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SOME FACTORIZATIONS OF MATRIX FUNCTIONS IN SEVERAL VARIABLES

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. We establish some criteria for a nonsingular square matrix depending on several parameters to be represented in the form of a matrix product of factors which depend on the single parameters.

The purpose of the present work is to find functional and differential equations for matrix-valued functions H that admit factorization

$$H(x,y) = F(x) \cdot G(y) \tag{1}$$

or, more generally,

$$H(x_1, x_2, \dots, x_k) = F_1(x_1) \cdot F_2(x_2) \cdot \dots \cdot F_k(x_k) \ , \tag{2}$$

where $\cdot$ stands for the usual matrix multiplication. The history of the scalar version of this problem goes back to the year 1747, when J. d'Alembert [d'Al] recognized that each (smooth) scalar function $h(x,y) = f(x)g(y)$ has to satisfy the following partial differential equation

$$h_{xy}h - h_x h_y = 0 \ . \tag{3}$$

In 1904, C. Stéphanos [St] announced a significant generalization of d'Alembert's result: *scalar functions of the type*

$$h(x,y) = \sum_{k=1}^{n} f_k(x) g_k(y) \tag{4}$$

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form the space of all solutions of the partial differential equation with the "Wronskian" of order $n+1$

$$\begin{vmatrix} h & h_y & \dots & h_{y^n} \\ h_x & h_{xy} & \dots & h_{xy^n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{x^n} & h_{x^n y} & \dots & h_{x^n y^n} \end{vmatrix} = 0 \tag{5}$$

(for the more precise statement, applications, further extensions and related results see [Neu 1], [Ra], [GR], [Neu 2], [CS 1], [NR] and [CS 2]).

Suppose that matrices H, F and G in (1) are of type $n \times n$ and denote their entries by h_{ij}, f_{ij} and g_{ij}, respectively, where $i, j \in \{1, 2, \dots, n\}$. Then (1) represents a system of n^2 scalar equalities

$$h_{ij}(x, y) = \sum_{k=1}^{n} f_{ik}(x) g_{kj}(y) \ ,$$

each of them is of type (4). Consequently, the above mentioned result of Stéphanos yields a necessary (but not sufficient) condition for a (smooth) matrix function H to have factorization (1): *each entry* h_{ij} *is a solution of the Wronski equation* (5). We will show here that criteria for factorization (1) can be stated in terms of matrix operations, without taking single entries of the matrix H and without using equations like (5). However, our procedure is not applicable unless the values of H are nonsingular, i.e. $\det H(x, y) \neq 0$ for all x and y. Let us finish this introductory part by remarking that a smooth matrix H of type (1) need not satisfy the equation

$$H_{xy} \cdot H - H_x \cdot H_y = 0 \,, \tag{6}$$

a formal matrix analogy of (3). (Equation (6) holds if the matrices F and G in (1) commute, which is rather an *exceptional* case.) The correct version of (6) is given in Theorem 3 below.

Throughout the paper, $GL_n(\mathbb{K})$ denotes the group of all $n \times n$ nonsingular matrices with elements from the field $\mathbb{K}$, where $\mathbb{K}$ stands for $\mathbb{R}$ (reals) or $\mathbb{C}$ (complex numbers). First we derive a functional equation that characterizes functions (1) without any smoothness condition.

Theorem 1. *Let* $H : X \times Y \to GL_n(\mathbb{K})$, *where* X *and* Y *are arbitrary nonempty sets. Choose elements* $x_1 \in X$ *and* $y_1 \in Y$. *Then the mapping* H *has a factorization* (1) *if and only if it satisfies the functional equation*

$$H(x, y) = H(x, y_1) \cdot H^{-1}(x_1, y_1) \cdot H(x_1, y) \quad \text{for each } x \in X \text{ and } y \in Y \,. \tag{7}$$

Moreover, the factors $F : X \to GL_n(\mathbb{K})$ *and* $G : Y \to GL_n(\mathbb{K})$ *from any representation* (1) *are exactly pairs of the form*

$$F(x) = H(x, y_1) \cdot C \quad \text{and} \quad G(y) = D \cdot H(x_1, y) \,, \tag{8}$$

where $C, D \in GL_n(\mathbb{K})$ are arbitrary constant matrices satisfying $C \cdot D = H^{-1}(x_1, y_1)$.

Proof. Let H be as in (1). Setting first $y = y_1$ and then $x = x_1$ in (1), we find that

$$F(x) = H(x, y) \cdot G^{-1}(y_1) \text{ and } G(y) = F^{-1}(x_1) \cdot H(x_1, y)$$

for each $x \in X$ and $y \in Y$. Multiplying these equalities and taking in account that

$$G^{-1}(y_1) \cdot F^{-1}(x_1) = (F(x_1) \cdot G(y_1))^{-1} = H^{-1}(x_1, y_1),$$

we conclude that H satisfies (7) and (8) holds. Conversely, if H satisfies (7) and if $C, D \in GL_n(\mathbb{K})$ are arbitrary matrices satisfying $C \cdot D = H^{-1}(x_1, y_1)$, then

$$\big(H(x, y_1)\, C\big) \cdot \big(D\, H(x_1, y)\big) = H(x, y_1) \cdot H^{-1}(x_1, y_1) \cdot H(x_1, y) = H(x, y)$$

and the proof is complete.

Let us add to Theorem 1 a simple but important rule

$$\text{(9)} \qquad H \text{ is of type (1)} \Rightarrow H(x_1, y) \cdot H^{-1}(x_2, y) \text{ does not depend on } y\,,$$

which will be used in next proofs.

Now we turn our attention to the matrix functions H of type (1) which are differentiable in one of both variables, say x in Theorem 2 (for the case of the variable y see Remark 1). We show that such functions are characterized by a *mixed* functional differential equation.

Theorem 2. *Let $H : X \times Y \to GL_n(\mathbb{K})$, where X is an interval in $\mathbb{R}$ and Y is a nonempty set. Suppose that the partial derivative H_x exists at each point of $X \times Y$. Then the mapping H has a factorization* (1) *if and only if it satisfies*

$$\text{(10)} \quad H_x(x, y) \cdot H^{-1}(x, y) = H_x(x, y_1) \cdot H^{-1}(x, y_1) \text{ for each } x \in X \text{ and } y, y_1 \in Y\,.$$

Proof. (i) If H is as in (1), then

$$H_x(x, y) \cdot H^{-1}(x, y) = \big(F'(x) G(y)\big) \cdot \big(G^{-1}(y) F^{-1}(x)\big) = F'(x) \cdot F^{-1}(x)$$

for each $y \in Y$, hence the both sides of (10) are equal to $F'(x) \cdot F^{-1}(x)$.

(ii) If H satisfies (10), then

$$\begin{aligned}
&\frac{\partial}{\partial x}\big(H^{-1}(x, y_1) H(x, y)\big) = \\
&- H^{-1}(x, y_1) H_x(x, y_1) H^{-1}(x, y_1) H(x, y) + H^{-1}(x, y_1) H_x(x, y) = \\
&H^{-1}(x, y_1)\big[-H_x(x, y_1) H^{-1}(x, y_1) + H_x(x, y) H^{-1}(x, y)\big] H(x, y) = 0\,.
\end{aligned}$$

Thus $H^{-1}(x,y_1)\cdot H(x,y)$ does not depend on $x\in X$, i.e.

$$H^{-1}(x,y_1)\cdot H(x,y)=H^{-1}(x_1,y_1)\cdot H(x_1,y) \text{ for each } x\in X\,,$$

where $x_1\in X$ is a chosen point. Multiplying the last equality by $H(x,y_1)$ from the left, we obtain factorization (7).

Remark 1. The reader can easily verify that

$$\text{(11)}\qquad H^{-1}(x,y)\cdot H_y(x,y)=H^{-1}(x_1,y)\cdot H_y(x_1,y)\quad (x,x_1\in X,\ y\in Y)$$

is the analogy of (10) for functions H differentiable in the variable y.

Now we state a differential criterion of (1) for mappings H which are smooth in both variables x and y.

Theorem 3. *Let $H\colon X\times Y\to GL_n(\mathbb{K})$, where X and Y are two intervals in $\mathbb{R}$. Suppose that the partial derivatives H_x, H_y and $H_{xy}=(H_x)_y$ exist at each point of $X\times Y$. Then the mapping H has a factorization* (1) *if and only if it solves the differential equation*

$$\text{(12)}\qquad H_{xy}=H_x\cdot H^{-1}\cdot H_y\quad \textit{on the rectangle } X\times Y.$$

Proof. If H is as in (1) and the derivatives H_x and H_y exist, then (8) implies that the derivatives $F'=\frac{dF}{dx}$ and $G'=\frac{dG}{dy}$ exist too. So we can write

$$\begin{aligned}H_x\cdot H^{-1}\cdot H_y=(F'G)\cdot(FG)^{-1}\cdot(FG')&=F'GG^{-1}F^{-1}FG'=\\&=F'G'=H_{xy}\,,\end{aligned}$$

which means that H satisfies (12). Conversely, let H be such that the derivatives H_x, H_y, $H_{xy}=(H_x)_y$ exist and satisfy (12). Then the product $H_x\cdot H^{-1}$ is differentiable in y and

$$\begin{aligned}&\frac{\partial}{\partial y}\big(H_x\cdot H^{-1}\big)=\\&=H_{xy}H^{-1}-H_xH^{-1}H_yH^{-1}=(H_{xy}-H_xH^{-1}H_y)H^{-1}=0\end{aligned}$$

on the set $X\times Y$. Hence $H_x\cdot H^{-1}$ does not depend on $y\in Y$, i.e. the mapping H satisfies (10). In view of Theorem 2, H has a factorization (1).

Remark 2. In the statement of Theorem 3, the mixed derivative $(H_x)_y$ can be replaced by $(H_y)_x$, because any solution of $(H_y)_x=H_x\cdot H^{-1}\cdot H_y$ satisfies (11).

Now we will solve the problem when a smooth nonsingular matrix function H in $p+q$ variables is factorizable into the form

$$H(x_1,\dots,x_p;y_1,\dots,y_q)=F(x_1,\dots,x_p)\cdot G(y_1,\dots,y_q)\ . \tag{13}$$

Let us emphasize that if $H:(X_1\times\ldots\times X_p)\times(Y_1\times\ldots\times Y_q)\to GL_n(\mathbb{K})$, then Theorem 1 with vector variables $\boldsymbol{x}=(x_1,\dots,x_p)$ and $\boldsymbol{y}=(y_1,\dots,y_q)$ yields the following conclusion: *the factors F and G from any factorization* (13) *of the function H are given by*

$$F(x_1,\dots,x_p)=H(x_1,\dots,x_p;v_1,\dots,v_q)\cdot C$$
$$G(y_1,\dots,y_q)=D\cdot H(u_1,\dots,u_p;y_1,\dots,y_q)$$

where the elements $u_i\in X_i$ and $v_j\in Y_j$ are chosen arbitrarily and the matrices $C,D\in GL_n(\mathbb{K})$ satisfy $C\cdot D=H^{-1}(u_1,\dots,u_p;v_1,\dots,v_q)$.

Theorem 4. *Let $X=X_1\times\ldots\times X_p$ and $Y=Y_1\times\ldots\times Y_q$ be the Cartesian products of real intervals $X_1,\dots,X_p$ and $Y_1,\dots,Y_q$, respectively. Suppose that a mapping $H:X\times Y\to GL_n(\mathbb{K})$ has the partial derivatives H_{x_i}, H_{y_j} and $H_{x_iy_j}$ (in some order of differentiation) on the set $X\times Y$, $1\le i\le p$ and $1\le j\le q$. Then the mapping H has a factorization* (13) *if and only if it satisfies the system of pq differential equations*

$$H_{x_iy_j}=H_{x_i}\cdot H^{-1}\cdot H_{y_j}\ (1\le i\le p,\ 1\le j\le q) \text{ on the set } X\times Y. \tag{14}$$

Proof. Consider the *partial* functions $H_{ij}:X_i\times Y_j\to GL_n(\mathbb{K})$ defined by

$$H_{ij}(x_i,y_j)=H(x_1,\dots,x_p;y_1,\dots,y_q)$$

on condition that the other variables $x_1,\dots,x_{i-1}$, $x_{i+1},\dots,x_p$ and $y_1,\dots,y_{j-1}$, $y_{j+1},\dots,y_q$ are assumed to be fixed.

(i) If H is as in (13), then $H_{ij}(x_i,y_j)=F_i(x_i)\cdot G_j(y_j)$, where

$$F_i(x_i)=F(x_1,\dots,x_p) \text{ and } G_j(y_j)=G(y_1,\dots,y_q)\ .$$

Applying Theorem 3 (or Remark 2) to each function H_{ij}, we conclude that H satisfies (14).

(ii) Suppose that H solves (14). Then Theorem 3 (or Remark 2) implies that each partial function H_{ij} is of type (1) on the set $X_i\times Y_j$. Choose $u_1\in X_1$ and $\boldsymbol{v}\in Y$ and define a mapping $\Phi_1:X\to GL_n(\mathbb{K})$ by

$$\Phi_1(\boldsymbol{x})=H(x_1,\dots,x_p;\boldsymbol{v})\cdot H^{-1}(u_1,x_2,\dots,x_p;\boldsymbol{v})\ ,$$

for each $\boldsymbol{x}=(x_1,\dots,x_p)\in X$. According to the rule (9) applied to H_{1j}, where $1\le j\le q$, the matrix product

$$H(x_1,\dots,x_p;y_1,\dots,y_q)\cdot H^{-1}(u_1,x_2,\dots,x_p;y_1,\dots,y_q)$$

does not depend on any of the variables $y_1, \dots, y_q$, i.e. it equals to $\Phi(x_1, \dots, x_p)$. This leads to the factorization

$$H(\boldsymbol{x}; \boldsymbol{y}) = \Phi_1(\boldsymbol{x}) \cdot H(u_1, x_2, \dots, x_p; \boldsymbol{y}) ,$$

for each $\boldsymbol{x} = (x_1, \dots, x_p) \in X$ and $\boldsymbol{y} \in Y$. In the case when $p > 1$, we repeat the previous procedure to the function $\tilde{H}(x_2, \dots, x_p; \boldsymbol{y}) = H(u_1, x_2, \dots, x_p; \boldsymbol{y})$ to obtain the factorization

$$H(u_1, x_2, \dots, x_p; \boldsymbol{y}) = \Phi_2(x_2, \dots, x_p) \cdot H(u_1, u_2, x_3, \dots, x_p; \boldsymbol{y})$$

(with a chosen $u_2 \in X_2$), etc. After p repetitions we conclude that H is of the form (13) in which

$$F(x_1, \dots, x_p) = \Phi_1(x_1, \dots, x_p) \cdot \Phi_2(x_2, \dots, x_p) \dots \cdot \Phi_p(x_p)$$

and $G(y_1, \dots, y_q) = H(u_1, \dots, u_p; y_1, \dots, y_q)$. This completes the proof.

Now we start to deal with the factorization problem (2). To state an extension of Theorem 1 as Theorem 5, we introduce the following notation. Given a function $H : X_1 \times X_2 \times \dots \times X_k \to GL_n(\mathbb{K})$ and chosen k elements $u_i \in X_i$, $1 \le i \le k$, we define the k-tuple of *partial* functions $H_i : X_i \to GL_n(\mathbb{K})$ by

$$H_i(x) = H(u_1, \dots, u_{i-1}, x, u_{i+1}, \dots, u_k) \quad (x \in X_i,\ 1 \le i \le k) . \tag{15}$$

Theorem 5. *Let $H : X_1 \times \dots \times X_k \to GL_n(\mathbb{K})$, where $X_1, \dots, X_k$ are $k \ge 2$ nonempty sets. Consider partial functions (15) for a fixed k-tuple $u_1, \dots, u_k$. Then the mapping H has a factorization (2) if and only if it satisfies the equation*

$$H(\boldsymbol{x}) = H_1(x_1) \cdot H_0^{-1} \cdot H_2(x_2) \cdot H_0^{-1} \cdot \dots \cdot H_0^{-1} \cdot H_k(x_k) \quad \text{for any } \boldsymbol{x} = (x_1, \dots, x_k) \in X_1 \times \dots \times X_k , \tag{16}$$

where $H_0 = H(u_1, \dots, u_k)$. Moreover, the factors $F_i : X_i \to GL_n(\mathbb{K})$ from any factorization (2) are given by

$$F_1(x) = H_1(x) \cdot C_1,\ F_i(x) = D_{i-1} \cdot H_i(x) \cdot C_i \quad (1 < i < k) \quad \text{and } F_k(x) = D_{k-1} \cdot H_k(x) , \tag{17}$$

where $C_i, D_i \in GL_n(\mathbb{K})$ are arbitrary constant matrix satisfying

$$C_1 \cdot D_1 = C_2 \cdot D_2 = \dots = C_{k-1} \cdot D_{k-1} = H_0^{-1}. \tag{18}$$

Proof. If H is as in (2), then one can check inductively that

$$H_1(x_1) H_0^{-1} \dots H_0^{-1} H_i(x_i) = F_1(x_1) \dots F_i(x_i) F_{i+1}(u_{i+1}) \dots F_k(u_k),$$

for each $i=2,3,\ldots,k$. This equality with $i=k$ proves (16). Conversely, if the mapping H satisfies (16), then it is clearly of type (2), with factors $F_1=H_1$ and $F_i=H_0^{-1}H_i$, $i=2,3,\ldots,k$ (as well as with factors (17) under condition (18)). So it remains to show that the factors F_i from any factorization must be of type (17). Indeed, it follows from (2) that $F_1(x)=H_1(x)H_0^{-1}F_1(u_1)$,

$$F_i(x) = F_{i-1}^{-1}(u_{i-1})F_{i-2}^{-1}(u_{i-2})\ldots F_1(u_1)^{-1}H_i(x)H_0^{-1}F_1(u_1)F_2(u_2)\ldots F_i(u_i)$$

for $i=2,3,\ldots,k-1$ and $F_k(x)=F_{k-1}^{-1}(u_{k-1})\ldots F_1(u_1)^{-1}H_k(x)$. So F_i are of type (17), with matrices

$$C_i=H_0^{-1}F_1(u_1)F_2(u_2)\ldots F_i(u_i) \text{ and } D_i=F_i^{-1}(u_i)F_{i-1}^{-1}(u_{i-1})\ldots F_1^{-1}(u_1)$$

that satisfy (18). The proof is complete.

Comparing (16) with (7), the reader may analogously presume that

$$H_{x_1x_2\ldots x_k} = H_{x_1}\cdot H^{-1}\cdot H_{x_2}\cdot H^{-1}\cdot\ldots\cdot H^{-1}\cdot H_{x_k} \tag{19}$$

is a *good* generalization of (12) for the factorization problem (2). We disprove this conjecture in the following

Theorem 6. *Let $X_1,\ldots,X_k$ be $k\geq 2$ intervals in $\mathbb{R}$ and let the mapping $F_i : X_i\to GL_n(\mathbb{K})$ be differentiable at each point of X_i, $1\leq i\leq k$. Then the mapping H defined by (2) is a solution of (19) on the set $X_1\times\ldots\times X_k$. However, in the case when $k\geq 3$, equation (19) has such solutions which are not of the form (2).*

Proof. If H is as in (2), with differential factors F_i, then one can check inductively that

$$H_{x_1}H^{-1}H_{x_2}H^{-1}\ldots H^{-1}H_{x_i}H^{-1} = F_1'F_2'\ldots F_i'F_{i-1}^{-1}F_{i-2}^{-1}\ldots F_1^{-1}$$

for $i=1,2,\ldots,k-1$ and, in the last step,

$$H_{x_1}H^{-1}H_{x_2}H^{-1}\ldots H^{-1}H_{x_k} = F_1'F_2'\ldots F_k' = H_{x_1x_2\ldots x_k}.$$

Hence H solves (19). On the other side, a smooth mapping $H=H(x_2,\ldots,x_k)$ (which does not depend on x_1) is an example of a solution of (19), which is not of type (2) in general (provided that $k\geq 3$). The proof is complete.

We finish our paper by showing that the factorization problem (2) can be reduced to a family of problems (1). This reduction (described in Theorem 7) enables to formulate differential criteria of factorizations (2), based on the preceding results on factorizations (1) - see Remark 3. Given a mapping $H: X_1\times\ldots\times X_k\to GL_n(\mathbb{K})$, let us introduce the families of $\binom{k}{2}$ partial functions $H_{\alpha\beta}: X_\alpha\times X_\beta\to GL_n(\mathbb{K})$, where $1\leq\alpha<\beta\leq k$, defined by

$$H_{\alpha\beta}(x_\alpha,x_\beta) = H(x_1,\ldots,x_k) \quad \text{for any } x_\alpha\in X_\alpha \text{ and } x_\beta\in X_\beta \tag{20}$$

on condition that the other variables $x_i\in X_i$ ($1\leq i\leq k$, $i\neq\alpha$ and $i\neq\beta$) are assumed to be fixed.

Theorem 7. *Let $H : X_1 \times \ldots \times X_k \to GL_n(\mathbb{K})$, where $X_1, \ldots, X_k$ are $k \geq 3$ nonempty sets. The mapping H has a factorization (2) if and only if each partial function $H_{\alpha\beta}$ $(1 \leq \alpha < \beta \leq k$, see (20)) is of the form*

$$H_{\alpha\beta}(x_\alpha, x_\beta) = \Phi(x_\alpha) \cdot \Psi(x_\beta) \quad (x_\alpha \in X_\alpha \text{ and } x_\beta \in X_\beta)\,, \tag{21}$$

for each $(k-2)$-tuple of the other variables $x_i \in X_i$ $(1 \leq i \leq k,\ i \neq \alpha$ and $i \neq \beta)$.

Proof. We will proceed in a similar way as in the proof of Theorem 4. If H is as in (2) and $1 \leq \alpha < \beta \leq k$, then (21) holds with

$$\Phi(x_\alpha) = F_1(x_1) \cdot \ldots \cdot F_\alpha(x_\alpha) \text{ and } \Psi(x_\beta) = F_{\alpha+1}(x_{\alpha+1}) \cdot \ldots \cdot F_k(x_k)\,.$$

Conversely, suppose that each $\binom{k}{2}$-tuple of partial functions $H_{\alpha\beta}$ of a given function H satisfies (21). Choose fixed elements $u_i \in X_i$, $1 \leq i \leq k-1$. In view of the rule (9) applied to each $H_{\alpha\beta}$, the relations

$$F_i(x_i) = H(u_1, \ldots, u_{i-1}, x_i, \ldots, x_k) \cdot H^{-1}(u_1, \ldots, u_i, x_{i+1}, \ldots, x_k) \ (1 \leq i \leq k-1)$$

determine $k-1$ mappings $F_i : X_i \to GL_n(\mathbb{K})$ (in a correct way). Moreover, the identity $F_1(x_1) \cdot \ldots \cdot F_i(x_i) = H(x_1, \ldots, x_k) \cdot H^{-1}(u_1, \ldots, u_i, x_{i+1}, \ldots, x_k)$ holds for $i = 1, 2, \ldots, k-1$. So putting $F_k(x_k) = H(u_1, \ldots, u_{k-1}, x_k)$, we get factorization (2). This completes the proof.

Remark 3. A criterion for each factorization (21) can be stated by applying one of Theorems 1 – 3 (or even Theorem 4 if some of the variables $x_1, \ldots, x_k$ are multidimensional). For example, if $X_1, \ldots, X_k$ are real intervals, then (smooth) mappings $H : X_1 \times \ldots \times X_k \to GL_n(\mathbb{K})$ of type (2) form the set of all solutions of the differential system

$$H_{x_\alpha x_\beta} = H_{x_\alpha} \cdot H^{-1} \cdot H_{x_\beta} \quad (1 \leq \alpha < \beta \leq k) \text{ on } X_1 \times \ldots \times X_k\,.$$

Remark 4. The reader may ask whether the system of $\binom{k}{2}$ conditions (21) can be reduced to a subsystem, say the subsystem of $(k-1)$ conditions (21) with

$$(\alpha, \beta) \in \big\{(1,2), (2,3), \ldots, (k-1, k)\big\}\,.$$

The negative answer follows from the following example. Given indices p and q $(1 \leq p < q \leq k)$, define a mapping $H = H(x_1, \ldots, x_k) = \Phi(x_p, x_q)$, where Φ is a matrix-valued function in two variables which does not permit factorization (1). It is obvious that such a mapping H has $\binom{k}{2}$ factorizations (21) excepting the only one, that with $\alpha = p$ and $\beta = q$.

Remark 5. Let us mention an *open problem* which generalizes the subject of the present paper: *Given k surjective mappings $\varphi_i : X \to Y_i$, $1 \leq i \leq k$, find* *some necessary and sufficient conditions for a mapping $H: X \to GL_n(\mathbb{K})$ to be factorizable into*

$$H(x) = F_1(\varphi_1(x)) \cdot F_2(\varphi_2(x)) \cdot \ldots \cdot F_k(\varphi_k(x)), \tag{22}$$

with some factors $F_i : Y_i \to GL_n(\mathbb{K})$. We are able to solve it only in the case when the mapping $\varphi : X \to Y_1 \times Y_2 \times \ldots \times Y_k$ defined by

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)) \quad (x \in X)$$

is a bijection. Then (22) can be solved by transforming $\tilde{H}(y) = H(\varphi^{-1}(y))$ to a problem treated here: $\tilde{H}(y_1, y_2, \ldots, y_k) = F_1(y_1) \cdot F_2(y_2) \cdot \ldots \cdot F_k(y_k)$. As an example of this procedure, we derive functional and differential equations for matrix functions of the form

$$H(x, y) = F(x+y) \cdot G(x-y) \ . \tag{23}$$

Corollary 1. (i) *Let S be an abelian group divisible by 2. A given mapping $H : S \times S \to GL_n(\mathbb{K})$ has a factorization (23) if and only if it satisfies the equation*

$$H(x, y) = H\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \cdot H^{-1}(0,0) \cdot H\left(\frac{x-y}{2}, \frac{y-x}{2}\right)$$

for any $x, y \in S$.

(ii) *Suppose that a mapping $H : \mathbb{R} \times \mathbb{R} \to GL_n(\mathbb{K})$ has the second order differential d^2H at each point of the plane $\mathbb{R} \times \mathbb{R}$. Then H has a factorization (23) if and only if it satisfies the differential equation*

$$H_{xx} - H_{yy} = (H_x + H_y) \cdot H^{-1} \cdot (H_x - H_y) \quad \textit{on } \mathbb{R} \times \mathbb{R} \ .$$

Proof. Corollary 1 is an immediate consequence of Theorems 1 and 3 applied to the mapping

$$\tilde{H}(u, v) = H\left(\frac{u+v}{2}, \frac{u-v}{2}\right) \ .$$

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