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DIRECT FACTORS OF MULTILATTICE GROUPS, II

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Dedicated to Professor F. Šik on the occasion of his seventieth birthday

Abstract. Subgroups of a directed distributive multilattice group G are characterized which are direct factors of G. The main result is formulated in Theorem 2.

1. Basic notions and informations

This note is a supplement to the paper [1]. Its result is a corollary of Theorem 1.1 [1].

Let $\mathcal{P} = (P; \leqq)$ be a partially ordered set (p. o. set), A subset $A \subset P$ is said to be convex if $a, b \in A$, $c \in P$ and $a \leqq c \leqq b$ imply $c \in A$. A is connected if for each $a, b \in A$ there is a sequence $a = x_0, x_1, \dots, x_n = b, x_i \in A$, such that x_i and x_{i+1} are comparable for each $i \in \{0, 1, \dots, n-1\}$.

Given $a, b \in P$, denote $(a] = \{x \in P : x \leqq a\}$, $[a) = \{x \in P : a \geqq x\}$, $[a, b] = (a] \cap (b]$, $1(a, b) = (a] \cap (b]$ and $u(a, b) = [a) \cap [b)$. P is said to be directed if for any $a, b \in P$ the sets $1(a, b)$ and $u(a, b)$ are not empty. Call P a multilattice [2] if for any $a, b, c \in P$ such that $c \in u(a, b)$ the set $u(a, b) \cap (c]$ has a minimal element and dually for $c \in 1(a, b)$. Denote by $a \vee b$ the set of all minimal elements of $u(a, b)$; $a \wedge b$ has dual meaning.

A multilattice P is said to be distributive [3] if for each $a, b, c \in P$ the relations $(a \vee b) \cap (a \vee c) \neq 0$, $(a \wedge b) \cap (a \wedge c) \neq 0$ together imply $b = c$.

A partially ordered group [4] (p.o. group) $\mathcal{G} = (G; +, \leqq)$ is said to be a multilattice group if the p. o. set $(G; \leqq)$ is a multilattice. $\mathcal{G}$ is called distributive if the multilattice $(G; \leqq)$ is.

Let $\mathcal{G}$ be a p.o. group. We say that a subset C of G forms a direct factor of G whenever a direct product decomposition $f : G \cong A \times B$ exists such that $f^{-1}(\{(a, 0) : a \in A\}) = C$. A map $f : G \to G$ is called a retract mapping if it is an isomorphism and $f(x) = x$ for each $x \in f(G)$. The set $f(G)$ is called a retract.

In [1] the following theorem was proved (1.1 Theorem in [1]).

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Theorem 1. *Let $\mathcal{G}$ be a directed distributive multilattice group. A subset $C \subset G$ forms a direct factor of $\mathcal{G}$ iff it satisfies the following conditions*
(1) $(C;+)$ is a subgroup of $(G;+)$.
(2) C is convex and directed in $(G;+)$.
(3) for each $a \in G^+$ the set $C \cap [0,a]$ has a greatest element.

2 Main theorem

Theorem 2. *A subset C of a directed distributive multilattice group G forms a direct factor of G iff it fulfils the following conditions*
(i) C is a retract of $\mathcal{G}$
(ii) for each $a \in G^+$ the set $C \cap [0,a]$ has a greatest element.

Proof. 1. Suppose C is a direct factor of G. Then C forms a multilattice subgroup of G and there is a multilattice group $\mathcal{D}$ such that there is an isomorphism

$$f : \mathcal{G} \to \mathcal{C} \times \mathcal{D}.$$

Given $x \in G$, $f(x) = (x_1, x_2)$ where $x_1 \in C$, $x_2 \in D$. It is easy to verify that the map $x \mapsto x_1$ is a retraction map and C is a corresponding retract of $\mathcal{G}$.

2. Conversely, let C satisfy the conditions (i) and (ii). Then C trivially fulfils the conditions (1), (2), (3) of Theorem 1. Hence it is a direct factor of G.

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