t) = \lambda Q[x, x', x''(t)](t), \quad \lambda \in \mathbf{R},$$

admits for each $\lambda \ne 0$ at least two different solutions x_1, x_2, satisfying (2) with $x = x_j$ and (12).

4. Multiple solutions for BVP (3) - (4)

Since the proofs of results for BVP (3) - (4) are evident analogous to the ones for BVP (1) - (2), we state them without proofs.

Theorem 4. *Assume either (S_1), (S_2) or assumptions (S_3), (S_4) are fulfilled with constants $c_1 < c_2$. Then there exists a solution x of (3) satisfying (4) and $\|x\| \le A$, $c_1 \le x'(t) \le c_2$ for $t \in \langle 0,1\rangle$.*

Theorem 5. *Let $a_1, a_2 \in \mathbf{R}$, $a_1 < a_2$. Assume assumptions (S_1), (S_2) with $c_1 = 0$, $c_2 = a_2$ and assumptions (S_3), (S_4) with $c_1 = a_1$, $c_2 = 0$ are fulfilled. If $P[0,0](t) \not\equiv 0$ on $\langle 0,1\rangle$, then there exist at least two different solutions x_1, x_2 of (3) satisfying (4) with $x = x_j$ and*

$$\|x_j\| \le |a_j|, \quad a_1 \le x_1'(t) \le 0 \le x_2'(t) \le a_2 \ \text{for} \ t \in \langle 0,1\rangle, \quad (j = 1,2). \tag{14}$$

Corollary 2. *Assume equation (3) satisfies assumptions of Theorem 5 and $P[0,0](t) \not\equiv 0$ on $\langle 0,1\rangle$. Then the equation*

$$x''(t) = \lambda P[x, x'(t)](t), \quad \lambda \in \mathbf{R},$$

admits for each $\lambda \ne 0$ at least two different solutions x_1, x_2 satisfying (4) (with $x = x_j$) and (14).

Example 3. Consider the functional differential equation

$$x''(t) = \lambda(t - (x'(t))^2) \int_0^{t^{1/2}} (e^s + x^4(s^2))\, ds, \quad \lambda \in \mathbf{R} - \{0\}. \tag{15}$$

The assumptions of Corollary 2 are satisfied with $a_1 = -1$, $a_2 = 1$, and since $P[0,0](t) = t\int_0^{t^{1/2}} e^s\, ds \not\equiv 0$ on $\langle 0,1\rangle$, there exist solutions x_1, x_2 of (15), $x_1 \ne x_2$, such that $\alpha(x_j) = 0$, $\quad x_j'(1) - x_j'(0) = 0$, $\quad \|x_j\| \le 1$, $\quad -1 \le x_1'(t) \le 0 \le x_2'(t) \le 1$ for $t \in \langle 0,1\rangle$ and $j = 1,2$.

If for example $\alpha(x) = \sum_{j=1}^n a_j x(t_j)$ where a_j are positive constants ($j = 1,2,\dots,n$) and $0 \le t_1 < t_2 < \dots < t_{n-1} < t_n \le 1$, then there exist two different solutions y_1, y_2 of (15) satisfying $\sum_{j=1}^n a_j y_i(t_j) = 0$, $y_i'(1) - y_i'(0) = 0$, $\|y_i\| \le 1$ and $-1 \le y_1'(t) \le 0 \le y_2'(t) \le 1$ for $t \in \langle 0,1\rangle$ and $i = 1,2$.

Acknowledgement. The author is grateful to the refree for his useful suggetions, in particular for Lemma 1 and Lemma 2 and the associated references.

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