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ON FRAME CONGRUENCES GENERATED BY FRAME TOLERANCES

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Dedicated to Professor M. Novotný on the occasion of his seventieth birthday

Abstract. Nuclei of frame congruences generated by frame tolerances and by lattice congruences are constructed.

A *frame* is a complete lattice $(L, \leq)$ in which $x \wedge \bigvee\{x_i | i \in I\} = \bigvee\{x \wedge x_i | i \in I\}$ holds for any nonempty index set I and arbitrary elements $x, x_i \in L$ $(i \in I)$. Johnstone's book [1] is our general reference about frames. A *tolerance* of the frame $(L, \leq)$ is a reflexive and symmetric relation on L compatible with operations $\wedge$ and $\vee$ induced by the order $\leq$. A transitive tolerance is said to be a *congruence*. A *frame tolerance* is a tolerance T such that $[\bigvee\{x_i | i \in I\}, \bigvee\{y_i | i \in I\}] \in T$ whenever $[x_i, y_i] \in T$ $(i \in I)$. A *frame congruence* is a transitive frame tolerance. A mapping $f\colon (L, \leq) \to (L, \leq)$ is called *extensive* if $x \leq f(x)$ for each $x \in L$, *idempotent* if $f(f(x)) = f(x)$ for each $x \in L$, and a *meet-endomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ for each $x, y \in L$. An element $x \in L$ is a *fixpoint* of f if $f(x) = x$. Recall that a tolerance (congruence) of $(L, \leq)$ is a frame tolerance (frame congruence) if and only if each of its polars possesses a top element. Further, there is a one-to-one correspondence between frame tolerances of $(L, \leq)$ and extensive meet-endomorphisms of $(L, \leq)$ such that the extensive meet-endomorphism associated with a tolerance T, which will be referred to as the *nucleus* of T, assigns to each element $x \in L$ the top element of the polar $T(x) := \{y \in L | [x, y] \in T\}$ of the element x in the frame tolerance T. A frame tolerance is a frame congruence if and only if its nucleus is idempotent. We know that frame tolerances of a frame form a frame with respect to set inclusion, the same applies to frame congruences. The operator FC assigning to any relation R the least frame congruence that includes R is a frame homomorphism (that is a join-complete lattice homomorphism) of the frame of all frame tolerances onto the frame of all frame congruences. See [2], [3] and [4] for these and further results. The aim of this paper is to provide a

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more detailed analysis how FC works. The axiom of choice is a prerequisite of our considerations.

We know that the transitive closure of a tolerance is a congruence, see [5]. Therefore the transitive closure of a frame tolerance is a (lattice) congruence, but it may not be a frame congruence.

Example 1. Let α be an arbitrary infinite ordinal, consider $L := \alpha + 1$ with the usual order. Since it is a complete chain (with the greatest element α), it is a frame. Define $P := \{[\beta, \beta+1] | \beta < \alpha\}$, and $T := P \cup \breve{P} \cup \Delta_L$, where $\breve{P}$ is the reverse relation to P and Δ_L is the identity relation on L. It is obvious that T is a tolerance of $(L, \leq)$. As $T(\beta) = \{\beta - 1, \beta, \beta + 1\}$ for $0 < \beta < \alpha$, β being a successor ordinal, $T(\beta) = \{\beta, \beta + 1\}$ for $0 < \beta < \alpha$, β being a limit ordinal, $T(0) = \{0, 1\}$, $T(\alpha) = \{\alpha - 1, \alpha\}$, α being a successor ordinal, and $T(\alpha) = \{\alpha\}$, α being a limit ordinal, the polar of each element of L contains a greatest element, and we may conclude that T is a frame tolerance. Now, the polar of the element 0 in the smallest frame congruence $\mathrm{FC}(T)$ that includes T contains with each element $\beta \neq \alpha$ its successor, and simultaneously it has a greatest element. It is clear that this greatest element is exactly α, and $\mathrm{FC}(T)$ is the universal relation on L. In contrast, the polar of the element 0 in the smallest congruence $\Theta(T)$ that includes T consists of only (at most) finite ordinal numbers.

This example was not chosen by chance. In a similar manner we can "construct" frame congruences generated by frame tolerances in the general case. This fact follows from the well-known fixpoint theorems. We will utilize the construction of least fixpoints as "limits" of monotone transfinite sequences.

Lemma 1. *Let $(L, \leq)$ be a frame, $x \in L$ and $f \colon L \to L$ an extensive mapping. Then there exists precisely one order-preserving mapping f_x from the class of all ordinals to L such that $f_x(0) = x$, $f_x(\beta) = f(f_x(\beta - 1))$ if β is a successor ordinal, and $f_x(\beta) = \bigvee\{f_x(\gamma) | \gamma < \beta\}$ if β is a limit ordinal.*

Proof. Follows immediately from the transfinite recursion theorem.

Lemma 2. *Let $(L, \leq)$ be a frame, $x \in L$, and $f \colon L \to L$ an extensive mapping. Then there exists an ordinal β_x such that $f_x(\gamma) = f_x(\beta_x)$ whenever $\beta_x \leq \gamma$. Moreover, if f is order-preserving, $f_x(\beta_x)$ is the least fixpoint of the mapping f in the principal dual ideal $\langle x)$.*

Proof. Straightforward and therefore omitted.

Definition. The ordinal β_x just defined is said to be the *closure ordinal* of f in x.

Next we will investigate some properties of the mapping f^β defined by $f^\beta(x) := f_x(\beta)$, where f is an extensive meet-endomorphism of the frame $(L, \leq)$.

Lemma 3. *Let $(L, \leq)$ be a frame, let $f \colon L \to L$ be an extensive meet-endomorphism and β be an ordinal. Then the mapping f^β is an extensive meet-endomorphism of the frame $(L, \leq)$ and each fixpoint of f is also a fixpoint of f^β.*

Proof. We first show by transfinite induction on β that f^β is a meet-endomorphism for all ordinals β. For $\beta = 0$ we have $f^0(x) \wedge f^0(y) = f_x(0) \wedge f_y(0) = x \wedge y = f_{x\wedge y}(0) = f^0(x \wedge y)$. Suppose the assertion is valid for all $\gamma < \beta$. If β is a successor ordinal, we have $f^\beta(x) \wedge f^\beta(y) = f_x(\beta) \wedge f_y(\beta) = ff_x(\beta - 1) \wedge ff_y(\beta - 1) = f(f_x(\beta-1) \wedge f_y(\beta-1)) = f(f^{\beta-1}(x) \wedge f^{\beta-1}(y)) = ff^{\beta-1}(x \wedge y) = ff_{x\wedge y}(\beta-1) = f_{x\wedge y}(\beta) = f^\beta(x \wedge y)$, using the definition of f^β, the definition of $f_-(\beta)$, the meet-compatibility of f, the definition of $f^{\beta-1}$, the induction hypothesis, the definition of $f^{\beta-1}$, the definition of $f_-(\beta)$ and the definition of f^β. If β is a limit ordinal, we obtain $f^\beta(x) \wedge f^\beta(y) = f_x(\beta) \wedge f_y(\beta) = \bigvee\{f_x(\gamma)|\gamma < \beta\} \wedge \bigvee\{f_y(\gamma)|\gamma < \beta\} = \bigvee\{f_x(\gamma_1) \wedge f_y(\gamma_2)|\gamma_1, \gamma_2 < \beta\} = \bigvee\{f_x(\max(\gamma_1,\gamma_2)) \wedge f_y(\max(\gamma_1,\gamma_2))|\gamma_1,\gamma_2 < \beta\} = \bigvee\{f_x(\gamma) \wedge f_y(\gamma)|\gamma < \beta\} = \bigvee\{f^\gamma(x) \wedge f^\gamma(y)|\gamma < \beta\} = \bigvee\{f^\gamma(x \wedge y)|\gamma < \beta\} = \bigvee\{f_{x\wedge y}(\gamma)|\gamma < \beta\} = f_{x\wedge y}(\beta) = f^\beta(x \wedge y)$, using the definition of f^β, the definition of $f_-(\beta)$, the infinite distributivity, the monotonicity of f_x and f_y, the definition of f^γ, the induction hypothesis, the definition of f^γ, the definition of $f_{x\wedge y}$ and the definition of f^β. This completes the induction argument. Extensivity of f^β is obvious because $f^\beta(x) = f_x(\beta) \geq f_x(0) = x$. Now let z be a fixpoint of f. We show by transfinite induction that z is a fixpoint of f^β for all β. For $\beta = 0$ clearly $f^0(z) = f_z(0) = z$. Suppose $f^\gamma(z) = z$ for all ordinals $\gamma < \beta$. If β is a successor ordinal, $f^\beta(z) = f_z(\beta) = ff_z(\beta-1) = ff^{\beta-1}(z) = f(z) = z$. If β is a limit ordinal, $f^\beta(z) = f_z(\beta) = \bigvee\{f_z(\gamma)|\gamma < \beta\} = \bigvee\{f^\gamma(z)|\gamma < \beta\} = \bigvee\{z|\gamma < \beta\} = z$.

Notation. The set of all fixpoints of f will be denoted by $\mathrm{fix}(f)$.

Proposition 1. *Let $(L, \leq)$ be a frame, $f\colon L \to L$ an extensive meet-endomorphism of $(L, \leq)$. Define $g(x) := \min(\mathrm{fix}(f) \cap \langle x))$. Then g is the least idempotent extensive meet-endomorphism of $(L, \leq)$ such that $f \leq g$.*

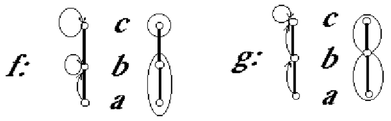
Proof. We first show that g is indeed an idempotent extensive meet-endomorphism of the frame $(L, \leq)$. The closure ordinal of f in x will be denoted by β_x. There exists an ordinal α such that $\beta_x \leq \alpha$ for each $x \in L$ because β_x-s form a set of ordinals. So we have $f^\alpha(x) = f_x(\alpha) = f_x(\beta_x) = \min(\mathrm{fix}(f) \cap \langle x))$. Hence $g(x) = f^\alpha(x)$. By the preceding lemma, g is an extensive meet-endomorphism. As $g(x)$ is a fixpoint of f for each $x \in L$, it is also a fixpoint of $f^\alpha = g$. Consequently g is idempotent. Clearly $f(x) \leq g(x)$. If h is also an idempotent extensive meet-endomorphism of $(L, \leq)$ such that $f \leq h$, then $x \leq h(x) \leq f(h(x)) \leq h(h(x)) = h(x)$, hence $h(x)$ is a fixpoint of f in $\langle x)$. Therefore $g(x) \leq h(x)$.

Consider the set $\mathrm{Ext}(L, \leq)$ of all extensive meet-endomorphisms of the frame $(L, \leq)$. For $f, g \in \mathrm{Ext}(L, \leq)$ define $L^f(g) := fg$. We know that $\mathrm{Ext}(L, \leq)$ is a frame with respect to the pointwise order (cf. [1]) and it is easy to see that L^f is an extensive meet-endomorphism of $\mathrm{Ext}(L, \leq)$. Hence there exists a least fixpoint of L^f in $\langle f)$ according to lemma 2. We can thus rewrite the statement of the preceding proposition as follows.

Corollary. *Let f be the nucleus of a frame tolerance T of $(L, \leq)$. Then the least fixpoint of L^f in $\langle f)$ is identical with the nucleus of* $\mathrm{FC}(T)$.

Note that fixpoints of L^f in $\langle f)$ are not idempotent in general.

Example 2. Consider the following situation:



Both f and g are nuclei, $f \leq g$, $fg = g$, $gg \neq g$.

Conversely, idempotents are obviously fixpoints of L^f.

In view of Proposition 1, we can construct $\mathrm{FC}(T)$ as follows. First, take the transitive closure of T, which is equal to $\Theta(T)$. Second, find congruence blocks of $\Theta(T)$ with top elements. Third, to each $x \in L$ assign the least top element $\tilde{x}$ of a block such that $x \leq \tilde{x}$. Fourth, take the quotient congruence of $\tilde{}$. It is $\mathrm{FC}(T)$. More precisely, $\mathrm{FC}(T) = \{[x,y] | \tilde{x} = \tilde{y}, \tilde{x} = \bigwedge\{t \in \langle x) | [s,t] \in \Theta(T) \implies s \leq t\}, \tilde{y} = \bigwedge\{t \in \langle y) | [s,t] \in \Theta(T) \implies s \leq t\}, \Theta(T) = \{[s,t] | \exists n \in \mathbb{N}\, \exists s_0, \ldots, s_n \in L, s = s_0, t = s_n, [s_{i-1}, s_i] \in T (i = 1, \ldots, n)\}\}$.

Not every congruence of a frame is a transitive closure of a frame tolerance.

Example 3. Take $L := \omega_1 + 1$. It is a frame. Let $C := \{[\omega_1, \omega_1]\} \cup \{[\alpha, \beta] | \alpha < \omega_1, \beta < \omega_1\}$. It is a congruence of $(L, \leq)$ with precisely two blocks ω_1 and $\{\omega_1\}$, which is not a frame congruence because ω_1 fails to have a top element. Suppose T is a frame tolerance on $(L, \leq)$ such that $C = \Theta(T)$, let f be its nucleus. Then for any $\alpha < \omega_1$ there exists $n < \omega$ such that $\alpha < f^n(0)$, hence ω_1 possesses a cofinal subset of type ω, which is a contradiction. (We of course accept the axiom of choice.)

Nevertheless, we are still able to show that the preceding construction works also in this more general case.

Definition. Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. We say that an element $x \in L$ is *C-good* if it is a top element of a block of C.

The set of all C-good elements forms a closure system in $(L, \leq)$.

Lemma 4. *Let $(L, \leq)$ be a frame, C a congruence on $(L, \leq)$. Then the top element in $(L, \leq)$ is C-good, and if x_i $(i \in I)$ are C-good, $\bigwedge\{x_i | i \in I\}$ is C-good as well.*

Proof. The top element $\top$ in L is obviously a top element in the block $\left[\top\right]_C$. Suppose that x_i $(i \in I)$ are C-good, that is x_i is a top element in $\left[x_i\right]_C$ for every index $i \in I$. But then $x_i \vee x \in \left[x_i\right]_C$ whenever $x \in \left[\bigwedge\{x_i | i \in I\}\right]_C$. This yields that $\bigwedge\{x_i | i \in I\}$ is a top element in $\left[\bigwedge\{x_i | i \in I\}\right]_C$.

Notation. The least C-good element of $(L, \leq)$ in $\langle x)$ will be denoted by $\nu_C(x)$.

Lemma 5. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$, $x, y \in L$, $x \leq y$, x being C-good. Then $z \leq \nu_C(y)$, $y \wedge z = x$ implies $z = x$.*

Proof. For each $a \in L$ put $A(a) := \{b \in L | a < b, [a,b] \in C\}$ if a is not C-good, and $A(a) := \{a\}$ otherwise. Let f be a choice function associated with A. It is obviously extensive, and its fixpoints are precisely C-good elements. We know that there exists a transfinite sequence $\{y_\beta\}$ and an element $\tilde{y}$ such that $y_0 = y$, $y_\beta = f(y_{\beta-1})$ if β is a successor ordinal, $y_\beta = \bigvee\{y_\gamma | \gamma < \beta\}$ if β is a limit ordinal, and $y_\beta = \tilde{y}$ for every ordinal β greater than the closure ordinal β_y of f in y. We first show by transfinite induction that $z \wedge y_\beta = x$ for any ordinal β. Since $y_0 = y$, $z \wedge y_0 = z \wedge y = x$. Now, suppose β is the least ordinal with $z \wedge y_\beta > x$. If β is a successor ordinal, we have $y_{\beta-1} < y_\beta$, and $[y_{\beta-1}, y_\beta] \in C$. Therefore $x < z \wedge y_\beta$ and $[x, z \wedge y_\beta] \in C$, which is a contradiction. If β is a limit ordinal, we have $y_\beta = \bigvee\{y_\gamma | \gamma < \beta\}$, hence $z \wedge y_\beta = z \wedge \bigvee\{y_\gamma | \gamma < \beta\} = \bigvee\{z \wedge y_\gamma | \gamma < \beta\} = \bigvee\{x | \gamma < \beta\} = x$, which is again a contradiction. This completes the induction argument. It is easy to see that $y_\beta \leq \nu_C(y)$ for all ordinals β, thus $\tilde{y} \leq \nu_C(y)$. The element $\tilde{y}$ is, however, a fixpoint of f, and therefore C-good. We may conclude that $\tilde{y} = \nu_C(y)$. It follows that $x = z \wedge \tilde{y} = z \wedge \nu_C(y) = z$.

Proposition 2. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. Then ν_C is an extensive meet-endomorphism of $(L, \leq)$.*

Proof. Extensivity is obvious. To show that ν_C is a meet-endomorphism, we shall use the preceding lemma twice. Clearly $\nu_C(a \wedge b) \leq \nu_C(a) \wedge \nu_C(b)$. Denote $a' = \nu_C(a \wedge b) \vee a$, $b' = \nu_C(a \wedge b) \vee b$. Then $\nu_C(a') = \nu_C(a)$, $\nu_C(b') = \nu_C(b)$ and $a' \wedge b' = \nu_C(a \wedge b)$. First, $\nu_C(a \wedge b) \leq b'$, $\nu_C(b) \wedge a' \leq \nu_C(b) = \nu_C(b')$ and $(\nu_C(b) \wedge a') \wedge b' = \nu_C(b) \wedge \nu_C(a \wedge b) = \nu_C(a \wedge b)$. By the lemma, $\nu_C(b) \wedge a' = \nu_C(a \wedge b)$. Second, $\nu_C(a \wedge b) \leq a'$, $\nu_C(a) \wedge \nu_C(b) \leq \nu_C(a) = \nu_C(a')$, $\nu_C(a) \wedge \nu_C(b) \wedge a' = a' \wedge \nu_C(b) = \nu_C(a \wedge b)$. By the lemma, $\nu_C(a) \wedge \nu_C(b) = \nu_C(a \wedge b)$.

Theorem. *Let $(L, \leq)$ be a frame, C a congruence of $(L, \leq)$. Then ν_C is the nucleus of* FC(C).

Proof. Immediate and therefore omitted.

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